Local conformal-invariance of the wave equation for finite-component fields. I. The conditions for invariance, and fully-reducible fields

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The conditions for local conformal-invariance of the wave equation are obtained for finite-component fields, of Types Ia and Ib [in the terminology of Mack and Salam, Ann. Phys. **53**, 174 (1969).] These conditions generate a set of locally invariant free massless field equations and restrict the relevant representation of the Lie algebra $[(k_4 \oplus d) \oplus' sl(2,C)]$ in the index space of the field to belong to a certain class. Those fully-reducible representations which are in this class are described in full. The corresponding Type Ia field equations include only the massless scalar field equation, neutrino equations, Maxwell's equations, and the Bargmann–Wigner equations for massless fields of arbitrary helicity, and no others. In particular, it is confirmed [Bracken, Lett. Nuovo Cimento **2**, 574 (1971)] that not all Poincaré-invariant sets of massless Type Ia field equations are conformal-invariant, contrary to some often-quoted results of McLennan [Nuovo Cimento **3**, 1360 (1956)], which are shown to be invalid. It is also shown that in the case of a potential, the wave equation is never conformal-invariant in the strong sense (excluding gauge transformations).

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1. INTRODUCTION

Much has been written on the theory and possible applications to particle physics of the conformal group of space-time transformations: for reviews, see Kastrup,[1] Fulton, Rohrlich, and Witten,[2] Barut,[3] Ferrara, Gatto, and Grillo,[4] and Bayen.[5] These ideas were largely stimulated by observations that the wave equations satisfied by certain free, massless fields are locally[6] conformal-invariant.

Bateman[7] and Cunningham[8] (see also Dirac[9]) showed that this is so for the free-field Maxwell equations; and according to Cunningham, Bateman knew then of the invariance of the wave equation

$$\Box\psi(x) = 0$$
$$x = (ct,\mathbf{x}) = (x^\mu) \quad \mu = 0,1,2,3 \tag{1.1}$$

in the case of a scalar field ψ. We do not know who first proved the invariance of the two- and four-component neutrino equations. (See, however, Schouten and Haantjes,[10] Pauli,[11] and Bludman.[12]) McLennan[13] claimed to prove the invariance of each of Gårding's[14] "irreducible sets" of wave equations for massless multi-spinor fields (at least, of each set which admits plane-wave solutions, the remainder being unsuitable as free-field equations.) These sets of first-order equations are rather general and include ones described earlier by Dirac[15] and Fierz.[16] Gross[17] showed the invariance of the Bargmann-Wigner[18] equations for massless fields of arbitrary spin. The invariance of particular sets of massless field equations has also been argued by Lomont,[19] Penrose,[20] Kursunoglu,[21] Mack and Todorov,[22] Bayen,[23] Barut and Haugen,[24] Lopuszanski and Oziewicz,[25] Post,[26] Fegan,[27] Jakobsen and Vergne,[28] and Budini.[29] Kotecky and Niederle[30] have found the conditions for conformal invariance of a Lorentz-invariant equation of the form

$$L_\mu \partial^\mu \psi(x) = 0,$$
$$\partial^\mu = \partial/\partial x_\mu, \tag{1.2}$$

where the L_μ are matrices (not necessarily square), and ψ is a multicomponent field. However, they did not specifically require that ψ be massless in the sense of Eq. (1.1).

It is clear that a body of opinion has developed to the effect that wave equations for free, massless fields are conformal-invariant in all possible cases [at least, in all cases where the fields have (manifestly) Lorentz-invariant helicity[31]—there are known subtleties in the case of equations satisfied by potentials[22,32,33]]. In the introductory remarks to many papers on the conformal group and its applications, one can find passing reference to "the well known fact that massless wave equations are conformal-invariant."

This opinion has no doubt been reinforced by the observation[22,34,35] that every zero-mass, discrete spin, unitary, irreducible representation of the Poincaré group ISL(2,C) can be extended to a unitary, irreducible representation of SU(2,2), a group locally isomorphic to the conformal group. Given a consistent set of field equations for a free, massless, classical field with Lorentz-invariant helicity, one should be able to exhibit a Hilbert space of solutions carrying a representation of ISL(2,C) of this type. This solution space will then be invariant under the action of an SU(2,2) group.

