A new set of coherent states for the isotropic harmonic oscillator: Coherent angular momentum states

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(Received 27 November 1979; accepted for publication 3 March 1980)

The Hamiltonian for the oscillator has earlier been written in the form

$$H = \hbar\omega(2\nu^\dagger\nu + \boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda} + \tfrac{3}{2}),$$

where $\nu^\dagger$ and ν are raising and lowering operators for $\nu^\dagger\nu$, which has eigenvalues k (the "radial" quantum number), and $\boldsymbol{\lambda}^\dagger$ and $\boldsymbol{\lambda}$ are raising and lowering 3-vector operators for $\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}$, which has eigenvalues l (the total angular momentum quantum number). A new set of coherent states for the oscillator is now defined by diagonalizing ν and $\boldsymbol{\lambda}$. These states bear a similar relation to the commuting operators H, $\mathbf{L}^2$, and L_3 (where $\mathbf{L}$ is the angular momentum of the system) as the usual coherent states do to the commuting number operators N_1, N_2, and N_3. It is proposed to call them coherent angular momentum states. They are shown to be minimum-uncertainty states for the variables ν, $\nu^\dagger$, $\boldsymbol{\lambda}$, and $\boldsymbol{\lambda}^\dagger$, and to provide a new quasiclassical description of the oscillator. This description coincides with that provided by the usual coherent states only in the special case that the corresponding classical motion is circular, rather than elliptical; and, in general, the uncertainty in the angular momentum of the system is smaller in the new description. The probabilities of obtaining particular values for k and l in one of the new states follow independent Poisson distributions. The new states are overcomplete, and lead to a new representation of the Hilbert space for the oscillator, in terms of analytic functions on $\mathbb{C}\times\mathbb{K}_3$, where $\mathbb{K}_3$ is the three-dimensional complex cone. This space is related to one introduced recently by Bargmann and Todorov, and carries a very simple realization of all the representations of the rotation group.

PACS numbers: 03.65.Fd, 03.65.Ca, 03.65.Ge

1. INTRODUCTION

The states of the isotropic harmonic oscillator with Hamiltonian

$$H = \frac{\mathbf{p}^2}{2M} + \tfrac{1}{2}M\omega^2\mathbf{x}^2 \tag{1}$$

are frequently described in terms of the basis vectors $|n_1,n_2,n_3\rangle$ which are eigenstates of H and also of the number operators

$$N_i = a_i^\dagger a_i, \quad i = 1,2,3 \quad (\text{no sum}), \tag{2}$$

where

$$a_i = (2M\hbar\omega)^{-1/2}(ip_i + M\omega x_i). \tag{3}$$

The occupation numbers n_i independently run over the nonnegative integers and a_i is a shift operator for N_i, lowering the corresponding eigenvalue n_i by 1.

An alternative approach[1-5] introduces the "coherent states" $|\mathbf{z}\rangle$, which are eigenvectors of the lowering operators

$$a_i|\mathbf{z}\rangle = z_i|\mathbf{z}\rangle, \quad z_i\in\mathbb{C}. \tag{4}$$

These vectors have many attractive properties. In particular, there is a well-defined sense in which one can say that when the system is progressing through a succession of coherent states its behavior is as close as possible to the behavior of its classical counterpart. The coherent states are therefore justifiably called "quasiclassical" states of the oscillator.

For any three dimensional system, the total angular momentum quantum number, denoted j in general, can take the values 0, 1, 2,$\cdots$ or the values $\frac{1}{2}$, $\frac{3}{2}$,$\cdots$. From this we are led to observe that it may be possible and useful to define, by diagonalizing suitable lowering operators for j, coherent angular momentum states for a variety of systems, including the isotropic oscillator. The latter is the subject of the present work. Several authors[6-12] have defined and discussed coherent angular momentum states, in particular for systems (such as the rigid rotor) for which the angular momentum operators $\mathbf{J}^2$ and J_3 provide a complete set of commuting operators. (These states, and those we define in this paper, are not to be confused with the so-called "coherent spin states" which have been widely discussed since their introduction by Radcliffe,[13] and which are superpositions of eigenvectors of J_3 for a *fixed* value of $\mathbf{J}^2$.) Atkins and Dobson[7] defined states by exploiting the Schwinger[14] boson calculus for SU(2), and diagonalizing the associated pair of boson annihilation operators. A difficulty here is that the states obtained are superpositions of states of integral and half-integral j, because the boson operators lower j by $\frac{1}{2}$ rather than 1. In order to obtain states which might apply to some physical system with only rotational degrees of freedom, the states with half-integral j had to be rather arbitrarily deleted from the superpositions. Bhaumik, Nag, and Dutta-Roy[8] avoided this difficulty by constructing, within the Schwinger calculus, two operators quadratic in the boson operators, which lower the value of j by 1. These operators were diagonalized to define coherent angular momentum states, which could then be identified as possible coherent angular momentum states for a system, again having only rotational degrees of freedom. The operators of Bhaumik *et al.* have algebraic properties similar to two components of the 3-vector operator $\boldsymbol{\lambda}$ we introduce below. However, the operator $\boldsymbol{\lambda}$

 0022-2488/81/040719-14$1.00

acts within an entirely different space, namely that within which the oscillator boson operators a_i and $a_i^\dagger$ act. The operators a_i, which form a 3-vector, should not be confused with the boson annihilation operators in the Schwinger calculus, which form a 2-spinor. At no stage in what follows do we work with the Schwinger calculus.

A suitable definition of coherent angular momentum states for a system such as the isotropic oscillator, which possesses translational as well as rotational degrees of freedom, is more difficult than for a system possessing only rotational degrees of freedom, because the dynamics will, in general, couple the degrees of freedom in the former case. Nevertheless, we may hope with the authors mentioned above that coherent angular momentum states can be defined in a suitable way for some systems, and that by analogy with the properties of the usual coherent states, these new states will have one or more of several nice properties. They may exhibit "quasiclassical" behavior, at least in their angular dependence, and they may be useful in examining the behavior of the angular momentum of the system as the classical limit is approached. They may also represent states of "minimum uncertainty" for certain noncommuting variables associated with the angular dependence in the problem and, as they will most likely be overcomplete, they may permit the construction of a representation in which, at the least, the angular dependence of the density matrix for the system can be put in a diagonal form.

With this motivation, we define in this paper a new set of quasiclassical states for the isotropic oscillator and call them "coherent angular momentum states." They bear a similar relation to the commuting operators H, $\mathbf{L}^2$, and L_3 (where $\mathbf{L} = \mathbf{x}\times\mathbf{p}$ is the angular momentum of the system) as the usual coherent states do to the commuting operators H, N_1, N_2, and N_3. In particular, these new states are eigenvectors of a 3-vector operator which lowers the value of the total angular momentum quantum number. We emphasize that the problem is *not* the straightforward one of expressing the usual coherent states for the three-dimensional oscillator as superpositions of the common eigenvectors of H, $\mathbf{L}^2$, and L_3, instead of superpositions of the vectors $|n_1, n_2, n_3\rangle$. Such expressions have been obtained and discussed by Mikhailov,[15] but those coherent states are not eigenvectors of any lowering operator for j (or, rather, l in this case). The states we shall define below are in general quite distinct from the usual coherent states, as we shall see. We shall demonstrate that they do have some of the attractive properties mentioned above.

We have shown in an earlier publication[16] (henceforth referred to as BL) that the operator H of Eq. (1) can also be written in the form

$$H = \hbar\omega(2\nu^\dagger\nu + \boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda} + \tfrac{3}{2}), \tag{5}$$

where $\nu^\dagger$ and ν are (boson) raising and lowering operators for $\nu^\dagger\nu$ (which we also write as K), while $\boldsymbol{\lambda}^\dagger$ and $\boldsymbol{\lambda}$ are raising and lowering operators for $\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}$ (which we also write as L). The eigenvalues k and l of K and L run over the nonnegative integers independently, and the eigenvalues of H appear in the form $\hbar\omega(2k + l + \frac{3}{2})$. Here k is the "radial" quantum number and l is the total angular momentum quantum number. Both are familiar from the treatment in the coordinate representation of the eigenvalue problem for H, $\mathbf{L}^2$, and L_3. Here we adopt no particular representation.

The basic algebraic relations satisfied by the operators ν, $\nu^\dagger$, $\boldsymbol{\lambda}$, and $\boldsymbol{\lambda}^\dagger$ are (see BL)

$$\begin{aligned}
&[\nu,\nu^\dagger] = 1\,,\\
&[\lambda_i,\nu] = 0 = [\lambda_i^\dagger,\nu^\dagger]\,,\\
&[\lambda_i,\nu^\dagger] = 0 = [\lambda_i^\dagger,\nu]\,,\\
&[\lambda_i,\lambda_j] = 0 = [\lambda_i^\dagger,\lambda_j^\dagger]\,,\\
&(2\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}+1)[\lambda_i,\lambda_j^\dagger] = (2\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}+1)\delta_{ij} - 2\lambda_i^\dagger\lambda_j\,,\\
&\boldsymbol{\lambda}\cdot\boldsymbol{\lambda} = 0 = \boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}^\dagger\,,\\
&L_i = -i\hbar\epsilon_{ijk}\lambda_j^\dagger\lambda_k\,.
\end{aligned}\tag{6}$$

With $K = \nu^\dagger\nu$ and $L = \boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}$, it follows that

$$\begin{aligned}
&L\,\boldsymbol{\lambda} = \boldsymbol{\lambda}(L-1), \quad L\,\boldsymbol{\lambda}^\dagger = \boldsymbol{\lambda}^\dagger(L+1)\,,\\
&[L,\nu] = 0 = [L,\nu^\dagger]\,,\\
&K\nu = \nu(K-1), \quad K\nu^\dagger = \nu^\dagger(K+1)\,,\\
&[K,\lambda] = 0 = [K,\boldsymbol{\lambda}^\dagger]\,,
\end{aligned}\tag{7}$$

and also that

$$\mathbf{L}^2 = L(L+1)\hbar^2\,, \tag{8}$$

so that when L has the eigenvalue l, $\mathbf{L}^2$ has the eigenvalue $l(l+1)\hbar^2$. The two alternative sets of dynamical variables for the isotropic oscillator $\{\nu, \nu^\dagger, \boldsymbol{\lambda}, \boldsymbol{\lambda}^\dagger\}$ and $\{\mathbf{a}, \mathbf{a}^\dagger\}$, are related by the equations

$$\begin{aligned}
\nu &= (\mathbf{a}\cdot\mathbf{a})(4K+4L+2)^{-1/2}\,,\\
\lambda_i &= (a_i L - i\hbar^{-1}\epsilon_{ijk}a_j L_k)\\
&\quad\times[(2L+1)(2K+2L+1)]^{-1/2}\,,\\
a_i &= \lambda_i[(2K+2L+1)/(2L+1)]^{1/2}\\
&\quad+\lambda_i^\dagger\nu[2/(2L+3)]^{1/2}\,,
\end{aligned}\tag{9}$$

and their conjugates.

As the four lowering operators ν and λ_i commute, we define the coherent angular momentum states as their common eigenvectors. Thus we seek vectors $|z, \boldsymbol{\zeta}\rangle$ satisfying

$$\begin{aligned}
&\nu|z,\boldsymbol{\zeta}\rangle = z|z,\boldsymbol{\zeta}\rangle\,,\\
&\lambda_i|z,\boldsymbol{\zeta}\rangle = \zeta_i|z,\boldsymbol{\zeta}\rangle\,,
\end{aligned}\tag{10}$$

where the eigenvalues z and ζ_i may be expected to be complex since ν and λ_i are not Hermitian. Noting from Eqs. (6) that $\boldsymbol{\lambda}^2 = 0$, we see that $\boldsymbol{\zeta}$ is confined to a complex cone

$$\boldsymbol{\zeta}^2 = 0\,. \tag{11}$$

The coherent angular momentum states will therefore be labelled by the four complex numbers z and ζ_i, of which only three are independent, whereas the usual coherent states are labelled by three complex numbers z_i.

Let us remark at this stage that although there is a certain so(2,1) ⊕ so(3,2) Lie algebra underlying the algebra of operators which we use for this system (see BL), the operators ν and $\boldsymbol{\lambda}$ are not actually in (the complexification of) this Lie algebra, so that the states we define are not coherent states for a Lie algebra or group in the sense of Barut and Girardello[17] or Perelomov,[18] although they are closely related to such states. We make some comments on this at the end of Sec. 5.

