A new definition of the Kurzweil integral

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Outline

History

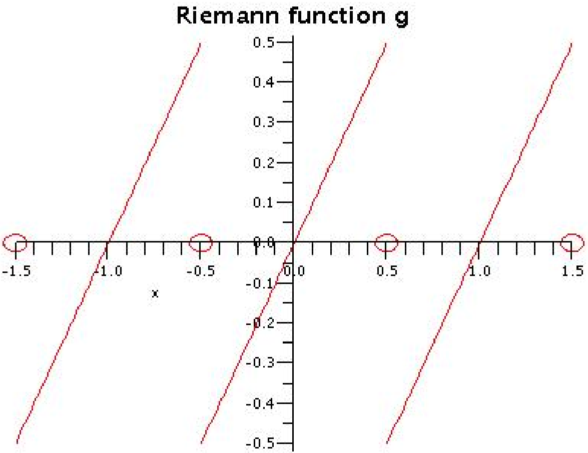
- B. Riemann 1867 [3], [2]
- H. Lebesgue 1901 [5]
- J.Kurzweil 1956 [6]

Riemann theory

- created a class of integrable functions
- characterized this class
- great success, still survives in classrooms

Riemann's example

- integrable function discontinuous at a dense subset, namely at points of the form $\frac{p}{2q}$ with p and $2q$ relatively prime
- $$f = \sum_{1}^{\infty} \frac{g(nx)}{n^2}$$


Riemann function g
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
x

Volterra's example

- function f differentiable everywhere
- f' bounded
- $$\int_a^b f' \neq f(b) - f(a)$$

Lebesgue theory

For all bounded functions

L1 For all a, b, h

$$\int_a^b f(x)\,dx = \int_{a+h}^{b+h} f(x-h)\,dx;$$

L2 For all a, b, c

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^a f(x)\,dx = 0;$$

L3

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx;$$

Lebesgue

L4 If $f \geq 0$ and $b \geq a$

$$\int_a^b f(x)\,dx \geq 0;$$

L5

$$\int_0^1 1\,dx = 1;$$

L6 For any convergent sequence $n \mapsto f_n$, tending to f, satisfying $f_1 \leq f_2 \leq \cdots$,

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Lebesgue

- Succeeding with *L1–L6* but not for all bounded functions, measurable, summable.
- Lebesgue said that he wanted to create theory in which Volterra’s example is impossible
- Lebesgue integral soon influenced many other branches of mathematics
- Is now THE INTEGRAL of the professional mathematician
- Acceptance slow in teaching, Wiener Cambridge 1931, UQ Honours 1968.

Kurzweil's integral

- In 1957 Kurzweil introduced generalized solution to (some) ODEs, for
$$y' = f(x)$$
this gave a new definition of an integral of f
- The great advantage of the Kurzweil theory is that the intuitive geometrical appeal of Riemann's theory is preserved but the theory has the Lebesgue 'power'.
- In fact Kurzweil theory contains Lebesgue theory as a special case.
- Every derivative is integrable.

