Compact quantum systems: Internal geometry of relativistic systems

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A generalization is presented of the kinematical algebra so(5), shown previously to be relevant for the description of the internal dynamics (Zitterbewegung) of Dirac’s electron. The algebra so(n + 2) is proposed for the case of a compact quantum system with n degrees of freedom. Associated wave equations follow from boosting these compact quantum systems. There exists a contraction to the kinematical algebra of a system with n degrees of freedom of the usual type, by which the commutation relations between n coordinate operators Q_i and corresponding momentum operators P_i, occurring within the so(n + 2) algebra, go over into the usual canonical commutation relations. The so(n + 2) algebra is contrasted with the sl(l,n) superalgebra introduced recently by Palev in a similar context; because so(n + 2) has spinor representations, its use allows the possibility of interpreting the half-integral spin in terms of the angular momentum of internal finite quantum systems. Connection is made with the ideas of Weyl on the possible use in quantum mechanics of ray representation of finite Abelian groups, and so also with other recent works on finite quantum systems. Possible directions of future research are indicated.

I. INTRODUCTION

Many years ago, Weyl considered the unitary representation of the Lie group defined by Heisenberg’s canonical commutation relations, and noted that it may also be considered as a ray representation of an infinite Abelian group. He speculated that unitary ray representations of finite Abelian groups might also prove important in quantum mechanics. Indeed, he gave the example of the unitary ray representation

\[ g_1 \rightarrow ig_1, \quad g_2 \rightarrow ig_2, \quad g_3 \rightarrow ig_3, \quad e \rightarrow I, \]

of the four-element Abelian group (Klein four-group), whose elements satisfy

\[ (g_1)^2 = (g_2)^2 = (g_3)^2 = e \quad (identity), \]

\[ g_2 g_3 = g_3 g_2 = g_1, \quad g_3 g_1 = g_1 g_3 = g_2, \]

in connection with the description of the electron’s spin. (Here the \( \sigma_i \) are Pauli matrices.)

Recent interest in “finite quantum systems” has approached the subject in three essentially different ways. Santhanam and co-workers have proceeded directly from Weyl’s position, writing the unitary ray representatives of finite Abelian groups in exponential form in order to define finite-dimensional Hermitian analogs of Heisenberg’s position and momentum variables, satisfying modified commutation relations. A related approach has been adopted by Gudder and Naroditsky, and also by Stoicek and Tolar.

Palev has considered a simple dynamical system, the isotropic harmonic oscillator in n dimensions, and adopted a noncanonical quantization (in the spirit of Wigner’s well-known work, but along different lines) in order to arrive at noncanonical position and momentum variables with finite-dimensional representations.

Our own work and continuing interest in this area has stemmed from the observation that Dirac’s equation for the electron may be regarded as providing the covariant description of a finite quantum oscillator—the Zitterbewegung. Associated with this equation, in the rest frame of the electron’s center of mass (or in any fixed frame with definite center of mass momentum), are internal coordinates \( Q_i \) and momenta \( P_i (i = 1, 2, 3) \), which satisfy noncanonical commutation relations and have a finite (four-) dimensional Hermitian representation. The kinematical algebra generated by these three \( Q \)’s and \( P \)’s under commutation is isomorphic to the Lie algebra so(5).

The authors mentioned above, together with many others (see Jagannathan and Saavedra and Uteras for references), have speculated on the possible utility of novel kinematics in the description of the internal dynamics of real systems, and in particular, of some relativistic “particles.” However, the so(5) algebra has the important distinguishing feature that it is known to be relevant to an important, real relativistic physical system, because of its association with Dirac’s equation.

Therefore, the structure of this particular kinematical algebra, its relation to the Heisenberg algebra and to Weyl’s ideas, and its generalization to the case of n degrees of freedom (that is, the Q’s and P’s) are of particular interest. This interest is heightened by the thought that the heavy leptons \( \mu \) and \( \tau \) may represent excited states of an internal electron dynamics. Furthermore, we show elsewhere that the cases \( n = 2 \) and \( n = 4 \), respectively, arise in the description of the internal dynamics of the neutrino, and of the electron in a proper time formalism.