In Sec. 2 we find, for arbitrary complex z and arbitrary complex $\boldsymbol{\zeta}$ satisfying Eq. (11), a nondegenerate normalized vector $|z, \boldsymbol{\zeta}\rangle$ satisfying Eqs. (10), in the form

$$|z, \boldsymbol{\zeta}\rangle = \exp(-\tfrac{1}{2}|z|^2 - \tfrac{1}{2}|\boldsymbol{\zeta}|^2) \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{klm}(z,\boldsymbol{\zeta})|klm\rangle , \tag{12}$$

with

$$a_{klm}(z,\boldsymbol{\zeta}) = z^k(\zeta_3)^{l-|m|}(-\epsilon\zeta_{-\epsilon})^{|m|} \times [(2l)!/k!2^l l!(l+m)!(l-m)!]^{1/2} , \tag{13}$$

where ϵ is the sign of m and $\zeta_{\pm} = \zeta_1 \pm i\zeta_2$. Here $|klm\rangle$ is the nondegenerate normalized common eigenvector of K, L, and L_3, as constructed in BL, which in the coordinate representation has the familiar form[19]

$$|klm\rangle = (-1)^k \left[\frac{2a^3 k!}{\Gamma(k+l+\frac{3}{2})} \right]^{1/2} \xi^l e^{-\frac{1}{2}\xi^2} \times L_k^{(l+\frac{1}{2})}(\xi^2) Y_{lm}(\theta,\phi) , \tag{14}$$

where $a = (M\omega/\hbar)^{1/2}$ and $\xi = ar$ (r, θ, and ϕ are the usual spherical polar coordinates), $L_k^{(l+\frac{1}{2})}$ is the generalized Laguerre polynomial defined as in Ref. (20), and the spherical harmonic Y_{lm} is defined as in Ref. (21). From Eqs. (12) and (14) one can deduce (see Appendix A) that in the coordinate representation

$$|z, \boldsymbol{\zeta}\rangle = \left[\frac{a}{\sqrt{\pi}} \right]^{3/2} \exp(-\tfrac{1}{2}|z|^2 - \tfrac{1}{2}|\boldsymbol{\zeta}|^2 - \tfrac{1}{2}a^2|\mathbf{x}|^2) \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[\frac{\Gamma(l+\frac{3}{2})}{\Gamma(k+l+\frac{3}{2})} \right]^{1/2} (-z)^k \times L_k^{(l+\frac{1}{2})}(a^2|\mathbf{x}|^2) \frac{(\sqrt{2}a\mathbf{x}\cdot\boldsymbol{\zeta})^l}{l!} , \tag{15}$$

a result we have not been able to express more simply, except in the special case $z = 0$, when it becomes

$$|0,\boldsymbol{\zeta}\rangle = \left[\frac{a}{\sqrt{\pi}} \right]^{3/2} \exp(-\tfrac{1}{2}|\boldsymbol{\zeta}|^2 - \tfrac{1}{2}a^2|\mathbf{x}|^2 + \sqrt{2}a\mathbf{x}\cdot\boldsymbol{\zeta}) . \tag{16}$$

In Sec. 3 we examine the expectation values of the dynamical variables when the system is in the state $|z, \boldsymbol{\zeta}\rangle$, and find in particular that the probability of obtaining the value l on measurement of the total angular momentum follows a Poisson distribution.

We go on to consider the sense in which the state $|z, \boldsymbol{\zeta}\rangle$ is a "minimum-uncertainty" state for the hermitian variables σ, τ, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$, where

$$\sqrt{2\hbar}\,\nu = \sigma + i\tau$$

and

$$\sqrt{2\hbar}\,\boldsymbol{\lambda} = \boldsymbol{\alpha} + i\boldsymbol{\beta} . \tag{17}$$

Letting $\langle A \rangle$ denote the expectation value of any observable A for a given state of the system, and defining the dispersions of σ and $\boldsymbol{\alpha}$ for that state by

$$\Delta\sigma = [\langle\sigma^2\rangle - \langle\sigma\rangle^2]^{1/2}$$

and

$$\Delta\alpha = [\langle\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\rangle - \langle\boldsymbol{\alpha}\rangle\cdot\langle\boldsymbol{\alpha}\rangle]^{1/2} , \tag{18}$$

with similar definitions for $\Delta\tau$ and $\Delta\beta$, we find that, in general,

$$\Delta\sigma\Delta\tau \geqslant \tfrac{1}{2}\hbar \tag{19a}$$

and

$$\Delta\alpha\Delta\beta \geqslant \hbar(1 + \tfrac{1}{2}\langle(2L+1)^{-1}\rangle) . \tag{19b}$$

In the state $|z, \boldsymbol{\zeta}\rangle$ both inequalities become equalities and, moreover,

$$(\Delta\sigma)^2 = (\Delta\tau)^2 = \tfrac{1}{2}\hbar \tag{20a}$$

and

$$(\Delta\alpha)^2 = (\Delta\beta)^2 = \hbar(1 + \tfrac{1}{2}\langle(2L+1)^{-1}\rangle) . \tag{20b}$$

We show in Sec. 4 that, if the system is in the state $|z_0,\boldsymbol{\zeta}_0\rangle$ at time $t = 0$, then at time t, in the Schrödinger picture, it is in the state $e^{-(3/2)i\omega t}|z(t), \boldsymbol{\zeta}(t)\rangle$, where

$$z(t) = e^{-2i\omega t}z_0$$

and

$$\boldsymbol{\zeta}(t) = e^{-i\omega t}\boldsymbol{\zeta}_0 . \tag{21}$$

We then deduce that the expectation values $\langle\sigma\rangle$, $\langle\tau\rangle$, $\langle\boldsymbol{\alpha}\rangle$, and $\langle\boldsymbol{\beta}\rangle$ reproduce the corresponding behavior in time of their classical counterparts $\hat{\sigma}$, $\hat{\tau}$, $\hat{\boldsymbol{\alpha}}$, and $\hat{\boldsymbol{\beta}}$, as discussed in BL. Moreover, the dispersions of the quantum-mechanical variables remain constant during the motion at their minimum values as in Eqs. (20), so that the coherent angular momentum states can properly be called quasiclassical states.

Corresponding to a given classical motion of the oscillator there are therefore (at least) two quasiclassical descriptions in quantum mechanics, which are distinct in general. One is provided by the usual coherent states, another by coherent angular momentum states. In the special case that the classical motion is circular rather than elliptical, these two quasiclassical descriptions are the same. We deduce this as a consequence of the identification of the coherent angular momentum state $|0, \boldsymbol{\zeta}\rangle$ with the usual coherent state $|\mathbf{z} = \boldsymbol{\zeta}\rangle$, an identification which follows from the result (16) and the known form for the states $|\mathbf{z}\rangle$ in the coordinate representation.[22]

It is important to note, however, that we find that, if the quasiclassical description of a given classical motion is given by coherent angular momentum states, the expectation values $\langle\mathbf{x}\rangle$ and $\langle\mathbf{p}\rangle$ do not exactly reproduce the corresponding behavior of the classical variables $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$, unless that classical motion is circular. Rather we find, for example, that $\langle\mathbf{x}\rangle$ follows an elliptical path different from, though in the same plane as, the path followed by $\hat{\mathbf{x}}$. Furthermore, the state $|z,\boldsymbol{\zeta}\rangle$ is not, in general, a minimum-uncertainty state for $\mathbf{x}$ and $\mathbf{p}$ and, when the system evolves in coherent angular momentum states, the dispersions Δx and Δp oscillate, but are always bounded. Therefore, in the coordinate representation (or the momentum representation) the state vector $e^{-3i\omega t/2} \times |z(t), \boldsymbol{\zeta}(t)\rangle$ would appear as a pulsating wavepacket which follows the classical motion approximately. This reflects the fact that the variables $\mathbf{x}$ and $\mathbf{p}$ bear a special relation to the usual coherent states, not the coherent angular momentum states. In other representations, the *usual* coherent states are also presumably represented by pulsating packets which follow the classical motion only approximately. Furthermore, in the quasiclassical description provided by the usual coherent states, the expectation values of σ, τ, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$ will not, in general, reproduce exactly the behavior of their classical counterparts.

The outstanding feature of the description by coherent angular momentum states, in the general case of an elliptic orbit, is that the uncertainty in the angular momentum of the system, as best measured, according to Delbourgo,[11] by $[\langle \mathbf{L}^2\rangle - \langle \mathbf{L}\rangle\cdot\langle \mathbf{L}\rangle]^{1/2}$, is smaller than it is for the description by the usual coherent states.

After a brief discussion of the classical limit which is obtained with $|z| \to \infty$, $|\boldsymbol{\zeta}| \to \infty$, and $\hbar \to 0$ we go on to Sec. 5, where we give a completeness relation for the states $|z, \boldsymbol{\zeta}\rangle$. They are, in fact, overcomplete and, just as for the usual coherent states, there is associated with these states a Hilbert space of analytic functions with a reproducing kernel. We briefly discuss this space and the associated elegant representation of the dynamical variables ν, $\nu^\dagger$, $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}^\dagger$, K, L, and H.

We conclude with some remarks in Sec. 6 about possible further developments.

2. THE COHERENT ANGULAR MOMENTUM STATES

We look for vectors satisfying Eqs. (10) in the form

$$|z, \boldsymbol{\zeta}\rangle = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} b_{klm}(z,\boldsymbol{\zeta})|klm\rangle . \tag{22}$$

In BL, Eqs. (53) and (58), we showed that

$$|klm\rangle = c_{klm}(\nu^\dagger)^k(\lambda_\epsilon^\dagger)^{|m|}(\lambda_3^\dagger)^{l-|m|}|0\rangle , \tag{23}$$

where

$$c_{klm} = (-\epsilon)^m\,[(2l)!/k!2^l l!(l-m)!(l+m)!]^{1/2}, \tag{24}$$

ϵ is the sign of m, $|0\rangle$ is a normalized vector on which ν and $\boldsymbol{\lambda}$ vanish and $\lambda_\pm^\dagger = \lambda_1^\dagger \pm i\lambda_2^\dagger$. Supposing that the vector $|z, \boldsymbol{\zeta}\rangle$ of Eq. (22) does satisfy Eqs. (10), then we must have

$$\begin{aligned} b_{klm}|z, \boldsymbol{\zeta}\rangle &= \langle klm|z, \boldsymbol{\zeta}\rangle \\ &= (c_{klm})^*\langle 0|\nu^k(\lambda_{-\epsilon})^{|m|}(\lambda_3)^{l-|m|}|z, \boldsymbol{\zeta}\rangle \\ &= c_{klm}z^k(\zeta_{-\epsilon})^{|m|}(\zeta_3)^{l-|m|}\langle 0|z, \boldsymbol{\zeta}\rangle , \end{aligned} \tag{25}$$

where $\lambda_\pm = \lambda_1 \pm i\lambda_2$ and $\zeta_\pm = \zeta_1 \pm i\zeta_2$.

