Mini paper on the Kurzweil integral

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Dedicated to Peter Lax

Abstract

This paper has a missionary zeal, convincing a professional mathematician to consider replacing traditional Riemann and Lebesgue integral with the Kurzweil integral. This integral is also known under following names: Henstock, Kurzweil-Henstock, the generalized Riemann integral.

1 Basic definitions

The set of couples

$$D \equiv \{(x_i, [t_{i-1}, t_i]); i = 1 \ldots n\}$$

is called a tagged division of a compact interval $[a, b]$ if

(a) the intervals $[t_{i-1}, t_i]$ are non-overlapping,

(b) $x_i \in [t_{i-1}, t_i]$ for $i = 1 \ldots n$,

(c) $\bigcup_{i=1}^{n} [t_{i-1}, t_i] = [a, b]$.

x_i is the tag of $[t_{i-1}, t_i]$. A positive function will be called gauge. In Riemann theory the fineness of a division is measured by a positive number, the length of the largest subinterval of the division. In Kurzweil theory it is measured by a gauge.

Definition 1 *If δ is a gauge then a tagged division D is said to be δ-fine if*

$$x_i - \delta(x_i) < t_{i-1} \le x_i \le t_i < x_i + \delta(x_i) \tag{1}$$

for $i = 1 \ldots n$.

It is possible to replace inequality (1) by a more intuitive one, namely by $t_i - t_{i-1} < \delta(x_i)$ but for technical reasons we prefer (1). The next theorem can be sometimes used with advantage instead of the Borel covering theorem.

Theorem 1 (Cousin's lemma) *For any gauge δ there always exits a δ-fine tagged division of $[a,\, b]$.*

An indirect proof is easy by a bisection argument.

Definition 2 (The Kurzweil integral) *The number A is said to be the Kurzweil integral of f on $[a,\, b]$ if for every positive ε there exists a gauge δ such that for every δ-fine tagged division D we have*

$$\left| \sum_{i=1}^{n} f(x_i)(t_i - t_{i-1}) - A \right| < \varepsilon. \tag{2}$$

Cousin's lemma is needed for this definition to be meaningful. Argument usual in establishing uniqueness of limits shows that the integral is uniquely determined. If the Kurzweil integral of f exists we call f Kurzweil integrable. If f is Riemann integrable then it is obviously Kurzweil integrable and the integrals agree. We also employ the usual notation like $\int_a^b f$ or $\int_a^b f(x)\, dx$.

2 Some graphs

Given a function f and a gauge δ there is *Maple* program displaying Riemann sums for f and δ-fine tagged division. This is discussed in [4]. Here we give only examples of two such graphs. In the first example the function is Riemann integrable, nevertheless a suitable choice of a gauge shows how nicely the Riemann sums adjust to the behavior of the function.

The basic interval is $[0,\, 1]$ and the integrand f is defined as

$$f \;:=\; x \mapsto 1 - x^4.$$

We choose δ as

$$\delta \;:=\; x \mapsto \mathrm{Min}\left((0.025)^{1/4}, \frac{0.025}{\left(|x| + (0.025)^{1/4}\right)^3} \right)$$

For our second graph we choose an unbounded integrand. The basic interval is $[-1, 1]$. The integrand f is defined as follows

$$f(x) = \begin{cases} 0.1 & \text{if } x \le -1 \\ \dfrac{1}{\sqrt{1 - x^2}} & \text{if } x < 1 \\ 0.1 & \text{if } x \ge 1 \end{cases}$$

We choose the gauge to be an ε multiple of the distance to the points where the integrand is unbounded. More precisely

$$\delta(x) = \begin{cases} \varepsilon & \text{if } |x| = 1 \\ \varepsilon(1 - |x|) & \text{if } |x| < 1 \end{cases}$$