One might be forgiven for thinking that there is little more to be said on this subject, at least in the case of fields having Lorentz-invariant helicity. On the other hand, it is

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clear that the conformal invariance of the wave equation (1.1), which is evidently scale- and Poincaré-invariant, will in general require further, non-trivial, conditions to be satisfied when $\psi(x)$ is a multicomponent field. After all, the Poincaré group extended by dilatations is a *proper* subgroup of the conformal group, and we recall that in the case of Lagrangian field equations,[32,36–40] scale- and Poincaré-invariance does not guarantee conformal-invariance. We assert that, contrary to the body of opinion mentioned above, the wave equations satisfied by free massless fields are not in general locally conformal-invariant, even for fields having Lorentz-invariant helicity.

Some years ago, one of us showed[41] that if the index space of a field $\psi(x)$ carries an irreducible, finite-dimensional representation of sl(2,C) labeled (m,n) (in the familiar scheme, where $2m$ and $2n$ are non-negative integers, as described in the next section), then if $mn \neq 0$ the wave equation (1.1) is not locally conformal-invariant. If this be so, then some of the results of McLennan[13] in particular must be false. Indeed, it is not immediately clear that this result of Ref. 41 can be reconciled with the invariance of the Bargmann–Wigner equations,[18] though it turns out that there is no contradiction there, as we show in Sec. 4, where we discuss the results of earlier works in relation to ours. There also we point out some errors in McLennan's work, invalidating some of his conclusions.

What of the second argument suggested above, concerning the extendability of massless representations of ISL(2,C) to representations of SU(2,2)? The reconciliation of this fact with the noninvariance (in some cases) of the equation (1.1), has been discussed earlier.[41] Essentially, the point is that the group SU(2,2) which arises in this way cannot always be associated even locally with the conformal group. Suppose, for example, we construct a realization of the zero-mass, discrete spin, helicity λ, positive energy, unitary representation of ISL(2,C) in a Hilbert space of multicomponent fields $\psi(x)$, which have Lorentz-invariant helicity and whose index space carries a single representation (m,n) of sl(2,C). According to a result of Weinberg,[42] (see also Lemma 3.2 below), it must be true that $m - n = \lambda$, though not necessarily that $mn = 0$. According to the results of Ref. 22, we can find in addition to the ISL(2,C) generators P_μ and $M_{\mu\nu}$, operators D' and K'_μ acting on this space. Together these operators generate a unitary irreducible representation of SU(2,2) in the so-called "ladder series." Now what happens is this: If $mn \neq 0$, then K'_μ can *not* be identified with the generators of special conformal transformations of the fields $\psi(x)$. Those generators have rather specific forms, as described by Mack and Salam.[43] (See Sec. 2.) In particular, the operators K'_μ are not local in space–time when $mn \neq 0$. What is more, in those cases they only satisfy the appropriate commutation relations within the representation space—this is, only weakly on the fields, as a consequence of the free-field equations. In contrast, in the cases when $mn = 0$, K'_μ (and D') *are* identifiable with generators of conformal transformations.[22,35] They are local operators, and they can be defined on all (sufficiently smooth) fields of the given type, in such a way that the appropriate commutation relations are satisfied, whether or not the fields satisfy the free-field equations. These properties are crucial if one is to be able to talk meaningfully about conformal-invariance being preserved in the presence of interactions, when the free-field equations cease to hold.

In short, when $mn = 0$, the conformal group is a space–time symmetry group of the field equations, while when $mn \neq 0$, SU(2,2) is only a dynamical symmetry group of the one-particular Hilbert space. The difference between these two concepts is quite fundamental, but in the present context it has not generally been fully appreciated.

In view of the fact that not all possible Poincaré-invariant massless field theories are conformal-invariant, the invariance of the equations governing the electromagnetic and neutrino fields assumes, perhaps, a greater significance. Unfortunately, Ref. 41 seems to have been largely unnoticed,[44–46] and passing remarks persist to "the well known fact that...." Indeed, after this work[41] appeared, a proof of the conformal-invariance of the field equations in the cases $mn \neq 0$ was presented by Post.[26] This proof is deficient, as we show in Sec. 4, and Post's conclusions in this regard are false.