II. THE KINEMATICAL ALGEBRA SO(\( n + 2 \))

In the description of the Zitterbewegung of the Dirac electron in the rest frame of its center of mass, the three Hermitian operators \( Q_i \) appear as the coordinate of the charge relative to the center of mass. The three Hermitian operators \( P_i \) have been introduced as the corresponding rela-
tive momentum variables. Together they generate the $so(5)$
kinematical algebra, with commutation relations

\begin{align}
[Q_i, Q_j] &= (i\hbar^2/2\lambda)S_{ij}, \\
[P_i, P_j] &= (4i\hbar/\lambda^2)S_{ij}, \\
[Q_i, P_j] &= i\hbar\delta_{ij} J, \\
[Q_i, S_{jk}] &= i\hbar\delta_{ij} Q_j - \delta_{ij} Q_k), \\
[P_i, S_{jk}] &= i\hbar\delta_{ij} P_j - \delta_{ij} P_k), \\
[Q_i, J] &= (i\hbar^2/2\lambda^2)J_i, \\
[P_i, J] &= (4i\hbar/\lambda^2)Q_i, \\
[J, S_{ij}] &= 0, \\
[S_{ij}, S_{kl}] &= i\hbar(\delta_{ik}S_{lj} + \delta_{lj}S_{ik} - \delta_{ij}S_{kl} - \delta_{kl}S_{ij}).
\end{align}

(2.1a) (2.1b) (2.1c) (2.1d) (2.1e) (2.1f) (2.1g) (2.1h) (2.1i)

Here $\lambda$ is a constant with the dimension of length. As has been
emphasized before, the appearance of at least one such constant is
inevitable in any finite quantum system incorporating Hermitian
coordinate variables, whose eigenvalues are necessarily discrete, with
dimensions of length. In the application of the $so(5)$ algebra to the
internal dynamics of the electron, $\lambda$ equals the Compton wavelength
of that particle. Furthermore, in that application the operators of the
algebra (2.1) can be expressed in terms of the more familiar Dirac
matrices as

\begin{align}
Q_i &= i\frac{\hbar}{\lambda}a_i \beta, \\
P_i &= (\hbar/\lambda)\gamma_i \alpha, \\
J &= -\beta,
\end{align}

(2.2a) (2.2b) (2.2c)

while $S_{ij}$ is the usual spin tensor

\[ S_{ij} = -i\hbar [\alpha_i, \alpha_j] = \epsilon_{ijk}S_{kj}. \]

(2.2d)

The relevant representation of $so(5)$ is then the four-dimen-
sional spinor representation, in which $J (= -\beta)$ is a trace-
less operator with unit square.

There is an obvious generalization of the algebra (2.1) to the case
of $n$ degrees of freedom: simply allow the indices there to run over $1, 2, \ldots, n$ instead of $1, 2, 3$. Then the Lie algebra $so(n+2)$ is
obtained. If one defines $J_{AB} = -J_{BA}$, $A, B = 1, 2, \ldots, n+2$ by
setting $J_{ij} = S_{ij}/\hbar$, $J_{i,n+2} = \lambda^{-1}Q_i - (1/2\hbar)P_i$, and
$J_{n+1,n+2} = 1/2J$, then the $J_{AB}$ satisfy the $so(n+2)$ commutation
relations in standard form

\[ J_{AB}J_{CD} = i\hbar\delta_{AC}J_{BD} + \delta_{BD}J_{AC} - \delta_{BC}J_{AD} - \delta_{AD}J_{BC}. \]

(2.3)

The fundamental spinor representations of $so(n+2)$, of dimen-
sion $2^n$, are of particular interest. [Here $p = (n + 1)/2$ $n$ is
odd, and $p = n/2$ if $n$ is even. In the latter case there are two
inequivalent representations.] The relations of such represen-
tations to Clifford algebras, and associated anticommutation
relations, are well known. Only in these representations does the
operator $J$, which is traceless in every representation, have unit square, so that its eigenvalues are $\pm 1$.