Riemann approximation

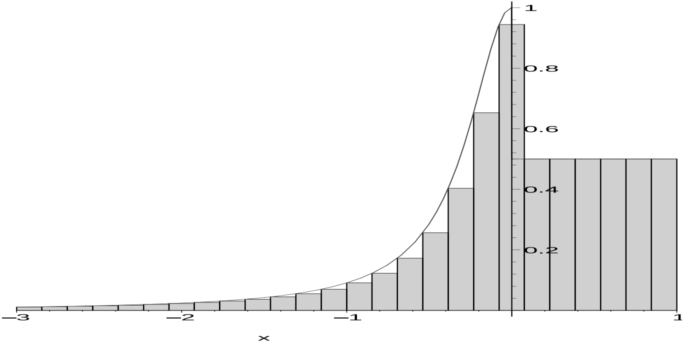


Figure: Typical Riemann approximation

Kurzweil approximation

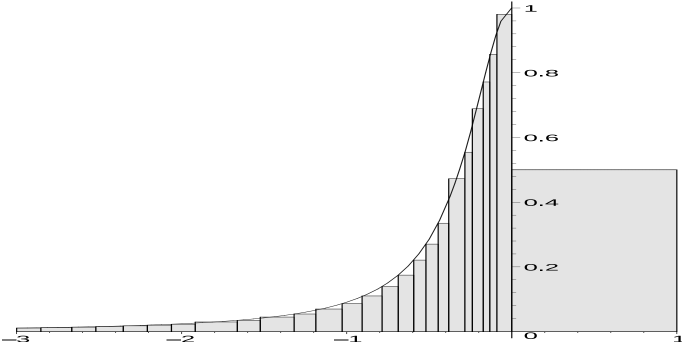


Figure: Typical Kurzweil approximation

Tagged divisions

- A set of couples $D \equiv \{((x_i, x_{i+1}), \xi_{i+1});\ i = 0, \ldots, n-1\}$ is called tagged partial division of a compact interval $[a, b]$ if, for $i = 0 \ldots n-1$, the points $\xi_i \in [x_i,\ x_{i+1}]$, the intervals $[x_i,\ x_{i+1}]$ are non-degenerate, non-overlapping and $[x_i,\ x_{i+1}] \subset [a,\ b]$. If

$$\cup_0^{n-1} [x_i,\ x_{i+1}] = [a,\ b]$$

  then the partial tagged division of $[a, b]$ becomes a tagged division of $[a, b]$.
- A positive function will be called a gauge, a non-negative function for which the set of zeros is countable (including the finite and the empty set) will be called countably closed gauge.

Fine divisions

- For a countably closed gauge ω a tagged partial division D is said to be ω-fine if

$$\xi_{i+1} - \omega(\xi_{i+1}) < x_i \le \xi_{i+1} \le x_{i+1} < \xi_{i+1} + \omega(\xi_{i+1})$$

for $i = 0, \ldots, n-1$.

- This define the concept of δ-fine for a tagged division D and a gauge δ.

Kurzweil's definition

Definition

A number I is said to be the Kurzweil (Kurzweil-Henstock) integral of f from a to b if for every $\epsilon > 0$ there exists a gauge δ such that

$$\left| \sum_{i=0}^{n-1} f(\xi_{i+1})(x_{i+1} - x_i) - I \right| < \epsilon$$

whenever the tagged division $D \equiv \{((x_i, x_{i+1}), \xi_{i+1});\ i = 0, \ldots, n-1\}$ is δ-fine.

The result

Theorem

A function $f : [a, b] \mapsto \mathbb{R}$ *is Kurzweil-Henstock integrable if and only if there exists a continuous function* F *such that for every* $\epsilon > 0$ *there is a countably closed gauge* η *with the property that*

$$\sum_{i=0}^{n-1} |f(\xi_{i+1})(x_{i+1} - x_i) - [F(x_{i+1}) - F(x_i)]| < \epsilon \qquad (1)$$

whenever the tagged partial division $D \equiv \{((x_i, x_{i+1}), \xi_{i+1});\ i = 0, \ldots, n-1\}$ *is* η*-fine. If the condition is satisfied then* $F(b) - F(a) = \int_a^b f$.

Proof of the FT

Let F be continuous on $[a, b]$ and $F'(x) = f(x)$ for $x \in [a, b]$ except a countable set N. For $\xi \notin N$ there exists, by the alternative definition of the derivative (see [2] p. 46) a positive $\eta(\xi)$ such that

$$|F(v) - F(u) - f(\xi)(v - u)| < \frac{\epsilon}{b - a}(v - u)$$

whenever $\xi - \eta(\xi) < u \leq \xi \leq v < \xi + \eta(\xi)$. Define $\eta(x) = 0$ for $x \in N$. Clearly η is a countably closed gauge and if D is tagged partial division of $[a, b]$ then it follows easily from this equation that inequality (1) in our Theorem holds for a η-fine tagged partial division D.

For Further Reading I

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Cambridge University Press 2000

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Sui principii del calcolo integrale.
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H. Lebesgue.
Intégrale, Longueur, Aire.
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