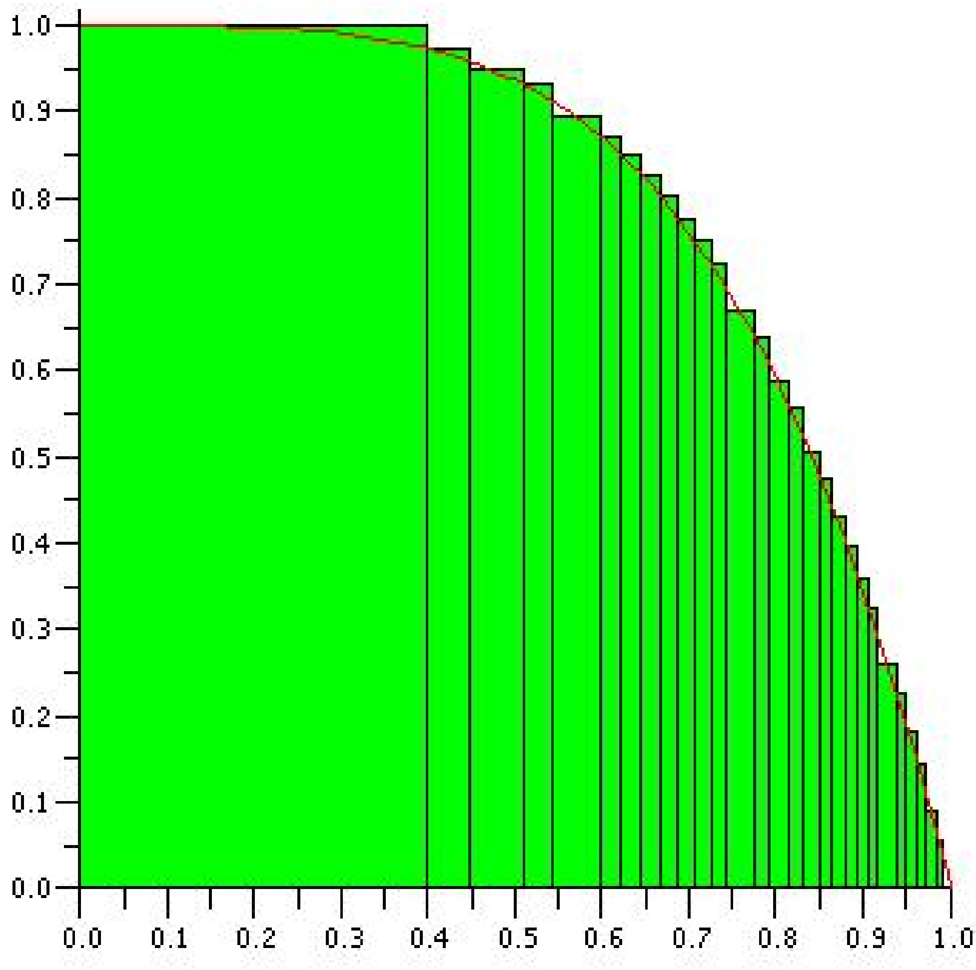

Figure 1: Riemann integrable function

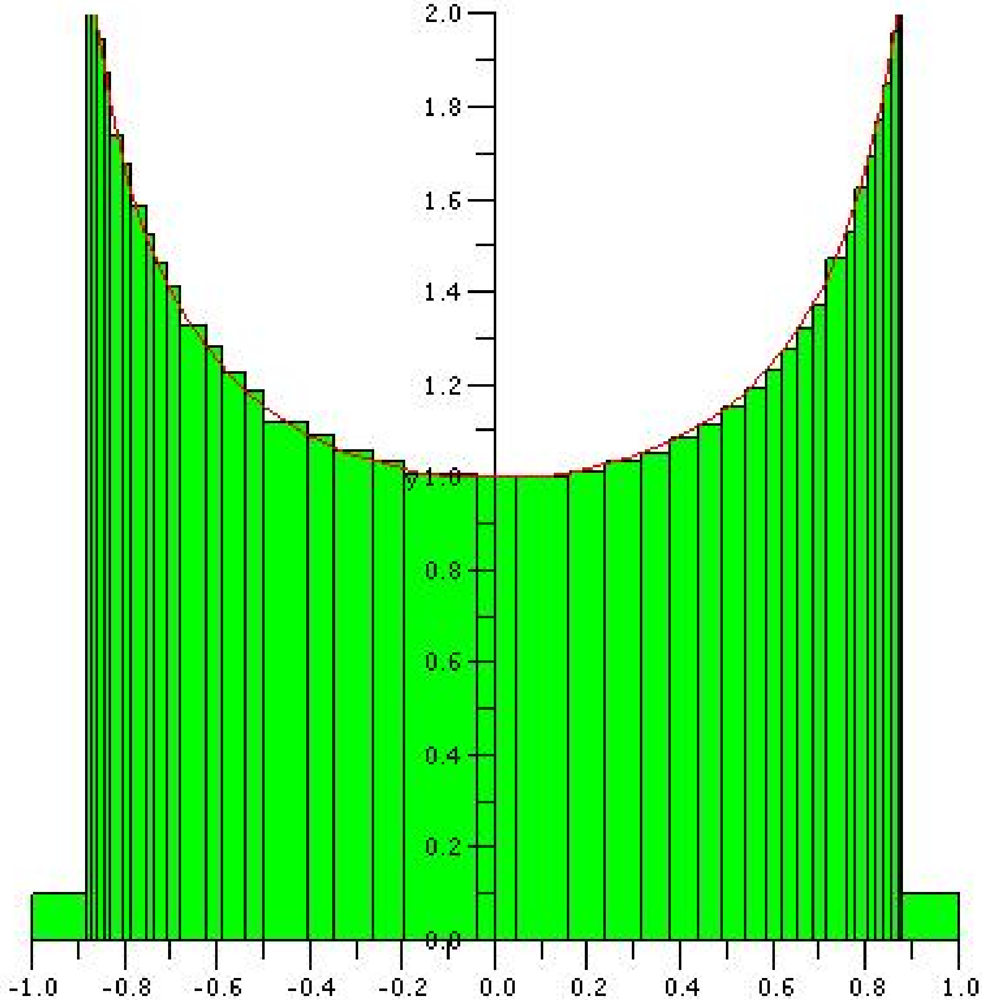

Figure 2: An unbounded function

For the display we have chosen $\varepsilon = 0.09$.

For detailed explanation for the choices of gauges, more examples and some animations we refer to web pages of Peter Adams and the author[1].

3 Two examples

Let $N = \{x; g(x) \neq 0\}$.

Example 1. If $N = \{c_1, c_2, \ldots\}$ then $\int_a^b g = 0$. For the proof it is sufficient to define the gauge δ to be 1 for $x \notin N$ and $\delta(c_n) = \dfrac{\varepsilon}{2^{n+1}|g(c_n)|}$.

Example 2. If N is of measure zero then g is K-integrable and $\int_a^b g = 0$. We additionally assume that g is bounded, $|g(x)| < L$. Let O be an open set, $N \subset O$ with the measure $m(O) < \dfrac{\varepsilon}{L}$. Let F be a complement of O and $d(x) = \operatorname{dist}(x, F)$. To complete the proof it is sufficient to define $\delta(x) = 1$ for $x \notin N$ and $\delta(x) = \dfrac{d(x)}{2}$ otherwise. The proof in the general case is not much harder.