Inspection of (2.1c) suggests that one is then, in an intuitive
sense, as close as possible to the canonical commutation relations

\[ [Q_i, P_j] = i\hbar\delta_{ij} J, \]

(2.4)

where $I$ is the unit operator. (Note that the commutator of
any $Q_i$ and $P_j$ represented by finite matrices must be trace-
less.)

Various dynamics are possible within the framework of the
$so(n+2)$ algebra, corresponding to various choices of
Hamiltonian operator $H$ in the enveloping algebra of the
particular representation at hand. In the case $n = 3$, when the
fundamental (Dirac) spinor representation is chosen, the
only true $so(3)$ scalars available (as distinct from pseudoscalars) are $J_1 (= -\beta)$ and $J_2$ (identity). With $H$ of the form $\lambda t + d\beta$, where $c$ and $d$ are numbers with dimensions of energy, the commutation relations (2.1f) and (2.1g), together
with Heisenberg’s equation of motion

\[ i\hbar\dot{A} = [A, H], \quad \dot{A} = \frac{dA}{dt}, \]

(2.5)

imply

\[ \dot{Q}_i = d(\lambda^2/2\hbar^2)P_i, \quad \dot{P}_i = -(4d/\lambda^2)Q_i, \]

(2.6)

so that

\[ \dot{Q}_i = -(4d^2/\hbar^2)Q_i, \quad \dot{P}_i = -(4d^2/\hbar^2)P_i. \]

(2.7)

Thus harmonic oscillator dynamics is singled out in this case. This
would not be true for other representations of $so(5)$, nor for larger values of $n$, even in the fundamental
spinor representations.

Nevertheless, because the constants $h$ and $\lambda$ are avail-
able, dimensionless creation and annihilation operators can
always be defined, whatever the representation and whatever
the dynamics, as

\begin{align}
A_i &= Q_i/\lambda + i\lambda/2\hbar P_i, \\
A^\dagger_i &= Q_i/\lambda - i\lambda/2\hbar P_i, \quad i = 1, 2, \ldots, n.
\end{align}

(2.8)

The $A_i^\dagger$ is Hermitian conjugate to $A_i$, and relations (2.1) become

\begin{align}
[A_i, A_j] &= 0 = [A_i^\dagger, A_j^\dagger], \\
[A_i, A_j^\dagger] &= \delta_{ij}J + (2i/\lambda)S_{ij}, \\
[A_i, J] &= -2A_i, \\
[A_i^\dagger, J] &= +2A_i^\dagger,
\end{align}

(2.9)

together with (2.1h) and (2.1i) and relations like (2.1d) and
(2.1e), which express the $n$-vector nature of $A_i$ and $A_i^\dagger$.

The relations (2.1) are also equivalent to

\begin{align}
[A_i, A_j^\dagger, A_k] &= 2(\delta_{ik}A_j^\dagger - \delta_{jk}A_i^\dagger - \delta_{ki}A_j), \\
[A_i^\dagger, A_j, A_k^\dagger] &= 2(\delta_{ik}A_j^\dagger - \delta_{jk}A_i^\dagger - \delta_{ki}A_j^\dagger), \\
[A_i^\dagger, A_j^\dagger, A_k] &= [A_k, A_i, A_j^\dagger] \\
&= 2(\delta_{ik}A_j^\dagger - \delta_{jk}A_i^\dagger) - \delta_{ij}J + \delta_{jk}[A_i, A_j^\dagger] - \delta_{ki}[A_i^\dagger, A_j^\dagger]), \\
[A_i, A_j^\dagger] &= 0 = [A_i^\dagger, A_j^\dagger],
\end{align}

(2.10)

in which form they show most clearly how these operators differ from the ones introduced for a finite quantum oscillator
by Palev. His operators satisfy

\begin{align}
[A_i^\dagger, A_j, A_k] &= -\delta_{ik}A_j + \delta_{jk}A_i, \\
[A_i^\dagger, A_j^\dagger, A_k] &= \delta_{ik}A_j^\dagger - \delta_{jk}A_i^\dagger, \\
[A_i^\dagger, A_j^\dagger, A_k^\dagger] &= \delta_{ik}A_j^\dagger - \delta_{jk}A_i^\dagger - \delta_{ki}A_j^\dagger), \\
[A_i, A_j] &= 0 = [A_i^\dagger, A_j^\dagger],
\end{align}

(2.11)
and define the Lie superalgebra $\mathfrak{sl}(l,n)$. Like $\mathfrak{so}(n + 2)$, this has infinitely many inequivalent irreducible Hermitian representations.