Conversely, taking the coefficients in Eq. (22) to have the form

$$b_{klm}(z,\boldsymbol{\zeta}) = n(z,\boldsymbol{\zeta})c_{klm}z^k(\zeta_{-\epsilon})^{|m|}(\zeta_3)^{l-|m|} , \tag{26}$$

with $n(z,\boldsymbol{\zeta})$ an arbitrary function of z and $\boldsymbol{\zeta}$, one can check that the vector $|z, \boldsymbol{\zeta}\rangle$ so defined does satisfy Eqs. (10). To verify this, one needs to use the equations

$$\begin{aligned} \nu|klm\rangle &= k^{1/2}|k-1lm\rangle \\ \lambda_3|klm\rangle &= \left[\frac{(l-m)(l+m)}{(2l-1)}\right]^{1/2}|kl-1m\rangle \\ \lambda_\pm|klm\rangle &= \pm\left[\frac{(l\mp m)(l\mp m-1)}{(2l-1)}\right]^{1/2} \\ &\quad\times|kl-1\,m\pm 1\rangle , \end{aligned} \tag{27}$$

as given in BL [Eqs. (59)], and also to use Eq. (11) in the form

$$\zeta_+\zeta_- = -(\zeta_3)^2 . \tag{28}$$

It is easily seen that this vector $|z, \boldsymbol{\zeta}\rangle$ is normalizable for arbitrary complex z and for arbitrary complex $\boldsymbol{\zeta}$ satisfying Eq. (11). Using the orthonormality of the vectors $|klm\rangle$, we have

$$\begin{aligned} \langle z',\boldsymbol{\zeta}'|z,\boldsymbol{\zeta}\rangle &= n(z',\boldsymbol{\zeta}')^*n(z,\boldsymbol{\zeta})\sum_{klm}(z'^*z)^k(\zeta_3'^*\zeta_3)^{l-|m|} \\ &\quad\times[(\zeta'_{-\epsilon})^*\zeta_{-\epsilon}]^{|m|}(c_{klm})^2 . \end{aligned} \tag{29}$$

Because $\boldsymbol{\zeta}^2 = 0 = \boldsymbol{\zeta}'^2$, we have, with the help of the binomial theorem,

$$\begin{aligned} &\sum_{m=-l}^{l}(\zeta_3'^*\zeta_3)^{l-|m|}[(\zeta'_{-\epsilon})^*\zeta_{-\epsilon}]^{|m|}\frac{(2l)!}{2^l(l+m)!(l-m)!} \\ &\quad= (\boldsymbol{\zeta}'^*\cdot\boldsymbol{\zeta})^l , \end{aligned} \tag{30}$$

so that Eq. (29) reduces to

$$\begin{aligned} \langle z',\boldsymbol{\zeta}'|z,\boldsymbol{\zeta}\rangle &= n(z',\boldsymbol{\zeta}')^*n(z,\boldsymbol{\zeta})\sum_{k,l}\frac{(z'^*z)^k}{k!}\frac{(\boldsymbol{\zeta}'^*\cdot\boldsymbol{\zeta})^l}{l!} \\ &= n(z',\boldsymbol{\zeta}')^*n(z,\boldsymbol{\zeta})\exp(z'^*z + \boldsymbol{\zeta}'^*\cdot\boldsymbol{\zeta}) . \end{aligned} \tag{31}$$

Thus $|z, \boldsymbol{\zeta}\rangle$ is normalized if we take

$$n(z,\boldsymbol{\zeta}) = \exp(-\tfrac{1}{2}|z|^2 - \tfrac{1}{2}|\boldsymbol{\zeta}|^2) \tag{32}$$

and we have, from Eqs. (22), (24), and (26), the final form (12) for the coherent angular momentum states.

With this normalization, we have

$$\begin{aligned} \langle z',\boldsymbol{\zeta}'|z, \boldsymbol{\zeta}\rangle = \exp[&-\tfrac{1}{2}(|z'|^2 + |z|^2 + |\boldsymbol{\zeta}'|^2 + |\boldsymbol{\zeta}|^2) \\ &+ z'^*z + \boldsymbol{\zeta}'^*\cdot\boldsymbol{\zeta}] , \end{aligned} \tag{33}$$

so that

$$|\langle z',\boldsymbol{\zeta}'|z, \boldsymbol{\zeta}\rangle|^2 = \exp[-\tfrac{1}{2}|z'-z|^2 - \tfrac{1}{2}|\boldsymbol{\zeta}'-\boldsymbol{\zeta}|^2] . \tag{34}$$

These vectors are, therefore, not orthogonal for $(z', \boldsymbol{\zeta}') \neq (z, \boldsymbol{\zeta})$, but they approximate orthogonality as $|z'-z| \to \infty$ and $|\boldsymbol{\zeta}'-\boldsymbol{\zeta}| \to \infty$. We shall see in Sec. 5 that they are overcomplete.

3. EXPECTATION VALUES OF PHYSICAL VARIABLES AND THE MINIMUM UNCERTAINTY PROPERTY

When the system is in the state $|z, \boldsymbol{\zeta}\rangle$ we have at once

$$\begin{aligned} &\langle\nu\rangle = z, \quad \langle\nu^\dagger\rangle = z^* , \\ &\langle\boldsymbol{\lambda}\rangle = \boldsymbol{\zeta}, \quad \langle\boldsymbol{\lambda}^\dagger\rangle = \boldsymbol{\zeta}^* , \end{aligned} \tag{35}$$

so that, using Eqs. (10) and (17),

$$\begin{aligned} &\langle\sigma\rangle = \sqrt{\hbar/2}(z+z^*), \quad \langle\tau\rangle = -i\sqrt{\hbar/2}(z-z^*) , \\ &\langle\boldsymbol{\alpha}\rangle = \sqrt{\hbar/2}(\boldsymbol{\zeta}+\boldsymbol{\zeta}^*), \quad \langle\boldsymbol{\beta}\rangle = -i\sqrt{\hbar/2}(\boldsymbol{\zeta}-\boldsymbol{\zeta}^*) . \end{aligned} \tag{36}$$

Since $K = \nu^\dagger\nu$ and $L = \boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda}$ we have

$$\langle K\rangle = |z|^2, \quad \langle L\rangle = |\boldsymbol{\zeta}|^2$$

and

$$\langle H\rangle = \hbar\omega(2|z|^2 + |\boldsymbol{\zeta}|^2 + \tfrac{3}{2}) , \tag{37}$$

and from the last of Eqs. (6) we have

$$\langle\mathbf{L}\rangle = -i\hbar\boldsymbol{\zeta}^*\times\boldsymbol{\zeta} . \tag{38}$$

The probability of obtaining the value k on measuring K in the state $|z, \boldsymbol{\zeta}\rangle$ is given by

$$p(k) = \sum_{l,m} |\langle klm|z,\zeta\rangle|^2$$
$$= |n(z,\zeta)|^2 \frac{|z|^{2k}}{k!} \sum_{l,m} |\zeta_3|^{2l-2m} |\zeta_-|^{2m} \times \frac{(2l)!}{2^l l!(l+m)!(l-m)!}$$
$$= \frac{|z|^{2k}}{k!} e^{-|z|^2}, \tag{39}$$

which corresponds to a Poisson distribution with mean $|z|^2$. Similarly, the probability of obtaining the value l on measuring L is given by

$$p(l) = \sum_{k,m} |\langle klm|z,\zeta\rangle|^2 = \frac{|\zeta|^{2l}}{l!} e^{-|\zeta|^2}, \tag{40}$$

corresponding to a Poisson distribution with mean $|\zeta|^2$.

Because the k values and l values are distributed in probability according to (independent) Poisson distributions, it follows that

$$\langle K^n\rangle = e^{-\gamma}\left(\gamma\frac{\partial}{\partial\gamma}\right)^n e^{\gamma}, \quad \gamma = |z|^2$$

and

$$\langle L^n\rangle = e^{-\gamma}\left(\gamma\frac{\partial}{\partial\gamma}\right)^n e^{\gamma}, \quad \gamma = |\zeta|^2. \tag{41}$$

In particular,

$$\langle \mathbf{L}^2\rangle = \hbar^2\langle L(L+1)\rangle = \hbar^2(|\zeta|^4 + 2|\zeta|^2). \tag{42}$$

According to Delbourgo,[11] the quantity $[\langle\mathbf{L}\rangle^2 - \langle\mathbf{L}\rangle\cdot\langle\mathbf{L}\rangle]^{1/2}$ provides the best measure of the uncertainty in the angular momentum of the system in a given state. In view of Eq. (38), we have in the state $|z,\zeta\rangle$

$$\langle\mathbf{L}\rangle\cdot\langle\mathbf{L}\rangle = \hbar^2|\zeta|^4 \tag{43}$$

and hence

$$\langle\mathbf{L}^2\rangle - \langle\mathbf{L}\rangle\cdot\langle\mathbf{L}\rangle = 2\hbar^2|\zeta|^2, \tag{44}$$

a result to which we shall refer in Sec. 4.

The conditional probability of obtaining the value $m\hbar$ for L_3, given that l has been observed for L, is given by

$$p(m;l) = \frac{1}{p(l)}\sum_k |\langle klm|z,\zeta\rangle|^2 = \frac{|\zeta_3|^{2l-2m}|\zeta_-|^{2m}}{|\zeta|^{2l}} \frac{(2l)!}{2^l(l+m)!(l-m)!}. \tag{45}$$

Now, because $\zeta^2 = 0$, we have

$$\sqrt{2}|\zeta| = |\zeta_+| + |\zeta_-|, \tag{46}$$

which, with Eq. (45), enables us to write

$$p(m;l) = \binom{2l}{l+m}\theta^{l+m}(1-\theta)^{l-m}, \tag{47}$$

with

$$\theta = \frac{|\zeta_-|}{\sqrt{2}|\zeta|} = 1 - \frac{|\zeta_+|}{\sqrt{2}|\zeta|}, \quad 0 \leqslant \theta \leqslant 1. \tag{48}$$

Thus the m values are distributed, for a given l, in accordance with a binomial distribution with mean

$$\sum_{m=-l}^{l} m\, p(m;l) = l(2\theta - 1) = l\left[\frac{|\zeta_-| - |\zeta_+|}{|\zeta_-| + |\zeta_+|}\right]. \tag{49}$$

[These results are not valid if $\zeta = 0$, when we get simply $p(0;0) = 1$ and $p(m;l) = 0$ otherwise.] The unconditional probability of obtaining the value $m\hbar$ on measuring L_3 in the state $|z,\zeta\rangle$ is given by

$$p(m) = \sum_{l=|m|}^{\infty} p(l)\,p(m;l)$$
$$= e^{-|\zeta|^2}\sum_{l=|m|}^{\infty} |\zeta_-|^{l+m}|\zeta_+|^{l-m} \times \frac{(2l)!}{2^l l!(l+m)!(l-m)!}$$
$$= \left|\frac{\zeta_-}{\zeta_+}\right|^m \left|\frac{\zeta_3^2}{2}\right|^{|m|} e^{-|\zeta|^2}\sum_{n=0}^{\infty}|\zeta_3|^{2n} \times \frac{(2n+2|m|)!}{2^n n!(n+|m|)!(n+2|m|)!}$$
$$= \left|\frac{\zeta_-}{\zeta_+}\right|^m \left|\frac{\zeta_3^2}{2}\right|^{|m|} \times \frac{e^{-|\zeta|^2}}{(|m|)!} M(|m| + \tfrac{1}{2}, 2|m|+1, 2|\zeta_3|^2)$$
$$= \left|\frac{\zeta_-}{\zeta_+}\right|^m \exp(-\tfrac{1}{2}|\zeta_+|^2 - \tfrac{1}{2}|\zeta_-|^2) \times I_m(|\zeta_3|^2), \tag{50}$$

where M is the confluent hypergeometric function and I_m is the modified Bessel function of order m.[20] Note that

$$\sum_{m=-\infty}^{\infty} x^m I_m(2y) = \exp[y(x + x^{-1})], \tag{51}$$

which ensures that $\Sigma_{m=-\infty}^{\infty} p(m) = 1$, as required. Note also that the result (50) can be written in the form

$$p(m) = |\zeta_{-\epsilon}|^{2|m|}|\zeta_+\zeta_-|^{-|m|} I_{|m|}(|\zeta_+\zeta_-|) \times \exp(-\tfrac{1}{2}|\zeta_+|^2 - \tfrac{1}{2}|\zeta_-|^2), \tag{52}$$

where ϵ is the sign of m, by using Eq. (28) and the properties of the modified Bessel functions. In this form $p(m)$ is well defined even if $\zeta_+\zeta_- = 0$, because $z^{-|m|}I_{|m|}(z)$ is well defined at $z = 0$.

Let us now consider the sense in which coherent angular momentum states are minimum-uncertainty states. As the operators v and $v^\dagger$ are boson operators, we know that the inequality (19a) holds in general and, from our experience with the usual coherent states, we know that in the state $|z,\zeta\rangle$, this inequality becomes an equality with $(\Delta\sigma)^2 = (\Delta\tau)^2 = \frac{1}{2}\hbar$, as in Eq. (20a). Thus the states $|z,\zeta\rangle$ are minimum-uncertainty states in the usual sense for the conjugate variables σ and τ.