4 Fundamental Theorem

It is remarkable that the next theorem needs no preparation.

Theorem 2 (Fundamental Theorem) *If $F'(x)$ exists for every $x \in [a, b]$ then*

$$\int_a^b F' = F(b) - F(a). \tag{3}$$

Proof: For every ε positive and $x \in [a, b]$ there exists a gauge δ such that

$$|F(t) - F(x) - F'(x)(t - x)| < \varepsilon|t - x| \tag{4}$$

whenever $|t - x| < \delta(x)$. Write[2]

$$F(b) - F(a) = \sum_{i=1}^{n}[F(t_i) - F(x_i) + \sum_{i=1}^{n}[F(x_i) - F(t_{i-1}) \tag{5}$$

$$F'(x_i)(t_i - t_{i-1}) = F'(x_i)(t_i - x_i) + F'(x_i)(x_i - t_{i-1}). \tag{6}$$

If D is δ-fine then by (4)

$$\left|\sum_1^n \left[F'(x_i)(t_i - t_{i-1})\right] - \big(F(b) - F(a)\big)\right| < \varepsilon \sum_1^n (t_i - x_i) + \varepsilon \sum_1^n (x_i - t_{i-1}) = \varepsilon(b - a). \tag{7}$$

[1] www.maths.uq.edu.au/˜pa www.maths.uq.edu.au/˜rv

[2] For convenience sake we assume $a = t_0 < t_1 \ldots t_n = b$.