Palev considered this algebra as a dynamical algebra associated with a particular Hamiltonian
\[
H = \left(\hbar \omega/(n - 1)\right) [A_1^+, A_j],
\]
for an isotropic oscillator. [Here the constant $\omega$ is introduced, with dimensions of $(\text{time})^{-1}$, but Palev also needs to introduce a constant with dimensions of a length in order to define coordinate and momentum operators.] In contrast, we view the $\mathfrak{so}(n + 2)$ algebra as kinematical. It always admits as a particular dynamics, that associated with the Hamiltonian
\[
H = (\hbar \omega/2n) [A_j^+, A_j],
\]
which leads to the harmonic oscillator equations
\[
A_j = - i \omega A_j, \quad A_j^+ = + i \omega A_j^+,
\]
or equivalently, to Eqs. (2.7) with
\[
d = \hbar \omega.
\]
As already remarked, this is the only dynamics permitted in the case of the fundamental spinor representation of $\mathfrak{so}(5)$ $(n = 3)$, when it is directly relevant to the description of the Zitterbewegung of the electron as a finite quantum oscillator. No doubt Palev’s algebra (without reference to $\omega$) could also be viewed more widely as a kinematical algebra admitting a variety of representations, and a variety of dynamics in most representation.

Another important distinction between the $\mathfrak{so}(n + 2)$ and $\mathfrak{sl}(l,n)$ algebras relates to the representations of the $\mathfrak{so}(n)$ subalgebra that can appear. This subalgebra is associated in both cases with the “angular momentum” of the finite quantum system. Since spinor representation of $\mathfrak{so}(n + 2)$ are allowed (as for the electron), then spinor representations of the $\mathfrak{so}(n)$ subalgebra can be accommodated. However, the $\mathfrak{so}(n)$ subalgebra of $\mathfrak{sl}(l,n)$ appears in the chain
\[
\mathfrak{so}(n) < \mathfrak{sl}(n) < \mathfrak{sl}(l,n),
\]
and only tensor representations of $\mathfrak{so}(n)$ appear in the representations of $\mathfrak{sl}(n)$. Thus Palev’s algebra can only describe finite quantum systems with integral angular momentum or spin.

We comment at the end about the noncompact versions of the $\mathfrak{so}(n + 2)$ algebras.

III. RELATIONSHIP TO THE HEISENBERG ALGEBRA AND TO WEYL’S IDEA

The $\mathfrak{so}(n + 2)$ algebra, which is of dimension $\frac{1}{2}(n + 1)(n + 2)$, is generated by the $n \mathcal{Q}$’s and $n \mathcal{P}$’s under commutation, as Eqs. (2.1) show. In contrast, $n$ canonical $\mathcal{Q}$’s and $\mathcal{P}$’s generate the Heisenberg algebra, which is of the smaller dimension $(2n + 1)$:
\[
[q_i, q_j] = 0 = [p_i, p_j], \quad [q_i, p_j] = i\hbar \delta_{ij} I, \quad [I, q_i] = 0 = [I, p_i].
\]
These may be compared with Eqs. (2.1a)–(2.1c), (2.1e), and (2.1f). However, it is more appropriate to compare the $\mathfrak{so}(n + 2)$ algebra with the kinematical Lie algebra $k_n$, also of dimension $\frac{1}{2}(n + 1)(n + 2)$, obtained by extending the Heisenberg algebra by the algebra $\mathfrak{so}(n)$ of rotations; introduce the $n(n - 1)/2 \mathfrak{so}(n)$ (angular momentum) operators
\[
1_{ij} = - l_{ij}, \quad i, j = 1, 2, \ldots, n
\]
satisfying
\[
[q_i, l_{jk}] = i\hbar [\delta_{ik} q_j - \delta_{ij} q_k],
\]
\[
[p_i, l_{jk}] = i\hbar [\delta_{ik} p_j - \delta_{ij} p_k],
\]
\[
[l_{ij}, l_{kl}] = i\hbar (\delta_{ik} l_{jl} + \delta_{jk} l_{il} - \delta_{jl} l_{ik} - \delta_{il} l_{jk}),
\]
which may be compared with Eqs. (2.1d), (2.1e), (2.1h), and (2.1i). Any representation of the Heisenberg algebra can be extended to a representation of $k_n$ by setting
\[
l_{jk} = q_j p_k - q_k p_j.
\]
However, there are also representations of $k_n$ in which the relation (3.3) does not hold. We may always add one or more “spin terms” to the right-hand side of Eq. (3.3), thus ensuring in particular that spinor representations of $\mathfrak{so}(n)$ can occur.