By a simple extension of a familiar argument[23] it is easily shown that, if $\Delta\alpha$ and $\Delta\beta$ are defined as in Eq. (18), then

$$(\Delta\alpha)^2 + c^2(\Delta\beta)^2 \geqslant -ic\langle[\alpha_i,\beta_i]\rangle = \hbar c\langle[\lambda_i,\lambda_i^\dagger]\rangle, \tag{53}$$

for arbitrary real c, with the equality holding if and only if

$$(\boldsymbol{\alpha} + ic\boldsymbol{\beta})|\psi\rangle = (\langle\boldsymbol{\alpha}\rangle + ic\langle\boldsymbol{\beta}\rangle)|\psi\rangle, \tag{54}$$

where $|\psi\rangle$ is the appropriate state vector. We know from the fifth of Eqs. (6) that

$$[\lambda_i, \lambda_i^\dagger] = \frac{4L+3}{2L+1}, \tag{55}$$

which is positive definite. The strongest inequality of the type (53) is therefore obtained by taking that positive value of c which makes $(\Delta\alpha)^2/c + c(\Delta\beta)^2$ a minimum, viz.

$$c = \Delta\alpha/\Delta\beta, \tag{56}$$

and the inequality then has the form

$$\Delta\alpha\Delta\beta \geqslant \tfrac{1}{2}\hbar\langle[\lambda_i, \lambda_i^\dagger]\rangle = \hbar(1 + \tfrac{1}{2}\langle(2L+1)^{-1}\rangle). \tag{57}$$

From Eq. (54) we see that this becomes an equality if and only if

$$[(\Delta\beta)\boldsymbol{\alpha} + i(\Delta\alpha)\boldsymbol{\beta}]|\psi\rangle = [(\Delta\beta)\langle\boldsymbol{\alpha}\rangle + i(\Delta\alpha)\langle\boldsymbol{\beta}\rangle]|\psi\rangle. \tag{58}$$

Now if $|\psi\rangle = |z,\zeta\rangle$, we have

$$\begin{aligned}\langle\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\rangle &= \tfrac{1}{2}\hbar\langle(\boldsymbol{\lambda}+\boldsymbol{\lambda}^\dagger)\cdot(\boldsymbol{\lambda}+\boldsymbol{\lambda}^\dagger)\rangle \\ &= \tfrac{1}{2}\hbar\langle(\boldsymbol{\lambda}^\dagger\cdot\boldsymbol{\lambda} + \boldsymbol{\lambda}\cdot\boldsymbol{\lambda}^\dagger)\rangle \\ &= \tfrac{1}{2}\hbar\langle 2L + [\lambda_i, \lambda_i^\dagger]\rangle \\ &= \tfrac{1}{2}\hbar\left\langle\frac{4L^2+6L+3}{2L+1}\right\rangle.\end{aligned} \tag{59}$$

The same value is obtained for $\langle\boldsymbol{\beta}\cdot\boldsymbol{\beta}\rangle$. Furthermore, according to Eqs. (36) and (37),

$$\begin{aligned}\langle\boldsymbol{\alpha}\rangle\cdot\langle\boldsymbol{\alpha}\rangle &= \hbar|\boldsymbol{\zeta}|^2 \\ &= \hbar\langle L\rangle,\end{aligned} \tag{60}$$

and the same value is obtained for $\langle\boldsymbol{\beta}\rangle\cdot\langle\boldsymbol{\beta}\rangle$. Combining Eqs. (59) and (60), we get

$$\langle\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}\rangle - \langle\boldsymbol{\alpha}\rangle\cdot\langle\boldsymbol{\alpha}\rangle = \tfrac{1}{2}\hbar\left\langle\frac{4L+3}{2L+1}\right\rangle, \tag{61}$$

and the same value for $\langle\boldsymbol{\beta}\cdot\boldsymbol{\beta}\rangle - \langle\boldsymbol{\beta}\rangle\cdot\langle\boldsymbol{\beta}\rangle$. Thus we have

$$(\Delta\alpha)^2 = (\Delta\beta)^2 = \hbar(1 + \tfrac{1}{2}\langle(2L+1)^{-1}\rangle), \tag{62}$$

and the inequality (57) becomes an equality. That Eq. (58) is satisfied when $|\psi\rangle = |z,\zeta\rangle$ is also now evident. Since $\Delta\alpha = \Delta\beta$, it reduces to the equation

$$\boldsymbol{\lambda}|z,\boldsymbol{\zeta}\rangle = \langle\boldsymbol{\lambda}\rangle|z,\boldsymbol{\zeta}\rangle, \tag{63}$$

which is satisfied because $|z,\boldsymbol{\zeta}\rangle$ is an eigenvector of $\boldsymbol{\lambda}$.

In this sense then, the states $|z,\boldsymbol{\zeta}\rangle$ are minimum-uncertainty states for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, as well as for σ and τ. However, this is a somewhat weaker notion of minimum-uncertainty than that applying to σ and τ, in two respects. First, the inequality (57) does not place restrictions on the uncertainty products for individual components of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such as the product $\Delta\alpha_1\Delta\beta_1$. While $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be regarded in a certain sense as conjugate variables, the components α_i and β_i cannot be regarded as three pairs of independent conjugate variables, because $[\alpha_i, \alpha_j]$, $[\alpha_i, \beta_j]$ and $[\beta_i, \beta_j]$ are nonzero for $i \neq j$. Second, the right-hand side of the inequality (57) is not constant. Since $(2L+1)^{-1}$ has eigenvalues $1, \frac{1}{3}, \frac{1}{5}, \ldots$, one sees that the greatest lower bound of $\langle(2L+1)^{-1}\rangle$ is 0, but also that there are no states in which this bound is attained. Thus, in addition to Eq. (57), one can say that in general

$$\Delta\alpha\Delta\beta > \hbar \tag{64}$$

and that there are *no* states of the system in which $\Delta\alpha\Delta\beta$ is minimized in an absolute sense. In the state $|z,\boldsymbol{\zeta}\rangle$, it is not hard to show from the result (40) that

$$\langle(2L+1)^{-1}\rangle = |\boldsymbol{\zeta}|^{-1}e^{-|\boldsymbol{\zeta}|^2}\int_0^{|\boldsymbol{\zeta}|} e^{y^2}\,dy. \tag{65}$$

What one can properly say, then, is that of all states for which $\langle(2L+1)^{-1}\rangle$ has a particular value say, A, (note that it then follows that $0 < A \leqslant 1$), some of the states in which $\Delta\alpha\Delta\beta$ is minimized are the states $|z,\boldsymbol{\zeta}\rangle$, with

$$|\boldsymbol{\zeta}|^{-1}e^{-|\boldsymbol{\zeta}|^2}\int_0^{|\boldsymbol{\zeta}|} e^{y^2}\,dy = A. \tag{66}$$

In the introduction we remarked that the states $|z,\boldsymbol{\zeta}\rangle$ are not minimum-uncertainty states for $\mathbf{x}$ and $\mathbf{p}$. This can be seen most simply by observing[24] that any minimum-uncertainty state for $\mathbf{x}$ and $\mathbf{p}$ is an eigenvector of

$$(1+\mu)\mathbf{a} + (1-\mu)\mathbf{a}^\dagger \tag{67}$$

for some real $\mu > 0$. (The usual coherent states have $\mu = 1$.) It is reasonably obvious from the expressions (9) that no operator of the form (67) is diagonalized on the coherent angular momentum state $|z,\boldsymbol{\zeta}\rangle$ in general. (In the special case that $z = 0$, $|z,\boldsymbol{\zeta}\rangle$ becomes equal to one of the usual coherent states, as we saw in the introduction. Thus $|0,\boldsymbol{\zeta}\rangle$ is an eigenvector of $\mathbf{a}$.)

Let us now consider the uncertainties in position and momentum of the oscillator in the state $|z,\boldsymbol{\zeta}\rangle$. We note from Eq. (9) that

$$a_i = u(K,L)\lambda_i + \lambda_i^\dagger w(K,L)\nu, \tag{68}$$

where

$$\begin{aligned}u(K,L) &= [(2K+2L+3)/(2L+3)]^{1/2}, \\ w(K,L) &= [2/(2L+3)]^{1/2}.\end{aligned} \tag{69}$$

With the help of Eqs. (6) and (7) we then deduce that

$$\begin{aligned}a_i a_j = {} & u(K,L)u(K,L+1)\lambda_i\lambda_j \\ & + \lambda_i^\dagger w(K,L)u(K+1,L)\lambda_j\nu \\ & + \lambda_i^\dagger\lambda_j^\dagger w(K,L+1)w(K+1,L)\nu^2 \\ & + \lambda_j^\dagger u(K,L+1)w(K,L+1)\lambda_i\nu \\ & + \delta_{ij}u(K,L)w(K,L)\nu \\ & - 2\lambda_i^\dagger u(K,L+1)w(K,L+1)(2L+3)^{-1}\lambda_j\nu\end{aligned} \tag{70}$$

and that

$$\begin{aligned}a_i^\dagger a_j = {} & \lambda_i^\dagger u(K,L)u(K,L)\lambda_j + \lambda_i^\dagger\lambda_j^\dagger u(K,L+1)w(K,L)\nu \\ & + \nu^\dagger w(K,L)u(K,L+1)\lambda_i\lambda_j \\ & + \nu^\dagger\lambda_j^\dagger w(K,L+1)w(K,L+1)\lambda_i\nu \\ & + \delta_{ij}\nu^\dagger w(K,L)w(K,L)\nu \\ & - 2\nu^\dagger\lambda_i^\dagger w(K,L+1)w(K,L+1)(2L+3)^{-1}\lambda_j\nu.\end{aligned} \tag{71}$$

Then we have, in the state $|z,\boldsymbol{\zeta}\rangle$,

$$\begin{aligned}\langle a_i\rangle &= \langle u(K,L)\rangle\zeta_i + \langle w(K,L)\rangle\zeta_i^*z \\ &= u\zeta_i + w\zeta_i^*z,\end{aligned} \tag{72}$$

say, and

$$\begin{aligned}\langle a_i a_j\rangle = {} & c_1\zeta_i\zeta_j + c_2\zeta_i^*\zeta_j z + c_3\zeta_i^*\zeta_j^*z^2 \\ & + c_4\zeta_j^*\zeta_i z + c_5\delta_{ij}z,\end{aligned} \tag{73}$$

$$\langle a_i^\dagger a_j\rangle = d_1\zeta_i^*\zeta_j + d_2\zeta_i^*\zeta_j^* z + d_3\zeta_i\zeta_j z^* + d_4\zeta_j^*\zeta_i z^* z + d_5\delta_{ij} z^* z + d_6\zeta_i^*\zeta_j z^* z\,, \quad (74)$$

where, for example,

$$c_1 = \langle u(K,L)u(K,L+1)\rangle\,. \quad (75)$$

Then

$$\begin{aligned}(2M\omega/\hbar)^{1/2}\langle x_i\rangle &= \langle a_i + a_i^\dagger\rangle \\ &= \langle a_i\rangle + \langle a_i\rangle^* \\ &= u(\zeta_i + \zeta_i^*) + w(\zeta_i^* z + \zeta_i z^*)\end{aligned} \quad (76)$$

and

$$\begin{aligned}(2M\omega/\hbar)\langle x_i x_j\rangle &= \langle (a_i + a_i^\dagger)(a_j + a_j^\dagger)\rangle \\ &= \langle a_i a_j\rangle + \langle a_i^\dagger a_j\rangle \\ &\quad + \langle a_j^\dagger a_i\rangle + \langle a_i a_j\rangle^* + \delta_{ij}\,.\end{aligned} \quad (77)$$

We introduce Δx, an overall measure of uncertainty in positions, by

$$\begin{aligned}\Delta x &= (\langle \mathbf{x}\cdot\mathbf{x}\rangle - \langle \mathbf{x}\rangle\cdot\langle \mathbf{x}\rangle)^{1/2} \\ &= [(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2]^{1/2}\,,\end{aligned} \quad (78)$$

and using Eqs. (73), (74), (76) and (77) we deduce that

$$\begin{aligned}(2M\omega/\hbar)(\Delta x)^2 &= (c_2 + c_4 - 2uw)|\zeta|^2(z + z^*) \\ &\quad + 2(d_1 - u^2)|\zeta|^2 \\ &\quad + 2(d_4 + d_6 - w^2)|\zeta|^2|z|^2 \\ &\quad + 3c_5(z + z^*) + 6d_5|z|^2 + 3\,.\end{aligned} \quad (79)$$

In a similar way, we deduce that

$$(2/M\omega\hbar)^{1/2}\langle p_i\rangle = -iu(\zeta_i - \zeta_i^*) - iw(\zeta_i^* z - \zeta_i z^*), \quad (80)$$

$$\begin{aligned}(2/M\omega\hbar)(\Delta p)^2 &= (2uw - c_2 - c_4)|\zeta|^2(z + z^*) \\ &\quad + 2(d_1 - u^2)|\zeta|^2 \\ &\quad + 2(d_4 + d_6 - w^2)|\zeta|^2|z|^2 \\ &\quad - 3c_5(z + z^*) + 6d_5|z|^2 + 3\,.\end{aligned} \quad (81)$$

We shall make further reference to these results in Sec. 4. They are not particularly revealing as they stand, but they do make it obvious that $|z,\zeta\rangle$ is not in general one of the usual coherent states, for which one always has

$$\begin{aligned}\Delta x &= (3\hbar/2M\omega)^{1/2}\,, \\ \Delta p &= (3M\omega\hbar/2)^{1/2}\,.\end{aligned} \quad (82)$$