Recently there has been renewed interest in massless, higher-spin fields,[47–50] and fields of spin $\frac{3}{2}$ and $\frac{5}{2}$ in particular have been discussed in connection with "supergravity." The question now arises as to whether or not the theories proposed are conformal-invariant. While we do not examine this question specifically, it seems timely to investigate in detail the conditions under which the wave equation (1.1) is locally conformal-invariant when ψ is a finite-component field, and that is our main object here. We do not restrict ourselves to the cases where the index space carries an irreducible representation of sl(2,C), but rather consider the most general possible situation, according to Mack *et al.*,[43] where the field may be of Type Ib in their notation. (See Sec. 2.) Such fields have received comparatively little attention in the literature.[9,24,25,29,43,51–53] As free fields, their main interest lies in the possibility that one might be able to use them to describe spin multiplets of massless particles.[43] There are discouraging difficulties in attempting to describe such fields in any generality, because of the nature of the finite-dimensional index-space representations of the Lie algebra $\mathscr{W}$,

$$\mathscr{W} = (k_4 \oplus d) \oplus \mathrm{sl}(2,C), \tag{1.3}$$

which are involved. (See Sec. 2.) These representations are not in general completely reducible, and no classification of them is available. However, we find that only a certain class of representations is directly involved in the case of free massless fields obeying conformal-invariant equations.

Our main results are summarized in Theorems 3.1, 3.2, 3.3, 2.1, 3.4, 3.5, 4.1, and 4.2 below. In particular, we find that when Eq. (1.1) is locally conformal-invariant, then the field ψ must satisfy certain other equations. For example, if ψ is an antisymmetric tensor field $F_{\mu\nu}(x)$, then conformal-invariance of Eq. (1.1) requires that $F_{\mu\nu}$ satisfy *all* of Maxwell's free-field equations. Thus the imposition of conformal-invariance of the "mass condition" (1.1) can be a means of defining complete sets of conformal-invariant free-field equations. This fact leads us not only to well-known sets of wave equations, but also to new sets of locally conformal-invariant equations for massless fields of Type Ib with arbitrary helicity.

In general the extra equations which ψ must satisfy place severe restrictions on the representation of $\mathscr{W}$ carried by the index space of ψ. Furthermore, they imply that in every case ψ is a direct sum of fields having Lorentz-invariant helicity. Thus when ψ is a potential, the wave equation (1.1) is never conformal-invariant in the strong sense (i.e., excluding the possibility of gauge transformations to supplement the conformal transformations). This generalizes a well-known result[22,32,38] for the electromagnetic potential $A_\mu(x)$. We do not address the problem of classifying for potentials those equations which are conformal-invariant in the weak sense, i.e., up to a change of gauge.

Notation: We adopt the diagonal metric tensor $g_{\mu\nu} = g^{\mu\nu}$, with $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$. The alternating tensor $\epsilon_{\mu\nu\rho\sigma}$ is defined with $\epsilon^{0123} = -\epsilon_{0123} = 1$.

2. Preliminaries. Index space representations of sl(2,C) and $\mathscr{W}$

Consider infinitesmal conformal transformations of space–time,

$$x'^\mu = x^\mu + \epsilon x^\mu + \epsilon^\mu + \epsilon^{\nu\mu} x_\nu + (2\theta^\nu x_\nu x^\mu - \theta^\mu x^\nu x_\nu)$$

(symbolically,

$$x' = x + \delta g x), \tag{2.1}$$

where ϵ, ϵ^μ,$\epsilon^{\nu\mu}(= -\epsilon^{\mu\nu})$ and θ^ν are real infinitesimal parameters characterizing dilatations, translations, homogeneous Lorentz transformations, and special conformal transformations, respectively. Suppose we are given classical fields $\psi(x)$, with a fixed finite number of complex-valued components $\psi_a(x)$, and a cotransformation law of the general form

$$\psi'_a(x') = \psi_a(x) + \sum_b \delta S(\delta g, x)_{ab} \psi_b(x). \tag{2.2}$$

Mack *et al.*[43] (see also Flato *et al.*[38] and Kotecky and Niederle[54]) have shown that there is no loss of generality if the following statements are assumed to follow:

(1) The index space of the fields carries a finite-dimensional representation of the 11-dimensional Lie algebra[55] $\mathscr{W}$ of Eq. (1.3), with basis $\Sigma_{\mu\nu}(= -\Sigma_{\nu\mu})$,$\Delta$ and κ_μ, satisfying

$$i[\Sigma_{\mu\nu},\Sigma_{\rho\sigma}] = g_{\mu\rho}\Sigma_{\nu\sigma} + g_{\nu\sigma}\Sigma_{\mu\rho} - g_{\nu\rho}\Sigma_{\mu\sigma} - g_{\mu\sigma}\Sigma_{\nu\rho}, \tag{2.3a}$$

$$i[\kappa_\mu,\Sigma_{\nu\rho}] = g_{\mu\rho}\kappa_\nu - g_{\mu\nu}\kappa_\rho, \tag{2.3b}$$