It is noteworthy that, although there is (up to equivalence) only one unitary representation of the (Weyl) group associated with the Heisenberg Lie algebra, by von Neumann’s theorem, there are evidently infinitely many inequivalent unitary representations (with various spin content) of the group $K_n$ whose Lie algebra is $k_n$. Corresponding to this in our case is the fact that there are infinitely many inequivalent unitary representations of the group $\mathfrak{so}(n + 2)$.

There is a contraction of the algebra $\mathfrak{so}(n + 2)$ to $k_n$; this emphasizes the naturalness of the choice of $\mathfrak{so}(n + 2)$ as an appropriate kinematical algebra for finite quantum systems. To see this without going into details, define
\[
q_i = \epsilon_i Q_i, \quad p_i = \epsilon_i P_i, \quad \tilde{I} = \epsilon_i \epsilon_j I_{ij}, \quad \tilde{I}_{ij} = S_{ij},
\]
with $Q_i, P_i$, etc., as in (2.1) and $\epsilon_i, \epsilon_j$ real parameters. Then
\[
[q_i, q_j] = i\lambda^2/\hbar \epsilon_i \epsilon_j l_{ij},
\]
\[
[p_i, p_j] = (4\hbar/\lambda^2) \epsilon_i \epsilon_j l_{ij},
\]
\[
[q_i, p_j] = i\hbar \delta_{ij} \tilde{I},
\]
while the remaining relations are as in Eqs. (3.2), with $q_i$ replacing $q_j$, etc. When $\epsilon_i$ and $\epsilon_j$ are set to zero, Eqs. (3.5) reduce to Eqs. (3.1). If $\epsilon_i$ is set to zero but not $\epsilon_j$ (or vice versa), the Lie algebra obtained can be seen to be that of the Euclidean group $E(n + 1)$. (These cases correspond physically to an oscillator or free particle.) This indicates that the contraction from $\mathfrak{so}(n + 2)$ to $k_n$ can proceed in two stages, via $e(n + 1)$ (and that there are two distinct routes along which this may be accomplished).

There is also a close relationship between the fundamental spinor representations of the $\mathfrak{so}(n + 2)$ algebra, and unitary ray representations of finite Abelian groups, so that contact can be made with Weyl’s idea, and also the work of Santhanam, mentioned in the Introduction. Consider, for example, the case $n = 1$ (one $Q$ and one $P$) and the fundamental spinor representation of $\mathfrak{so}(3)$, which is two dimensional. We may take in this case
\[
Q = \gamma_1 \sigma_1, \quad P = (\gamma_2 \lambda) \sigma_2,
\]
where $\sigma_1$ and $\sigma_2$ are Pauli matrices. Then Eqs. (2.1) show
\[
J = \sigma_3.
\]
while there are no so(n) operators in this case. Define the unitary operators
\[ A (\theta) = \exp \left( i \theta / \lambda \right) |Q\rangle, \quad B (\phi) = \exp \left( i \phi / \lambda / P / 2 \hbar \right) \]
(3.8)

Then
\[ A (\pi) = i \sigma_1, \quad B (\pi) = i \sigma_2, \]
(3.9)
and it can be seen that \( A (\pi) \) and \( B (\pi) \) generate under multiplication the unitary ray representation of the four-element Abelian group defined by Eqs. (1.1). In contrast, the set of all unitary operators \( A (\theta), B (\phi) \), with \( \theta, \phi \in [0, \pi] \), generate under multiplication a two-valued representation of SO(3) [that is, a true representation of SU(2)].