4. QUASICLASSICAL BEHAVIOR AND THE CLASSICAL LIMIT

Classicaly, one may define the state of the oscillator at any time by giving the values of the classical variables $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$, or equivalently, by giving the value of the complex variable $\hat{\mathbf{a}}$, which is the classical counterpart of the usual lowering operator $\mathbf{a}$

$$\hat{\mathbf{a}} = (2M\omega)^{-1/2}(i\hat{\mathbf{p}} + M\omega\hat{\mathbf{x}})\,. \quad (83)$$

Alternatively, as shown in BL, one may give the values of the complex variables $\hat{\nu}$ and $\hat{\boldsymbol{\lambda}}$ (with $\hat{\boldsymbol{\lambda}}\cdot\hat{\boldsymbol{\lambda}} = 0$), which are the classical counterparts of ν and $\boldsymbol{\lambda}$. For if one knows the values of $\hat{\nu}$ and $\hat{\boldsymbol{\lambda}}$, one can calculate that of $\hat{\mathbf{a}}$, the vice versa [cf Eqs. (9)]. One can think of $\hat{\nu}$ as the coordinate of a point in the complex plane $\mathbb{C}$, and of $\hat{\boldsymbol{\lambda}}$ as the coordinates of a point on the complex cone $\mathbb{K}_3$, whose equation is $\hat{\boldsymbol{\lambda}}\cdot\hat{\boldsymbol{\lambda}} = 0$. Then the space $\mathbb{C}\times\mathbb{K}_3$ can be regarded as a sort of complex phase space for the oscillator, with $(\hat{\nu},\hat{\boldsymbol{\lambda}})$ being the coordinates of the representative point for the state of the system. As time t varies, this point moves in accordance with the classical equations of motion, with

$$\begin{aligned}\hat{\nu}(t) &= \hat{\nu}(0)e^{-2i\omega t}\,, \\ \hat{\boldsymbol{\lambda}}(t) &= \hat{\boldsymbol{\lambda}}(0)e^{-i\omega t}\,.\end{aligned} \quad (84)$$

In the quantum mechanics problem, we have

$$\begin{aligned}[H,\boldsymbol{\lambda}] &= -\hbar\omega\boldsymbol{\lambda}\,, \\ [H,\nu] &= -2\hbar\omega\nu\,.\end{aligned} \quad (85)$$

If $|z_0,\zeta_0\rangle$ is the state vector of the system at $t = 0$, then in the Schrödinger picture the state vector at time t is

$$|\psi(t)\rangle = e^{-iHt/\hbar}|z_0,\zeta_0\rangle\,, \quad (86)$$

and from Eqs. (85) we deduce that

$$\begin{aligned}\nu|\psi(t)\rangle &= z_0 e^{-2i\omega t}|\psi(t)\rangle\,, \\ \boldsymbol{\lambda}|\psi(t)\rangle &= \boldsymbol{\zeta}_0 e^{-i\omega t}|\psi(t)\rangle\,.\end{aligned} \quad (87)$$

From the fact that H has the value $\hbar\omega(2k + l + \frac{3}{2})$ on $|klm\rangle$, we readily deduce from Eqs. (86) and (12) that, in fact,

$$|\psi(t)\rangle = e^{-3i\omega t/2}|z(t),\boldsymbol{\zeta}(t)\rangle\,, \quad (88)$$

with

$$z(t) = z_0 e^{-2i\omega t}, \quad \boldsymbol{\zeta}(t) = \boldsymbol{\zeta}_0 e^{-i\omega t}\,. \quad (89)$$

We see that if the system is in a coherent angular momentum state at one time, it is so at all times.

The expectation values of ν and $\boldsymbol{\lambda}$ as functions of time are now given, according to Eqs. (35) and (89), by

$$\begin{aligned}\langle\nu\rangle(t) &= z(t)\,, \\ \langle\boldsymbol{\lambda}\rangle(t) &= \boldsymbol{\zeta}(t)\,,\end{aligned} \quad (90)$$

and we see by comparing Eqs. (84) and (89) that these expectation values are solutions of the classical equation of motion. Classical and quantum-mechanical descriptions which correspond are obtained by taking

$$\begin{aligned}\hat{\nu}(t) &= \sqrt{\hbar}\,z(t)\,, \\ \hat{\boldsymbol{\lambda}}(t) &= \sqrt{\hbar}\,\boldsymbol{\zeta}(t)\,.\end{aligned} \quad (91)$$

The factors of $\sqrt{\hbar}$ appear here because of a difference of a factor of $\sqrt{\hbar}$ in the definitions of classical and quantum-mechanical variables like $\hat{\mathbf{a}}$ in Eq. (83) and $\mathbf{a}$ in Eq. (3). (The quantum-mechanical variables are dimensionless; the classical ones are not.)

We see also from Eqs. (20) that the values of $\Delta\sigma, \Delta\tau, \Delta\alpha$, and $\Delta\beta$ remain constant during the motion, with the products $\Delta\sigma\Delta\tau$ and $\Delta\alpha\Delta\beta$ at their minimum values. (In the case of $\Delta\alpha\Delta\beta$, this minimum value is $\hbar(1 + \frac{1}{2}\langle(2L+1)^{-1}\rangle)$, which remains constant because $(2L+1)^{-1}$ is a constant of the motion.)

We may say that if the system is evolving through a succession of coherent states, as in Eq. (88), its state at time t may be defined approximately by specifying a point $(\sqrt{\hbar}\,z(t), \sqrt{\hbar}\,\boldsymbol{\zeta}(t))$ in the classical phase space $\mathbb{C}\times\mathbb{K}_3$. However, there is a "volume of uncertainty" of size $\approx\Delta\sigma\Delta\tau = \frac{1}{2}\hbar$ associated with the position of $\sqrt{\hbar}\,z$ in $\mathbb{C}$, and a volume of uncertainty defined by $\Delta\alpha\Delta\beta = \hbar(1 + \frac{1}{2}\langle(2L+1)^{-1}\rangle)$ associated with the position of $\sqrt{\hbar}\,\zeta$ in $\mathbb{K}_3$. This representative point follows a classical trajectory, and these volumes of uncertainty do not change with time. In this sense the coherent angular momentum states are justifiably called quasiclassical states of the oscillator. Consider a typical classical trajectory, with

$$\hat{\mathbf{x}} = (A\ \cos\omega t, B\ \sin\omega t, 0), \quad A \geqslant B \geqslant 0$$
$$\hat{\mathbf{p}} = M\omega(-A\ \sin\omega t, B\ \cos\omega t, 0),$$
$$\hat{\mathbf{a}} = (\tfrac{1}{2}M\omega)^{1/2} e^{-i\omega t}(A, iB, 0)\,, \tag{92}$$

or equivalently, (see BL, Sec. 4)

$$\hat{\nu} = \tfrac{1}{2}(M\omega)^{1/2}(A-B)e^{-2i\omega t}\,,$$
$$\hat{\boldsymbol{\lambda}} = (\tfrac{1}{2}M\omega AB)^{1/2} e^{-i\omega t}(1, i, 0)\,. \tag{93}$$

The description in terms of the coherent angular momentum states corresponding to this classical trajectory is provided by taking the state vector at time t to be as in Eqs. (88) and (89), with

$$z_0 = \tfrac{1}{2}(M\omega/\hbar)^{1/2}(A-B)\,,$$
$$\zeta_0 = (M\omega AB/2\hbar)^{1/2}(1, i, 0)\,. \tag{94}$$

According to Eq. (76), in this state the expectation value of $\mathbf{x}$ in particular is given by

$$\begin{aligned}\langle \mathbf{x}\rangle(t) &= u\sqrt{AB}\,(\cos\omega t, \sin\omega t, 0)\\ &\quad + \tfrac{1}{2}w(A-B)(M\omega AB/\hbar)^{1/2}(\cos\omega t, -\sin\omega t, 0)\\ &= (A'\cos\omega t, B'\sin\omega t, 0)\,.\end{aligned} \tag{95}$$

Therefore $\langle \mathbf{x}\rangle$ follows an elliptical path which is in the same plane, with the same center and the same orientation as the elliptical path followed by $\hat{\mathbf{x}}$, but which has different sized axes. The ratios A'/A and B'/B involve the expectation values u and w, which are constants of the motion, but which are not simply functions of A and B. One can show from Eqs. (69) and (37) that as $|z_0|$ and $|\zeta_0|$ are increased, u tends towards $\frac{1}{2}(A+B)/\sqrt{(AB)}$ and W tends towards $(\hbar/M\omega AB)^{1/2}$, so that A' and B' tend to A and B, respectively, as the classical limit is approached (see below).

From the expression (79) we see that, because c_2, c_4, u etc. are constants of the motion, as are $|\zeta|^2$ and $|z|^2$, and because

$$z + z^* = (z_0 + z_0^*)\cos 2\omega t - i(z_0 - z_0^*)\sin 2\omega t,$$

the value of $(\Delta x)^2$ makes bounded oscillations with angular frequency 2ω about a fixed mean value. A similar remark applies to $(\Delta p)^2$. In the coordinate representation (or the momentum representation), the wavefunction must therefore pulsate while it only approximately follows the classical motion, but it does not disperse.[25]

The reader should not hasten to conclude that the description corresponding to the classical motion, as provided by the coherent angular momentum states, is in any sense "less quasi-classical" than the description provided by the usual coherent states. In the latter case one would take the state vector to be (up to a phase factor) $|\mathbf{z}(t)\rangle$ [cf. Eq. (4)], where

$$\mathbf{z}(t) = (M\omega/2\hbar)^{1/2} e^{-i\omega t}(A, iB, 0) \tag{96}$$

for the particular motion described above. Then, as is well-known, $\langle \mathbf{x}\rangle$ and $\langle \mathbf{p}\rangle$ reproduce the behavior of $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ and, moreover, Δx_i and Δp_i are constants, with $\Delta x_1 \Delta p_1 = \frac{1}{2}\hbar$, etc. However, it is evident from the definition in Eqs. (9) that $\langle \nu\rangle$ and $\langle \boldsymbol{\lambda}\rangle$ will now not exactly reproduce the behavior of $\hat{\nu}$ and $\hat{\boldsymbol{\lambda}}$ and $\Delta\sigma$, $\Delta\tau$, $\Delta\alpha$, and $\Delta\beta$ will presumably now be oscillatory.

(In the special case of a circular orbit, the two descriptions become the same. We have $A = B$ above and thus $z_0 = 0 = z(t)$. Because $\nu|0,\zeta\rangle = 0$ implies $K|0,\zeta\rangle = 0$, we deduce that $u = 1$ in this situation and so Eq. (95) reduces to $\langle \mathbf{x}\rangle = \hat{\mathbf{x}}$. More generally, corresponding to any circular orbit, we have $z(t) = 0$, $\mathbf{z}(t)\cdot\mathbf{z}(t) = 0$, and $\mathbf{z}(t) = \zeta(t)$; and as explained in the Introduction, the coherent angular momentum state $|0,\zeta\rangle$ is equal to the usual coherent state $|\mathbf{z} = \zeta\rangle$.)

The most important distinction between the two corresponding quasi-classical descriptions is brought out by a consideration of the uncertainty in the angular momentum. If the system is in the state $|\mathbf{z}\rangle$, we readily deduce that

$$\langle \mathbf{L}\rangle^2 - \langle \mathbf{L}\rangle\cdot\langle \mathbf{L}\rangle = 2\hbar^2\mathbf{z}^*\cdot\mathbf{z}, \tag{97}$$

after noting that

$$L_i = -i\hbar\epsilon_{ijk}a_j^{\dagger}a_k\,,$$
$$\mathbf{L}^2 = \hbar^2\big[a_j^{\dagger}a_k^{\dagger}a_j a_k + 2\mathbf{a}^{\dagger}\cdot\mathbf{a} - (\mathbf{a}^{\dagger}\cdot\mathbf{a}^{\dagger})(\mathbf{a}\cdot\mathbf{a})\big]\,. \tag{98}$$

On the other hand, if the system is in the $|z,\zeta\rangle$ state, we have the result (44). Corresponding to the particular classical trajectory described above, at time t we have z and ζ as in Eqs. (89) and (94), and $\mathbf{z}$ as in Eq. (96). For the description provided by the coherent angular momentum states, we then have

$$\langle \mathbf{L}^2\rangle - \langle \mathbf{L}\rangle\cdot\langle \mathbf{L}\rangle = 2\hbar M\omega AB \tag{99}$$

at all times, while for the description using the usual coherent states we obtain

$$\langle \mathbf{L}^2\rangle - \langle \mathbf{L}\rangle\cdot\langle \mathbf{L}\rangle = \hbar M\omega(A^2 + B^2)\,. \tag{100}$$

The latter is greater, by an amount $\hbar M\omega(A-B)^2$.