$$[\kappa_\mu,\kappa_\nu] = 0, \tag{2.3c}$$

$$[\Delta,\Sigma_{\mu\nu}] = 0, \tag{2.3d}$$

$$i[\kappa_\mu,\Delta] = \kappa_\mu. \tag{2.3e}$$

(2) The infinitesimal field transformation (2.2) corresponding to (2.1) can be written in the form

$$\psi'(x) = \psi(x) + i[\epsilon D + \epsilon^\mu P_\mu + \tfrac{1}{2}\epsilon^{\mu\nu} M_{\mu\nu} + \theta^\mu K_\mu]\psi(x), \tag{2.4}$$

where

$$P_\mu = i\partial/\partial x^\mu, \quad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + \Sigma_{\mu\nu},$$
$$D = x^\mu P_\mu + \Delta, \quad K_\mu = 2x_\mu D - x^\nu x_\nu P_\mu + 2\Sigma_{\mu\nu}x^\nu + \kappa_\mu. \tag{2.5}$$

These statements (1) and (2) form the starting point of our analysis.

When we refer to "the field $\psi(x)$" we always have in mind the general element of the complex vector space $\mathscr{D}$ of smooth fields of a given type, i.e., fields whose components have partial derivatives of all orders, and which correspond to a given finite-dimensional representation of $\mathscr{W}$. This space $\mathscr{D}$ is the tensor product of the index space, with operators $\Sigma_{\mu\nu}$,Δ, etc., and the space of smooth functions $f(x)$, with operators x^μ,∂_μ etc. In Eqs. (2.5) the operators Δ and x^μ, for example, really denote the extensions in the obvious way to the tensor product space, of the index-space operator Δ and the function-space operator x^μ. We abuse the notation in this way and rely on context to make precise what we mean in any given case. We remark also that a complex numerical multiple of the identity operator on any of these spaces will be denoted by the appropriate complex number; again we rely on context to make the meaning precise.

It can be seen that $\mathscr{D}$ is a common, invariant domain for the operators P_μ,$M_{\mu\nu}$,D, and K_μ. On this space, the following commutation relations hold[43]:

$$i[D,P_\mu] = P_\mu, \tag{2.6a}$$

$$i[K_\mu,D] = K_\mu, \tag{2.6b}$$

$$[D,M_{\mu\nu}] = 0, \tag{2.6c}$$

$$i[P_\mu,M_{\nu\rho}] = g_{\mu\rho}P_\nu - g_{\mu\nu}P_\rho, \tag{2.6d}$$

$$i[K_\mu,M_{\nu\rho}] = g_{\mu\rho}K_\nu - g_{\mu\nu}K_\rho, \tag{2.6e}$$

$$i[M_{\mu\nu},M_{\rho\sigma}] = g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\nu\rho}, \tag{2.6f}$$

$$[P_\mu,P_\nu] = 0, \tag{2.6g}$$

$$[K_\mu,K_\nu] = 0, \tag{2.6h}$$

$$i[P_\mu,K_\nu] = 2M_{\mu\nu} - 2g_{\mu\nu}D. \tag{2.6i}$$

It follows that the operators D, P_μ,K_μ, and $M_{\mu\nu}$ (of which 15 are linearly independent) span a Lie algebra $\mathscr{A}$, which provides a representation in $\mathscr{D}$ of the Lie algebra of the conformal group.

The representation of $\mathscr{W}$ in the index space may also be regarded as a representation of the sl(2,C) subalgebra of $\mathscr{W}$, with basis $\Sigma_{\mu\nu}$. As such it will not in general be irreducible, but like any other finite-dimensional representation of sl(2,C) it will be fully reducible to a direct sum of irreducible representations. In any representation of sl(2,C), with basis $\Sigma_{\mu\nu}$, we can introduce the two Casimir operators[56]

$$C_1 = \tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}$$
$$C_2 = \tfrac{1}{8}i\epsilon_{\mu\nu\rho\sigma}\Sigma^{\mu\nu}\Sigma^{\rho\sigma}. \tag{2.7}$$