Note that if we started with the unitary ray representation of the Abelian group, and hence with \( A (\pi) \) and \( B (\pi) \), we could define
\[ Q = - (i \lambda / \pi) \log A (\pi), \quad P = - (2i \hbar / \lambda) \log B (\pi), \]
(3.10)
and recover the so(3) algebra generated under commutation by \( Q \) and \( P \). On the other hand, if we started with a unitary representation of su(2), we would more naturally identify \( Q \) and \( P \) by setting
\[ Q = - i \lambda \left. \frac{dA (\theta)}{d\theta} \right|_{\theta = \pi}, \quad P = - i \lambda \left. \frac{dB (\phi)}{d\phi} \right|_{\phi = \pi}. \]
(3.11)

IV. CONCLUDING REMARKS

Of various approaches to the description of a finite quantum system with \( n \) degrees of freedom, the one using the so\((n + 2)\) kinematical algebra is distinguished primarily by the fact that it is known to be relevant to real relativistic systems.\(^{7,10,11}\) Furthermore, it has been shown that there is a well-defined relationship between the so\((n + 2)\) algebra and the kinematical algebra \( k_n \) of a system with \( n \) degrees of freedom of the usual (noncompact) type. This relationship is defined by a group contraction.

Of course, we do not claim that so\((n + 2)\) is the only algebra which could have such a relationship with \( k_n \). However, the existence of this relationship suggests the possibility of studying a class of finite quantum systems which are well-defined analogs of infinite quantum systems, and also the connection between the two, through the contraction process. One could start with the finite quantum oscillator, as in Eqs. (2.7), for example, but it would be interesting also to construct finite analogs of other well-known dynamical systems, such as the Kepler system, and to investigate their symmetry and dynamical algebras.

Another important distinguishing feature of the so\((n + 2)\) algebra which has been emphasized above is the existence of spinor representations. This makes possible the "explanation" of the half-integral spin of "elementary" particles as the angular momentum of internal finite quantum systems. Such an idea dates back to Schrödinger’s work on Dirac’s electron,\(^{13}\) and has been further brought out in our own recent efforts.\(^{17}\)

Finite systems can be accommodated naturally in the vector space setting of quantum mechanics—we merely need to consider finite-dimensional subspaces of Hilbert space. On the other hand, one might suppose that they have no classical counterparts. That this is not necessarily the case is shown, for example, by the recent construction of a classical analog of Dirac’s spinning electron.\(^{14}\) (In this connection, we mention also the earlier work by Grossmann and Peres.\(^{15}\))

There is clearly more to be done towards understanding the relationship of finite quantum systems to the more familiar dynamical systems of classical and quantum mechanics. The use of the so\((n + 2)\) kinematical algebra defines a class of finite systems for which some possible directions of future research seem reasonably well defined.

Once the commutation relations of the internal dynamical variables have been recognized, we can also take the infinite-dimensional representations of the internal algebra so\((n + 2)\). These then represent many-body systems with \( n \) degrees of freedom in the center of mass frame. Relativistic theories of composite atoms or hadrons,\(^{16}\) or relativistic oscillator and rotator,\(^{17}\) belong to this category. The boosting of such a system (i.e., induced representations of the Poincaré group) gives relativistic finite-component wave equations in the case of finite-dimensional representations, and infinite-component wave equations in the case of compact systems.

In the infinite-dimensional case one can use perhaps more appropriately the unitary representations of the noncompact form of the algebras so\((p,q)\). The exact form of the noncompact form depends on the physical interpretation of the generators as Hermitian operators. For example, the so(3,2) form of so(5) has been used extensively.\(^{11,16-18}\)