In the case of a circular orbit ($A = B$), the results agree, as they must in view of our earlier remarks, but in the general case of an elliptical orbit we see that the uncertainty in the angular momentum is greater in the usual quasi-classical description.

We conclude this section with some brief comments on the classical limit. In the usual treatment this is reached by considering the system in a succession of states $|\mathbf{z}\rangle$, with $|\mathbf{z}|\to\infty$, $\hbar\to 0$, and $(\sqrt{\hbar})\mathbf{z}$ finite [and equal to $\sqrt{(M\omega/2)} \times e^{-i\omega t}(A, iB, 0)$ for the particular orbit described above. In a similar way, we can approach the limit very simply by considering a succession of states $|z,\zeta\rangle$, with $|z|\to\infty$, $|\zeta|\to\infty$, $\hbar\to 0$, and $(\sqrt{\hbar})z$ and $(\sqrt{\hbar})\zeta$ finite and equal to $\frac{1}{2}\sqrt{(M\omega)}(A-B)e^{-2i\omega t}$ and $\sqrt{(M\omega AB/2)}e^{-i\omega t}(1, i, 0)$ in the particular orbit]. Note that the case of a circular orbit is special and corresponds always to $z = 0$. It is evident from Eqs. (20) and (36) that as the limit is approached,

$$\frac{\Delta\sigma}{\langle\sigma\rangle}\to 0, \quad \frac{\Delta\tau}{\langle\tau\rangle}\to 0\,,$$
$$\frac{\Delta\alpha}{(\langle\boldsymbol{\alpha}\rangle\cdot\langle\boldsymbol{\alpha}\rangle)^{1/2}}\to 0, \quad \frac{\Delta\beta}{(\langle\boldsymbol{\beta}\rangle\cdot\langle\boldsymbol{\beta}\rangle)^{1/2}}\to 0\,, \tag{101}$$

and also, from Eqs. (43) and (44), that

$$\frac{\langle \mathbf{L}^2\rangle - \langle \mathbf{L}\rangle\cdot\langle \mathbf{L}\rangle}{\langle \mathbf{L}\rangle\cdot\langle \mathbf{L}\rangle} \to 0 . \tag{102}$$

Thus the relative widths of the probability distributions go to zero for all these variables, and one can easily see from Eqs. (37) that the same is true for K, L, and H.

5. COMPLETENESS AND A HILBERT SPACE OF ANALYTIC FUNCTIONS

In this section we find it more convenient to work with the unnormalized vectors

$$\begin{aligned} |z,\zeta) &= \exp\{\tfrac{1}{2}|z|^2 + \tfrac{1}{2}|\zeta|^2\}\,|z^*,\zeta^*\rangle \\ &= \sum_{klm} a_{klm}(z^*,\zeta^*)|klm\rangle , \end{aligned} \tag{103}$$

rather than with the $|z,\zeta\rangle$ themselves. According to Eq. (33) we then have

$$(z',\zeta'|z,\zeta) = \exp(z'z^* + \zeta'\cdot\zeta^*) . \tag{104}$$

The coefficients a_{klm} appearing in Eq. (103) were defined in Eq. (13).

We first note that

$$\int d\mu\,(z,\zeta) a_{klm}(z^*,\zeta^*)^* a_{k'l'm'}(z^*,\zeta^*) = \delta_{kk'}\delta_{ll'}\delta_{mm'} . \tag{105}$$

(A derivation appears in Appendix B.) In this equation

$$d\mu(z,\zeta) = \frac{2}{\pi^3}\, d^2z\, d^6\zeta\,\delta(\zeta\cdot\zeta)(2|\zeta|^2 - 1)\exp(-|z|^2 - |\zeta|^2) , \tag{106}$$

and the integration is over all possible complex z and ζ. Of course, the coefficients a_{klm} and vectors $|z,\zeta)$ have only been defined for (z,ζ) on $\mathbb{C}\times\mathbb{K}_3$, but that is all that is needed in integrals like that in Eq. (105) and those below, because of the delta function in $d\mu$. The meaning of the notation is as in Ref. 26: If $z = x + iy$, $\zeta = \mathbf{u} + i\mathbf{v}$, where x, y, $\mathbf{u}$ and $\mathbf{v}$ are real, then

$$d^2z\, d^6\zeta\,\delta(\zeta\cdot\zeta) = dx\,dy\,d^3u\,d^3v\,\delta(\mathbf{u}^2 - \mathbf{v}^2)\delta(2\mathbf{u}\cdot\mathbf{v}) . \tag{107}$$

It follows that

$$d\mu(z,\zeta) = d\mu(z^*,\zeta^*). \tag{108}$$

Now consider

$$\begin{aligned} &\int d\mu\,(z,\zeta)|z,\zeta)(z,\zeta| \\ &= \sum_{klm}\sum_{k'l'm'} \int d\mu\, a_{klm}(z^*,\zeta^*)^* a_{k'l'm'}(z^*,\zeta^*)|k'l'm'\rangle\langle klm| \\ &= \sum_{klm}|klm\rangle\langle klm| , \end{aligned} \tag{109}$$

using Eq. (105). As the vectors $|klm\rangle$ are complete and orthonormal, we have the result

$$\int d\mu\,(z,\zeta)|z,\zeta)(z,\zeta| = I , \tag{110}$$

expressing the completeness of the vectors $|z,\zeta)$ (and hence of the vectors $|z,\zeta\rangle$).

They are, in fact, overcomplete. For example, we can deduce from Eq. (110) an expression of linear dependence:

$$|z',\zeta') = \int d\mu\,(z,\zeta)(z,\zeta|z',\zeta')|z,\zeta) , \tag{111}$$

with $(z,\zeta|z',\zeta')$ as in Eq. (104).

Now consider an arbitrary vector $|\phi\rangle$ and write

$$|\phi\rangle = \sum_{klm}\phi_{klm}|klm\rangle , \tag{112}$$

with

$$\phi_{klm} = \langle klm|\phi\rangle . \tag{113}$$

Using Eq. (103), we see that

$$\begin{aligned} (z,\zeta|\phi\rangle &= \sum_{klm}\phi_{klm} a_{klm}(z^*,\zeta^*)^* \\ &= \sum_{klm}\phi_{klm}\left[\frac{(2l)!}{k!2^l l!(l-m)!(l+m)!}\right]^{1/2} \\ &\quad\times z^k(\zeta_3)^{l-|m|}(-\epsilon\zeta_\epsilon)^{|m|} \\ &= \phi(z,\zeta) , \end{aligned} \tag{114}$$

say. Because $(z,\zeta|\phi\rangle$ is finite for all (z,ζ) on $\mathbb{C}\times\mathbb{K}_3$, it follows that the series in Eq. (114) converges for all (z,ζ) on $\mathbb{C}\times\mathbb{K}_3$ and that $\phi(z,\zeta)$ is analytic there. Noting from Eq. (104) that

$$(z,\zeta|z,\zeta) = \exp(|z|^2 + |\zeta|^2),$$

we have from Eq. (114) and Schwartz's inequality, that

$$|\phi(z,\zeta)| \leqslant A\exp\{\tfrac{1}{2}|z|^2 + \tfrac{1}{2}|\zeta|^2\} , \tag{115}$$

with $A = \langle\phi|\phi\rangle^{1/2}$. Thus the growth of ϕ with $|z|$ and $|\zeta|$ is limited. We note also from Eq. (110) that

$$\begin{aligned} \langle\phi|\phi\rangle &= \int d\mu\,(z,\zeta)\langle\phi|z,\zeta)(z,\zeta|\phi\rangle \\ &= \int d\mu\,(z,\zeta)|\phi(z,\zeta)|^2 , \end{aligned} \tag{116}$$

so that, in addition to the inequality (115), ϕ satisfies

$$\int d\mu\,(z,\zeta)|\phi(z,\zeta)|^2 < \infty . \tag{117}$$

We see in this way that any vector $|\phi\rangle$ defines a function $\phi(z,\zeta)$, analytic on $\mathbb{C}\times\mathbb{K}_3$, and satisfying there the conditions (115) and (117). From Eqs. (110) and (114) we have

$$|\phi\rangle = \int d\mu\,(z,\zeta)\phi(z,\zeta)|z,\zeta) . \tag{118}$$

Conversely, suppose that a function $\phi(z,\zeta)$ of this type is given. Then we can define a vector $|\phi\rangle$ by Eq. (118) and check that it is normalizable and that $\phi(z,\zeta) = (z,\zeta|\phi\rangle$. To do this we first note, from Eqs. (103) and (110), that

$$(z',\zeta'|klm\rangle = \int d\mu\,(z,\zeta)(z',\zeta'|z,\zeta)(z,\zeta|klm\rangle ,$$

i.e.,

$$a_{klm}(z'^*,\zeta'^*)^* = \int d\mu\,(z,\zeta)\exp(z'z^* + \zeta'\cdot\zeta^*)a_{klm}(z^*,\zeta^*)^* . \tag{119}$$

Since ϕ is by assumption analytic, it can be expanded as a convergent series on $\mathbb{C}\times\mathbb{K}_3$:

$$\phi(z,\zeta) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} d_{ki_1i_2\cdots i_l} z^k\zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_l} , \tag{120}$$

and one can use Eq. (28) to bring this expansion to the form

$$\phi(z,\zeta) = \sum_{klm} \phi_{klm} a_{klm}(z^*,\zeta^*)^* \quad (121)$$

for suitable coefficients ϕ_{klm}. Since ϕ satisfies conditions (115) and (117), one can employ term by term integration to deduce from Eqs. (119) and (121) that

$$\int d\mu(z,\zeta)\exp(z'z^* + \zeta'\cdot\zeta^*)\phi(z,\zeta) = \phi(z',\zeta'). \quad (122)$$

Now we have from the definition (118) of $|\phi\rangle$ that, using Eqs. (104) and (122),

$$\langle\phi|\phi\rangle = \int d\mu(z,\zeta)\int d\mu(z',\zeta')\phi(z,\zeta)\phi(z',\zeta')^*(z',\zeta'|z,\zeta)$$
$$= \int d\mu(z',\zeta')|\phi(z',\zeta')|^2. \quad (123)$$

Therefore $|\phi\rangle$ is normalizable. Furthermore, we can now deduce from Eq. (118), with the help of Eq. (122), that

$$(z',\zeta'|\phi\rangle = \int d\mu(z,\zeta)\phi(z,\zeta)(z',\zeta'|(z,\zeta)$$
$$= \phi(z',\zeta'), \quad (124)$$

as claimed above.