Let (m,n) denote the irreducible representation, of dimension $(2m+1)(2n+1)$, in which these Casimir operators have the form

$$C_1 = 2m(m+1) + 2n(n+1)$$
$$C_2 = m(m+1) - n(n+1). \tag{2.8}$$

Here $2m$ and $2n$ are non-negative integers. Any given finite-dimensional representation $\mathscr{R}$ of sl(2,C), with representation space $\mathscr{V}$, will be a direct sum of such irreducible representations, for various *distinct* ordered pairs (m,n) in a finite set S

determined by $\mathscr{R}$, and with various positive integral multiplicites r_{mn}, determined by $\mathscr{R}$. Symbolically,

$$\mathscr{R} = \sum_{(m,n)\in S}^{\oplus} r_{mn}(m,n) \tag{2.9}$$

Let P_{mn} denote the projector onto that subspace $\mathscr{V}_{mn}$ of $\mathscr{V}$ which carries all the r_{mn} multiples of the irreducible representation (m,n). Then

$$\sum_{(m,n)\in S} P_{mn} = 1,$$
$$P_{mn}P_{kl} = P_{mn}\delta_{mk}\delta_{nl}, \quad (m,n),(k,l)\in S$$
$$[P_{mn},\Sigma_{\mu\nu}] = 0. \tag{2.10}$$

The space $\mathscr{V}$ is a direct sum of the subspaces $\mathscr{V}_{mn}$. Now define the operators

$$M = \sum_{(m,n)\in S} mP_{mn}, N = \sum_{(m,n)\in S} nP_{mn}, \tag{2.11}$$

and note from Eqs. (2.10) that

$$[M,\Sigma_{\mu\nu}] = 0 = [N,\Sigma_{\mu\nu}]$$
$$[M,N] = 0. \tag{2.12}$$

Thus M and N are commuting sl(2,C) scalars. We note also from Eqs. (2.8) and (2.11) that on all of $\mathscr{V}$,

$$C_1 = 2M(M+1) + 2N(N+1),$$
$$C_2 = M(M+1) - N(N+1). \tag{2.13}$$

These operators M and N are more convenient than C_1 and C_2 as labeling operators for the subspace $\mathscr{V}_{mn}$ of $\mathscr{V}$. While M and N by definition are functions of the projectors P_{mn}, it is important to see that, because the eigenvalues (m,n) of the pair (M,N) distinguish the subspaces $\mathscr{V}_{mn}$ onto which the P_{mn} project, these projectors can be regarded as functions of M and N. Any operator which commutes with M and N must commute with all the P_{mn}, and *vice versa*. A basis in $\mathscr{V}$ can be adopted, in which (the matrices of) all the operators $\Sigma_{\mu\nu}$ have the same block diagonal structure, each block corresponding to an irreducible representation of sl(2,C). In such a basis, the operators M, N, and P_{mn} are diagonal. Within any one of the blocks mentioned, M and N are multiples of the identity by the appropriate m and n values. We shall call such a basis an sl(2,C) basis, although it must be noted that M and N do not form a complete set of commuting operators on $\mathscr{V}$ if some of the r_{mn} are greater than unity.

In the case of interest, where $\mathscr{V}$ is the index space of the field ψ, and we have therein a representation of $\mathscr{W}$ which is being regarded as a representation $\mathscr{R}$ of sl(2,C), we see from Eqs. (2.3d) and (2.13) that

$$[\Delta, M(M+1)] = 0 = [\Delta, N(N+1)]. \tag{2.14}$$

It follows that Δ commutes with the positive, diagonalizable operators $(M+\frac{1}{2})^2$ and $(N+\frac{1}{2})^2$. But if a matrix A commutes with a diagonal, positive matrix B, then A commutes also with the positive, diagonal, square root of B. Thus Δ commutes with $(M+\frac{1}{2})$ and $(N+\frac{1}{2})$, and we have

$$[\Delta,M] = 0 = [\Delta,N], \tag{2.15}$$

and hence

$$[\Delta,P_{mn}] = 0. \tag{2.16}$$

It is *not* possible to prove that Δ can be taken to be diagonal in an sl(2,C) basis, as Mack *et al.*[43] claim to do in their Lemma 1, using Schur's lemma. The possible occurrence of repeated irreducible represenations of sl(2,C) causes the difficulty. A simple example counter to their result is provided by the representation of $\mathscr{W}$ on two-component fields ψ with[57]

$$\Sigma_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \kappa_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix}, \tag{2.17}$$

which shows, indeed, that we cannot *a priori* assume the diagonalizability of Δ, and also that representations of $\mathscr{W}$ exist, more complicated than those described in Ref. 43.