Since no assumption beyond the mere existence of F' is needed this version of the Fundamental Theorem makes it possible to prove *very simply* extensions of several classical theorems in elementary analysis. Two examples are the Bernoulli-Hospital rule and the remainder in the Taylor formula. Combining the ideas of the proofs of the Theorem 2 and Example 1 leads to the extension of Theorem 2 in which it is assumed that F is continuous in $[a, b]$ and F' exists except a countable set. Similarly, combining the ideas of the proofs of the Theorem 2 and Example 2 leads to a theorem which asserts that equation (3) holds for the Kurzweil integral if F is absolutely continuous. Consequently if f is Lebesgue integrable then it is Kurzweil integrable and the integrals agree.

5 Extension of the Definition and basic Theorems

The modification of Definition 1 for infinite intervals reads

Definition 3 *The number A is called the Kurzweil integral of f over an interval I if for every positive ε there exist a gauge δ and a compact interval K such that for every δ-fine tagged division D of a bounded interval $[a, b]$ with $K \subset [a, b] \subset I$ the inequality (2) holds.*

The proofs of basic theorems are easy and similar to the proofs in Riemann theory. The Lebesgue theory is so important because it has powerful convergence theorems which are easily applicable. However, the monotone convergence theorem and the dominated convergence theorem can be proved for the K-integral and the difficulty is comparable, if not smaller, than for the proofs in Lebesgue theory. As soon as the monotone convergence is established it is easy to prove that the indefinite Kurzweil integral of an absolutely integrable function is absolutely continuous. Consequently a function f is Lebesgue integrable if and only if both f and $|f|$ are Kurzweil integrable. The Kurzweil integral contains Lebesgue integral as a special case and also the Riemann improper integral as well.

The definition in several dimension translates almost verbatim from one dimension. Naturally, some proofs are not as easy. The Fubini theorem can be proved but some important theorems, like the change of variables theorem, require absolute integrability. However, the fact that the definition is based on Riemann sums is an advantage. For example, the following theorem has a relatively easy proof.

6 Some applications

Theorem 3 (Green's Theorem) *Let ∂K be the positively oriented boundary of*

$$K = [a, b] \times [c, d]$$

and K° its interior, i.e. $K^\circ = (a, b) \times (c, d)$. If

(i) *P, Q are continuous on K,*

(ii) *the set $S \subset K$ is countable, the functions P, Q are differentiable in $K^\circ \setminus S$,*

(iii) *the function $Q_x - P_y$ is absolutely KH-integrable on K*

then

$$\int_{\partial K} P(x,y)\,dx + Q(x,y)\,dy = \iint_K \big(Q_x(x,y) - P_y(x,y)\big)\,dxdy.$$

It is worth noting that no assumption has been made regarding continuity of the partial derivatives. This Theorem can be used to prove the homotopy version of the Cauchy Theorem *without* the assumption of continuity of the derivative. However, a direct proof using methods of Kurzweil's integration is also possible and not difficult. Example of a result which goes beyond Lebesgue theorem is: A K-integrable function f has a Fourier series and this series is Abel summable a.e. to f.

7 Conclusion

For proofs and details we refer to [1] or [3] where further references can be found. For instance the result concerning Abel summability is Theorem 7.5.2 of [1]. A treatment of the Kurzweil integral at very elementary level accessible to undergraduates is in [2].

References

[1] Lee Peng Yee Rudolf Výborný *The Integral: An Easy Approach after Kurzweil and Henstock* Cambridge University Press, 2000

[2] P. Adams K. Smith R. Výborný *Introduction to Mathematics with Maple* World Scientific 2004

[3] Robert G. Bartle *A Modern Theory of Integration* AMS 2001

[4] P.Adams R. Výborný *Maple tools for the Kurzweil integral* Matematica Bohemica. **131.** No. 4, (2006), 337–346.