We see that there is a 1–1 correspondence between vectors in our abstract Hilbert space and functions $\phi(z,\zeta)$ of the type described, and that this correspondence is defined by Eqs. (118) and (124). Supposing $|\phi\rangle$ and $|\psi\rangle$ are any two vectors corresponding to functions $\phi(z,\zeta)$ and $\psi(z,\zeta)$ in this way; then, using Eq. (122), we have from Eq. (118) and the corresponding equation for $|\psi\rangle$ that,

$$\langle\psi|\phi\rangle = \int d\mu(z,\zeta)\int d\mu(z',\zeta')\psi(z',\zeta')^*\phi(z,\zeta)(z',\zeta'|z,\zeta)$$
$$= \int d\mu(z',\zeta')\psi(z',\zeta')^*\phi(z',\zeta'). \quad (125)$$

We are now in a position to establish a realization of the abstract Hilbert space and algebra of operators (essentially, a $(\nu^\dagger,\lambda^\dagger)$-representation) by taking $\phi(z,\zeta)$ as the representative of $|\phi\rangle$ in a Hilbert space $\mathscr{H}$ of such functions with scalar product

$$(\phi,\psi) = \int d\mu(z,\zeta)\phi(z,\zeta)^*\psi(z,\zeta), \quad (126)$$

so that

$$(\phi,\psi) = \langle\phi,\psi\rangle. \quad (127)$$

We see from Eq. (110) that (for $|\phi\rangle$ in the domain of $\nu^\dagger$)

$$\nu^\dagger|\phi\rangle = \int d\mu(z,\zeta)(z,\zeta|\nu^\dagger|\phi\rangle|z,\zeta)$$
$$= \int d\mu(z,\zeta)z\phi(z,\zeta)|z,\zeta), \quad (128)$$

so that in $\mathscr{H}$, $\nu^\dagger$ is represented by the operator which sends $\phi(z,\zeta)$ into $z\phi(z,\zeta)$. Similarly, $\lambda^\dagger$ is represented by the operator which sends $\phi(z,\zeta)$ into $\zeta\phi(z,\zeta)$. From our experience with the usual coherent states we know that the representative in $\mathscr{H}$ of ν is $\partial/\partial z$, but the representative of λ is more difficult to determine. We note first that, according to Eqs. (103) and (124), the representative in $\mathscr{H}$ of $|klm\rangle$ is the function $a_{klm}(z^*,\zeta^*)^*$, which is homogeneous of degree k in z and homogeneous of degree l in the variables ζ_i. Equation (105) expresses the orthonormality of these functions with respect to the scalar product (126) and of course, as the representatives of the $|klm\rangle$, they must be complete in $\mathscr{H}$. In particular, we note that the vacuum vector $|0\rangle$ ($= |k=0\ l=0\ m=0\rangle$) is represented by the function 1 and, accordingly, the vector

$$(\nu^\dagger)^k\lambda^\dagger_\alpha\lambda^\dagger_\beta\cdots\lambda^\dagger_\tau|0\rangle, \quad (129)$$

where there are l subscripts $\alpha,\beta,\ldots,\tau$, is represented by

$$T^{kl}_{\alpha\beta\cdots\tau}(z,\zeta) = z^k\zeta_\alpha\zeta_\beta\cdots\zeta_\tau. \quad (130)$$

These functions also form a complete set in $\mathscr{H}$, when k and l run over the nonnegative integers independently and $\alpha,\beta,\ldots,\tau$ run over 1, 2, 3 independently, and they also represent eigenvectors of K and L corresponding to eigenvalues k and l. Note that $T^{kl}_{\alpha\beta\cdots\tau}$ is completely symmetric and traceless in the subscripts. Since it is evident that

$$z\frac{\partial}{\partial z}T^{kl}_{\alpha\beta\cdots\tau} = kT^{kl}_{\alpha\beta\cdots\tau},$$
$$\zeta\cdot\frac{\partial}{\partial\zeta}T^{kl}_{\alpha\beta\cdots\tau} = lT^{kl}_{\alpha\beta\cdots\tau}, \quad (131)$$

we may conclude that, because of the completeness of these functions, K is represented by $z\partial/\partial z$ (a result already evident since we know that $K = \nu^\dagger\nu$) and that L is represented by $\zeta\cdot\partial/\partial\zeta$. Furthermore, we know from Eq. (56) of BL that the representative of λ_i must send $T^{kl}_{\alpha\beta\gamma\cdots\sigma\tau}$ into

$$\begin{aligned}(\delta_{i\alpha}T^{kl-1}_{\beta\gamma\cdots\sigma\tau} + \delta_{i\beta}T^{kl-1}_{\alpha\gamma\cdots\sigma\tau} + \ldots + \delta_{i\tau}T^{kl-1}_{\alpha\beta\gamma\cdots\sigma}\\ - \frac{2}{(2l-1)}(\delta_{\alpha\beta}T^{kl-1}_{i\gamma\cdots\sigma\tau} + \delta_{\alpha\gamma}T^{kl-1}_{i\beta\cdots\sigma\tau} + \ldots + \delta_{\alpha\tau}T^{kl-1}_{i\beta\gamma\cdots\sigma}\\ + \delta_{\beta\gamma}T^{kl-1}_{i\alpha\cdots\sigma\tau} + \ldots + \delta_{\beta\tau}T^{kl-1}_{i\alpha\gamma\cdots\sigma}\\ + \ldots\\ \cdots\\ + \delta_{\sigma\tau}T^{kl-1}_{i\alpha\beta\gamma\cdots}),\end{aligned} \quad (132)$$

which enables us to deduce that $(2L+1)\lambda_i$ is represented by the operator

$$\left(2\zeta\cdot\frac{\partial}{\partial\zeta}+1\right)\frac{\partial}{\partial\zeta_i} - \zeta_i\frac{\partial^2}{\partial\zeta\cdot\partial\zeta}. \quad (133)$$

One can check directly that this operator is hermitian conjugate to $\zeta_i(2\zeta\cdot\partial/\partial\zeta+1)$ [the representative of $\lambda^\dagger_i(2L+1)$] with respect to the scalar product (126), but it is not a straightforward matter because of the delta function in $d\mu$. One needs to use some results on differentiation of the delta function of a complex variable, as described in Ref. (26). Formally, the operator conjugate to ζ_i is now seen to be $\partial/\partial\zeta_i - (2\zeta\cdot\partial/\partial\zeta+1)^{-1}\zeta_i\partial^2/\partial\zeta\cdot\partial\zeta$, which is not a differential operator on $\mathscr{H}$. However, L_i ($= i\hbar\epsilon_{ijk}\lambda^\dagger_j\lambda_k$) is seen to be represented by $-i\hbar\epsilon_{ijk}\zeta_j\,\partial/\partial\zeta_k$.

Summarizing, the realization of our abstract algebra is provided in a Hilbert space $\mathscr{H}$ of analytic functions $\phi(z,\zeta)$ on $\mathbb{C}\times\mathbb{K}_3$, satisfying

$$|\phi(z,\zeta)| \leqslant A\{\exp(\tfrac{1}{2}|z|^2 + \tfrac{1}{2}|\zeta|^2)\}$$

and

$$\int d\mu\,(z,\zeta)|\phi\,(z,\zeta)|^2 < \infty\,. \tag{134}$$

The scalar product in $\mathscr{H}$ is

$$(\phi,\psi) = \int d\mu\,(z,\zeta)\phi\,(z,\zeta)^*\psi(z,\zeta)\,, \tag{135}$$

and the relevant operators are represented as

$$\nu = \frac{\partial}{\partial z}, \quad \nu^\dagger = z, \quad K = z\frac{\partial}{\partial z},$$

$$(2L+1)\lambda = \left(2\zeta\cdot\frac{\partial}{\partial\zeta} + 1\right)\frac{\partial}{\partial\zeta} - \zeta\,\frac{\partial^2}{\partial\zeta\cdot\partial\zeta},$$

$$\lambda^\dagger = \zeta, \quad L = \zeta\cdot\frac{\partial}{\partial\zeta}$$

$$L_i = -i\hbar\epsilon_{ijk}\zeta_j\frac{\partial}{\partial\zeta_k}\,. \tag{136}$$

The common eigenvalues of K, L, and L_3 are represented as

$$|klm\rangle = \left[\frac{(2l)!}{k!2^l l!(l+m)!(l-m)!}\right]^{1/2} \times z^k(\zeta_3)^{l-|m|}(-\epsilon\zeta_\epsilon)^{|m|}\,. \tag{137}$$

This Hilbert space has a reproducing kernel given by

$$K(z',\zeta';z,\zeta) = \exp(z'^*z + \zeta'^*\cdot\zeta)\,, \tag{138}$$

since Eqs. (122) and (126) together give

$$(K(z',\zeta';\ \ ),\phi) = \phi\,(z',\zeta')\,. \tag{139}$$

The function $K(z',\zeta';z,\zeta)$ can be seen to be the representative in $\mathscr{H}$ of $|z',\zeta'\rangle$. The space $\mathscr{H}$ is especially attractive as a carrier space for SO(3). If we restrict out attention to functions $f(\zeta)$ analytic on the cone $\mathbb{K}_3$ (in effect, we consider those ϕ in $\mathscr{H}$ satisfying $K\phi = 0$), then we have a Hilbert subspace $\mathscr{H}_0$ with scalar product

$$(f_1,f_2) = \frac{2}{\pi^2}\int d^6\zeta\,\delta(\zeta\cdot\zeta)(2|\zeta|^2-1)\exp(-|\zeta|^2) \times f_1(\zeta)^* f_2(\zeta)\,, \tag{140}$$

carrying a reducible unitary representation of SO(3), with hermitian generators

$$\hbar^{-1}L_i = -i\epsilon_{ijk}\zeta_j\,\frac{\partial}{\partial\zeta_k}\,. \tag{141}$$

If we label as (l) the $(2l+1)$-dimensional irreducible representation of SO(3), we see that

$$\mathscr{H}_0 = \mathscr{H}_{00}\oplus\mathscr{H}_{01}\oplus\cdots, \tag{142}$$

where $\mathscr{H}_{0l}$ is $(2l+1)$-dimensional and carries the representation (l). A basis for $\mathscr{H}_{0l}$ is provided by the orthonormal functions

$$|0lm\rangle = \left[\frac{(2l)!}{2^l l!(l+m)!(l-m)!}\right]^{1/2}(\zeta_3)^{l-|m|}(-\epsilon\zeta_\epsilon)^{|m|} \tag{143}$$

of Eq. (137); or altenatively, by the $(2l+1)$ linearly independent elements of the traceless, symmetric rank-l tensor

$$T^{0l}_{\alpha\beta\cdots\tau}(\zeta) = \zeta_\alpha\zeta_\beta\cdots\zeta_\tau\,. \tag{144}$$

The decomposition (142) merely symbolizes the expansion of any of the analytic functions f in $\mathscr{H}_0$ in series form:

$$f(\zeta) = \sum_{l=0}^{\infty} d^{\,l;\alpha\beta\cdots\tau}\zeta_\alpha\zeta_\beta\cdots\zeta_\tau\,. \tag{145}$$

This space $\mathscr{H}_0$, which can be compared with the Bargmann space[27] for SU(2), can be said to provide a realization of the "modified boson" structure set up by Lohe and Hurst.[28] They introduced operators satsifying the same algebraic relations as our $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\dagger$, but nothing corresponding to our $\mathbf{v}$ and $\mathbf{v}^\dagger$. Recently Bargmann and Todorov[29] have described a very similar carrier space for SO(3). They also consider analytic functions $f(\zeta)$ on the cone $\mathbb{K}_3$, but choose a more complicated scalar product, and consequently find a more complicated reproducing kernel. In their space, however, the operator conjugate to ζ_i is simply $(\zeta\cdot\partial/\partial\zeta + \frac{1}{2})\,\partial/\partial\zeta_i - \frac{1}{2}\zeta_i\,\partial^2/\partial\zeta\cdot\partial\zeta$, and these two vector operators, together with our L and $\hbar^{-1}\mathbf{L}$, generate a unitary representation of SO(3,2). Their space was not derived from a consideration of coherent states of any kind, but we have mentioned in BL that there is an so(3,2) algebra associated with the oscillator, spanned by our operators L, $\hbar^{-1}\mathbf{L}$, $\boldsymbol{\Lambda}$, and $\boldsymbol{\Lambda}^\dagger$, where

$$\boldsymbol{\Lambda} = (2L+1)^{1/2}\boldsymbol{\lambda}, \quad \boldsymbol{\Lambda}^\dagger = \boldsymbol{\lambda}^\dagger(2L+1)^{1/2}\,. \tag{146}$$

Had we chosen to define coherent angular momentum states by diagonalizing the operators $\boldsymbol{\Lambda}$ rather than $\boldsymbol{\lambda}$, we would have arrived at the Hilbert space of Bargmann and Todorov. The coherent states so obtained would evidently be generalized coherent states for so(3,2) (in particular) in the sense of Barut and Girardello[17] and Perelomov.[18] We have already given in BL some reasons for our preference for the operators $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\dagger$. The main point is that with our definitions, the expectation values of H, K, L, and $\mathbf{L}$ in the coherent angular momentum states have simple properties—more simple than if we were to follow the alternative path. In particular, the nice property that in a coherent angular momentum state the l values are distributed in probability according to a Poisson distribution would be lost if we were to diagonalize the Λ_i.

6. CONCLUDING REMARKS

The existence of a second set of coherent or quasiclassical states for the oscillator places the usual ones in a new perspective. The two sets share several interesting properties, as we have seen, and we therefore hope to find interesting applications of our new states. In particular, we hope to be able to construct a new diagonal representation of the density matrix for the oscillator.

The usual coherent states are also quasiclassical states for the Hamiltonian

$$H' = H + \alpha L_3$$

corresponding to a charged oscillator in a uniform magnetic field. The same is true for coherent angular momentum states, since the diagonalized operators ν, λ_3, and $\lambda_\pm$, like a_3 and $a_\pm$ $(=a_1 \pm ia_2)$, are all shift operators for H'. The new states are also quasi-classical states for a Hamiltonian of the form

$$H' = H + \alpha L_3 + \beta L\,,$$

which is not an exact Hamiltonian for any physical system but may be of interest in the approximate description of some molecular spectra. In that connection, we may also expect the coherent angular momentum states to be of value in the analysis of Hamiltonians of the form

$$H' = H + \alpha L_3 + \beta L + \gamma \mathbf{L}^2,$$

although they will not then define states which are quasiclassical in the same sense as above.