A complete description of all finite-dimensional representations of $\mathscr{W}$ is not available. However, we shall see that only a subclass of such representations arises in connection with massless fields obeying locally conformal-invariant field equations. In particular, only representations[58] of Class $\mathscr{Q}$ (though not even all representations of this class) will arise:

Definition 2.1: A representation of $\mathscr{W}$ will be called of Class $\mathscr{Q}$ if it is finite-dimensional and its basis operators κ_μ, Δ and $\Sigma_{\mu\nu}$ satisfy

$$\kappa_\mu \kappa^\mu = 0, \tag{2.18a}$$
$$\Sigma_{\mu\nu}\kappa^\nu = (\Delta + i)\kappa_\mu \tag{2.18b}$$
$$\Delta^4 + (C_1+1)\Delta^2 + (C_2)^2 = 0 \tag{2.18c}$$

where C_1 and C_2 are the sl(2,C) invariants defined in terms of the $\Sigma_{\mu\nu}$ as in Eqs. (2.7). □

The representation defined by Eqs. (2.17) provides a rather simple example of a Class $\mathscr{Q}$ representation, although it is not one which arises in connection with locally conformal-invariant massless field equations, as we shall see.

It is important to show that this definition is a sensible one, to the extent that Eqs. (2.18) form a $\mathscr{W}$-invariant set. These equations are evidently invariant under transformations generated by $\Sigma_{\mu\nu}$ and Δ. For transformations generated by κ_μ, the invariance of Eq. (2.18a) follows because $[\kappa_\mu,\kappa_\nu] = 0$. Consider Eq. (2.18b) and the commutator

$$[\kappa_\mu, \Sigma_{\nu\rho}\kappa^\rho - (\Delta + i)\kappa_\nu]$$
$$= i(g_{\mu\nu}\kappa_\rho - g_{\mu\rho}\kappa_\nu)\kappa^\rho + i\kappa_\mu\kappa_\nu$$
$$= ig_{\mu\nu}\kappa_\rho\kappa^\rho. \tag{2.19}$$

When Eq. (2.18a) holds, this commutator vanishes, and the invariance of Eq. (2.18b) follows. Now consider Eq. (2.18c) and the commutator

$$[\kappa_\mu, \Delta^4 + (C_1+1)\Delta^2 + (C_2)^2]. \tag{2.20}$$

It can be deduced, using the commutation relations (2.3), that

$$[\kappa_\mu,\Delta^4] = (-4i\Delta^3 - 6\Delta^2 + 4i\Delta + 1)\kappa_\mu, \tag{2.21}$$
$$[\kappa_\mu,(C_1+1)\Delta^2]$$
$$= (\Delta - i)^2[\kappa_\mu, C_1 + 1] + (C_1+1)[\kappa_\mu,\Delta^2]$$
$$= (\Delta - i)^2(2i\Sigma_{\mu\nu}\kappa^\nu + 3\kappa_\mu) + (C_1+1)(-2i\Delta - 1)\kappa_\mu,$$
$$[\kappa_\mu,(C_2)^2] = 2i\Sigma_{\mu\nu}\Sigma^{\nu\rho}\Sigma_{\rho\tau}\kappa^\tau + 7\Sigma_{\mu\nu}\Sigma^{\nu\rho}\kappa_\rho \tag{2.22}$$
$$+ (2iC_1 - 6i)\Sigma_{\mu\nu}\kappa^\nu + 3C_1\kappa_\mu. \tag{2.23}$$

(In deriving the last of these equations, we found it helpful to use the identity

$$(C_2)^2 = \tfrac{1}{2}C_1(C_1+1) - \tfrac{1}{4}\Sigma_{\mu\nu}\Sigma^{\nu\rho}\Sigma_{\rho\tau}\Sigma^{\tau\mu}, \tag{2.24}$$

which follows from Eqs. (2.72) and (2.74) in Lemma 2.5 below.) When Eq. (2.18b) holds, Eqs. (2.22) and (2.23) reduce to

$$[\kappa_\mu,(C_1+1)\Delta^2] = \{(\Delta - i)^2(2i\Delta+1) - (C_1+1)(2i\Delta+1)\}\kappa_\mu \tag{2.22'}$$

$$[\kappa_\mu(C_2)^2] = \{2i(\Delta+i)^3 + 7(\Delta+i)^2 + (2iC_1 - 6i)(\Delta+i) + 3C_1\}\kappa_\mu, \tag{2.23'}$$

and, when combined with Eq. (2.21), enable us to see that the commutator (2.20) vanishes, so that Eq. (2.18c) is indeed $\mathscr{W}$-invariant.

Let us investigate something of the structure of an arbitrary representation $\mathscr{T}$ of class $\mathscr{D}$, with representation space $\mathscr{V}$. Regarded as a representation $\mathscr{R}$ of sl(2,C), it will have the form (2.9), for a finite set S and positive integers r_{mn} determined by $\mathscr{T}$. We introduce the projectors P_{mn} and the operators M and N as in the general discussion above. We first use the P_{mn} to write $\mathscr{V}$ as a direct sum of subspaces in two different ways:

(1) Let S_1 denote the set of *distinct* values θ of $|m-n|$ obtained as (m,n) runs over S. Every such number θ is a non-negative integer or semi-integer. For each θ in S_1, define the projector

$$P_{1\theta} = \sum_{\substack{(m,n)\in S\\ |m-n|=\theta}} P_{mn}. \tag{2.25}$$

It follows that

$$\sum_{\theta\in S_1} P_{1\theta} = 1, \quad P_{1\theta}P_{1\theta'} = P_{1\theta}\delta_{\theta\theta'}, \quad \theta,\theta'\in S_1. \tag{2.26}$$

Then $\mathscr{V}$ is a direct sum of the subspaces $\mathscr{V}_{1\theta}, \theta\in S_1$, where

$$\mathscr{V}_{1\theta} = P_{1\theta}\mathscr{V}. \tag{2.27}$$

It can be seen that on $\mathscr{V}_{1\theta}$ the operator $(M-N)^2$ has the value θ^2.

(2) Let S_2 denote the set of *distinct* values ν of $(m+n+1)$ obtained as (m,n) runs over S. Every such number ν is an integer or semi-integer, greater than or equal to 1. For each ν in S_2, define the projector

$$P_{2\nu} = \sum_{\substack{(m,n)\in S\\ m+n+1=\nu}} P_{mn}. \tag{2.28}$$

It follows that

$$\sum_{\nu\in S_2} P_{2\nu} = 1, \quad P_{2\nu}P_{2\nu'} = P_{2\nu}\delta_{\nu\nu'}, \quad \nu,\nu'\in S_2. \tag{2.29}$$

Then $\mathscr{V}$ is a direct sum of the subspaces $\mathscr{V}_{2\nu}, \nu\in S_2$, where

$$\mathscr{V}_{2\nu} = P_{2\nu}\mathscr{V}. \tag{2.30}$$

On $\mathscr{V}_{2\nu}$ the operator $(M+N+1)$ has the value ν.

We are not concerned with possible orthogonality or identity relations among the various $P_{1\theta}$ and $P_{2\nu}$. However, we note that as functions of the P_{mn}, they all commute with each other, and with Δ, $\Sigma_{\mu\nu}$, M, and N. Now consider Eq. (2.18c) which by definition holds in $\mathscr{T}$. Since the operators M and N satisfy Eqs. (2.13), we can write Eq. (2.18c) as

$$[\Delta^2 + (M-N)^2][\Delta^2 + (M+N+1)^2] = 0. \tag{2.31}$$

Define the operators

$$P_a = -[\Delta^2 + (M-N)^2][4MN + 2M + 2N + 1]^{-1}$$
$$P_b = [\Delta^2 + (M+N+1)^2][4MN + 2M + 2N + 1]^{-1}, \tag{2.32}$$

noting that $(4MN + 2M + 2N + 1)$ $[\equiv (M+N+1)^2 - (M-N)^2]$ has a well-defined inverse because M and N are commuting and non-negative. It follows from Eq. (2.31) that

$$P_a + P_b = 1, \; P_aP_b = P_bP_a = 0,$$
$$P_aP_a = P_a, \; P_bP_b = P_b. \tag{2.33}$$

Thus P_a and P_b are projectors, and with their help we can write $\mathscr{V}$ as a direct sum of two subspaces $\mathscr{V}_a$ and $\mathscr{V}_b$, where

$$\mathscr{V}_a = P_a\mathscr{V}, \mathscr{V}_b = P_b\mathscr{V}. \tag{2.34}$$

It follows from Eq. (2.31) that on $\mathscr{V}_a$, $[\Delta^2 + (M+N+1)^2]$ vanishes, while on $\mathscr{V}_b$, $[\Delta^2 + (M-N)^2]$ vanishes. We note that P_a and P_b as defined commute with all P_{mn}, and hence with Δ, $\Sigma_{\mu\nu}$, M, N, $P_{1\theta}$, and $P_{2\nu}$.