The analysis we have developed above and in BL can no doubt be extended to $n > 3$ dimensions. The Hilbert space for the isotropic oscillator in n dimensions carries only symmetric representations of so(n), labelled by a single nonnegative integer l. Accordingly, the first step in the generalization would be the introduction of a scalar operator L which has nonnegative integer eigenvalues l. One would then proceed to resolve n-vector boson operators $\mathbf{a}, \mathbf{a}^\dagger$ into shift operators for L. There appear to be various interesting alternative paths to follow from that point, corresponding to various chains of orthogonal subgroups of so(n).

In the case $n = 2$, the boson operators $a_\pm$ are shift operators for the so(2) scalar L_3, and accordingly, the usual coherent states and the coherent angular momentum states may be identified.

We recall that the eigenvalue problem for the three-dimensional isotropic oscillator can also be solved by separation of variables in a cylindrical-polar coordinate system. Are there then "cylindrical" coherent states, as well as "Cartesian" and "spherical" ones? The answer is "yes," but they can be taken to be the usual (Cartesian) ones. (Diagonalize a_3 and $a_\pm$, which are shift operators for N_3 and L_3.)

With regard to the definition of coherent angular momentum states for other Hamiltonians of the form

$$H = \mathbf{p}^2/2M + V(|\mathbf{x}|),$$

it is clear that one cannot, in general, hope to obtain quasiclassical states in the sense described above. The existence of this property for the oscillator depends on the special feature that the total angular momentum quantum number l appears *linearly* in the formula for the energy eigenvalue, when expressed in terms of l and the radial quantum number k:

$$E_{kl} = \hbar\omega(2k + l + \tfrac{3}{2}).$$

However, in the more general case one can still diagonalize vector lowering operators for L in order to define overcomplete sets of states for the "angular part" of the Hilbert space, and one may be able to use these to construct representations in which the angular part of the density matrix is diagonal.

ACKNOWLEDGMENT

We thank Professor L. Bass for several helpful discussions.

APPENDIX A

The combination of Eqs. (12) and (14) with the expanded form of $Y_{lm}(\theta,\phi)$ and property (28) gives

$$|z,\zeta\rangle = \exp(-\tfrac{1}{2}|z|^2 - \tfrac{1}{2}|\zeta|^2)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} z^k(\zeta_3)^{l-m}(-\zeta_-)^m\left[\frac{(2l)!}{k!2^l l!(l+m)!(l-m)!}\right]^{1/2}(-1)^k$$
$$\times\left[\frac{2a^3 k!}{\Gamma(k+l+\frac{3}{2})}\right]^{1/2}\xi^l e^{-\frac{1}{2}\xi^2}L_k^{(l+\frac{1}{2})}(\xi^2)(-1)^m\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2}P_l^m(\cos\theta)e^{im\phi} \quad (A1)$$
$$= \exp(-\tfrac{1}{2}|z|^2 - \tfrac{1}{2}|\zeta|^2 - \tfrac{1}{2}\xi^2)\times\left[\frac{a^3}{\pi^{3/2}}\right]^{1/2}\times\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\frac{\Gamma(l+\frac{3}{2})}{\Gamma(k+l+\frac{3}{2})}\right]^{1/2}(-z)^k L_k^{(l+\frac{1}{2})}(\xi^2)$$
$$\times\sum_{m=-l}^{l}\frac{1}{(l+m)!}(\sqrt{2}\xi)^l(\zeta_3)^{l-m}(\zeta_-)^m P_l^m(\cos\theta)e^{im\phi}, \quad (A2)$$

using the result that

$$\Gamma(l+\tfrac{3}{2}) = \frac{(2l+1)!\sqrt{\pi}}{2^{2l+1}l!}. \quad (A3)$$

We are able to evaluate the sum over m by noting that for any vector $\mathbf{v}$, with spherical polar coordinates v, θ, ϕ,

$$\mathbf{v}\cdot\boldsymbol{\zeta} = v\zeta_3\left\{\tfrac{1}{2}\left[\frac{\zeta_-}{\zeta_3}e^{i\phi} + \frac{\zeta_+}{\zeta_3}e^{-i\phi}\right]\sin\theta + \cos\theta\right\} = v\zeta_3\{\tfrac{1}{2}[A - 1/A]\sin\theta + \cos\theta\}, \quad (A4)$$

where $A = \zeta_- e^{i\phi}/\zeta_3$. We can further show, by induction, that

$$\{\tfrac{1}{2}[A - 1/A]\sin\theta + \cos\theta\}^l = l!\sum_{m=-l}^{l} A^m P_l^m(\cos\theta)/(l+m)!, \quad (A5)$$

from which it follows that

$$\frac{(\mathbf{v}\cdot\boldsymbol{\zeta})^l}{l!} = \sum_{m=-l}^{l}\frac{1}{(l+m)!}v^l(\zeta_3)^{l-m}(\zeta_-)^m P_l^m(\cos\theta)e^{im\phi}. \quad (A6)$$

With the use of this result in the case $\mathbf{v} = (\sqrt{2})a\mathbf{x}$, Eq. (A2) becomes

$$|z,\zeta\rangle = \exp(-\tfrac{1}{2}|z|^2 - \tfrac{1}{2}|\zeta|^2 - \tfrac{1}{2}\xi^2)\times\left[\frac{a}{\sqrt{\pi}}\right]^{3/2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\frac{\Gamma(l+\frac{3}{2})}{\Gamma(k+l+\frac{3}{2})}\right]^{1/2}(-z)^k L_k^{(l+1/2)}(\xi^2)\frac{(\sqrt{2}a\mathbf{x}\cdot\boldsymbol{\zeta})^l}{l!}, \quad (A7)$$

as in Eq. (15).

When $z=0$ only the $k=0$ term contributes, so that

$$|0,\zeta\rangle = \exp(-\tfrac{1}{2}|\zeta|^2 - \tfrac{1}{2}a^2|\mathbf{x}|^2)\left[\frac{a}{\sqrt{\pi}}\right]^{3/2}\sum_{l=0}^{\infty}\frac{(\sqrt{2}a\mathbf{x}\cdot\zeta)^l}{l!}, \tag{A8}$$

which is equivalent to Eq. (16).

APPENDIX B

Using the definitions (13) and (106) of a_{klm} and $d\mu$, respectively, and property (28), we can write

$$\int d\mu(z,\zeta)a_{klm}(z^*,\zeta^*)^*a_{KLM}(z^*,\zeta^*)$$
$$= \frac{2}{\pi^3}\left[\frac{(2l)!(2L)!}{k!K!l!L!2^{l+L}(l+m)!(L+M)!(l-m)!(L-M)!}\right]^{1/2}\int d^2z\exp(-|z|^2)z^k(z^*)^K$$
$$\times\int d^6\zeta\,(2|\zeta|^2-1)\exp(-|\zeta|^2)\delta(\zeta\cdot\zeta)(\zeta_3)^{l-m}(\zeta_3^*)^{L-M}(-\zeta_+)^m(-\zeta_-^*)^M. \tag{B1}$$

It is straightforward to show that

$$\int d^2z\exp(-|z|^2)z^k(z^*)^K = \pi k!\delta_{k,K}. \tag{B2}$$

However, the next integration is not so easy. Letting $\zeta = \mathbf{x}+i\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are the real vectors, the delta function becomes [see Ref. (26)]

$$\delta(\zeta\cdot\zeta) = \delta(|\mathbf{x}|^2-|\mathbf{y}|^2)\delta(2\mathbf{x}\cdot\mathbf{y}), \tag{B3}$$

and the integral becomes

$$\int d^6\zeta\,(2|\zeta|^2-1)\exp(-|\zeta|^2)\delta(\zeta\cdot\zeta)(\zeta_3)^{l-m}(\zeta_3^*)^{L-M}(-\zeta_+)^m(-\zeta_-^*)^M$$
$$= \int d^3x\,d^3y\exp(-|\mathbf{x}|^2-|\mathbf{y}|^2)\delta(|\mathbf{x}|^2-|\mathbf{y}|^2)\delta(2\mathbf{x}\cdot\mathbf{y})(2|\mathbf{x}|^2+2|\mathbf{y}|^2-1)$$
$$\times(-1)^{m+M}(x_3+iy_3)^{l-m}(x_3-iy_3)^{L-M}[x_1-y_2+i(x_2+y_1)]^m[x_1-y_2-i(x_2+y_1)]^M. \tag{B4}$$

We can introduce the new variables r, θ, ϕ, R, ψ, and u by the following:

$$[x_1,x_2,x_3] = [r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta]$$

and

$$\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}\cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi\\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}R\cos\psi\\R\sin\psi\\u\end{bmatrix}. \tag{B5}$$

The matrix is orthogonal, with determinant equal to 1. It is easily inverted to give

$$u = (\mathbf{x}\cdot\mathbf{y})/r. \tag{B6}$$

Now

$$\delta(2\mathbf{x}\cdot\mathbf{y}) = \delta(2ru) = \frac{1}{2r}\delta(u), \tag{B7}$$

and the integral (B4) becomes, after the u integration

$$\tfrac{1}{2}\int r\,dr\sin\theta\,d\theta\,d\phi\,R\,dR\,d\psi\exp(-r^2-R^2)\delta(r^2-R^2)(2r^2+2R^2-1)(-1)^{m+M}(r\cos\theta-iR\sin\theta\cos\psi)^{l-m}$$
$$\times(r\cos\theta+iR\sin\theta\cos\psi)^{L-M}[r\sin\theta\cos\phi-R\cos\theta\sin\phi\cos\psi-R\cos\phi\sin\psi$$
$$+i(r\sin\theta\sin\phi+R\cos\theta\cos\phi\cos\psi-R\sin\phi\sin\psi)]^m[r\sin\theta\cos\phi-R\cos\theta\sin\phi\cos\psi-R\cos\phi\sin\psi$$
$$-i(r\sin\theta\sin\phi+R\cos\theta\cos\phi\cos\psi-R\sin\phi\sin\psi)]^M. \tag{B8}$$

Since

$$\delta(r^2-R^2) = \frac{1}{r+R}\delta(r-R) \tag{B9}$$

here, we can perform the R integration and obtain

$$\tfrac{1}{4}\int r\,dr \sin\theta\, d\theta\, d\phi\, d\psi \exp(-2r^2) r^{l+L}(4r^2-1)(-1)^{m+M}(\cos\theta - i\sin\theta\cos\psi)^{l-m}(\cos\theta + i\sin\theta\cos\psi)^{L-M}$$

$$\times[\sin\theta - \sin\psi + i\cos\theta\cos\psi]^m[\sin\theta - \sin\psi - i\cos\theta\cos\psi]^M e^{i(m-M)\phi} \quad \text{(B10)}$$

$$= \tfrac{1}{2}\pi\delta_{m,M}\int r\,dr\sin\theta\, d\theta\, d\psi \exp(-2r^2) r^{l+L}(4r^2-1)(\cos\theta - i\sin\theta\cos\psi)^l \times (\cos\theta + i\sin\theta\cos\psi)^L$$

$$\times(1+\sin\theta\sin\psi)^{-m}(1-\sin\theta\sin\psi)^m . \quad \text{(B11)}$$

Here we make yet another change of variables from θ, ψ to α, β, where

$$[\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha] = [\cos\theta, \sin\theta\cos\psi, \sin\theta\sin\psi] \quad \text{(B12)}$$

and the integral becomes

$$\tfrac{1}{2}\pi\delta_{m,M}\int r\,dr\sin\alpha\, d\alpha\, d\beta \exp(-2r^2) r^{l+L}(4r^2-1)(1+\cos\alpha)^{-m}(1-\cos\alpha)^m(\sin\alpha)^{l+L}e^{i(L-l)\beta} \quad \text{(B13)}$$

$$= \pi^2\delta_{m,M}\delta_{l,L}\int r\,dr \exp(-2r^2) r^{2l}(4r^2-1)\int \sin\alpha\, d\alpha\,(1+\cos\alpha)^{l-m}(1-\cos\alpha)^{l+m} \quad \text{(B14)}$$

$$= \frac{\pi^2}{2}\frac{l!2^l(l-m)!(l+m)!}{(2l)!}\delta_{l,L}\delta_{m,M}\,, \quad \text{(B15)}$$

using the properties of the Gamma and Beta functions. Substitution of (B15) and (B2) into (B1) gives the required result, Eq. (105).

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