Finally, we define the projectors

$$P_{a\theta} = P_aP_{1\theta} = P_{1\theta}P_a, \quad \theta\in S_1$$
$$P_{b\nu} = P_bP_{2\nu} = P_{2\nu}P_b, \quad \nu\in S_2. \tag{2.35}$$

Since it follows that

$$\sum_{\theta\in S_1} P_{a\theta} = P_a, \quad \sum_{\nu\in S_2} P_{b\nu} = P_b, \tag{2.36}$$

we have

$$\sum_{\substack{\theta\in S_1\\ \nu\in S_2}} (P_{a\theta} + P_{b\nu}) = 1. \tag{2.37}$$

Furthermore, it is easily seen that

$$P_{a\theta}P_{a\theta'} = P_{a\theta}\delta_{\theta\theta'}, \quad \theta,\theta'\in S_1$$
$$P_{b\nu}P_{b\nu'} = P_{b\nu}\delta_{\nu\nu'}, \quad \nu,\nu'\in S_2$$
$$P_{a\theta}P_{b\nu} = P_{b\nu}P_{a\theta} = 0, \quad \theta\in S_1, \nu\in S_2. \tag{2.38}$$

We can therefore write $\mathscr{V}$ as a direct sum of subspaces $\mathscr{V}_{a\theta}, \mathscr{V}_{b\nu} (\theta\in S_1, \nu\in S_2)$ with

$$\mathscr{V}_{a\theta} = P_{a\theta}\mathscr{V}, \mathscr{V}_{b\nu} = P_{b\nu}\mathscr{V}. \tag{2.39}$$

Note that some of the projectors $P_{a\theta}, P_{b\nu}$ could vanish identically. (Indeed, this could even be true of P_a or P_b.) Then the corresponding $\mathscr{V}_{a\theta}$ or $\mathscr{V}_{b\nu}$ is the trivial subspace of $\mathscr{V}$.

It follows from what we have said above that on any vector in $\mathscr{V}_{a\theta}$,

$$[\Delta^2 + (M+N+1)^2] = 0 \tag{2.40a}$$

$$(M-N)^2 = \theta^2 \tag{2.40b}$$

and hence, in particular,

$$(\theta^2 - 1) = [\Delta^2 + 2M(M+1) + 2N(N+1)] = (\Delta^2 + C_1). \tag{2.41}$$

Similarly, on any vector in $\mathscr{V}_{b\nu}$

$$[\Delta^2 + (M-N)^2] = 0, \tag{2.42a}$$

$$(M+N+1) = \nu, \tag{2.42b}$$

and so

$$(\Delta^2 + C_1) = (\nu^2 - 1). \tag{2.43}$$

We shall now show that each of the subspaces $\mathscr{V}_{a\theta},\mathscr{V}_{b\nu}$ is $\mathscr{W}$-invariant. The operator $(\Delta^2 + C_1)$ commutes with Δ and $\Sigma_{\mu\nu}$. Consider the commutator

$$[\Delta^2 + C_1,\kappa_\rho] = 2i(\Delta + i)\kappa_\rho - 2i\Sigma_{\rho\nu}\kappa^\nu. \tag{2.44}$$

In the representation $\mathscr{T}$, the right-hand side vanishes by virtue of Eq. (2.18b). It follows that in $\mathscr{T}$, the operator $(\Delta^2 + C_1)$ is a $\mathscr{W}$-scalar. Since the subspaces $\mathscr{V}_{a\theta}$ correspond to distinct eigenvalues of this operator, they are not mixed together under the action of $\mathscr{W}$. Similarly, the subspaces $\mathscr{V}_{b\nu}$ are not mixed together, nor are the subspaces $\mathscr{V}_{a\theta}$ and $\mathscr{V}_{b\nu}$, with $\theta \neq \nu$. It remains to show that in a case with $\theta = \nu = \rho$, say, the subspaces $\mathscr{V}_{a\rho}$ and $\mathscr{V}_{b\rho}$ are not mixed together. Now on $\mathscr{V}_{a\rho}$ we have $(M-N)^2 = \rho^2$, so that any $v_a \in \mathscr{V}_{a\rho}$ can only have components belonging to irreducible representations (m,n) of sl(2,C) with $|m-n| = \rho$, i.e., the representations $(\rho,0)$, $(\rho+\frac{1}{2},\frac{1}{2}),\cdots$ and $(0,\rho)$, $(\frac{1}{2},\rho+\frac{1}{2}),\cdots$. Similarly, any $v_b \in \mathscr{V}_{b\rho}$ can only have components in representations (m,n) with $(m+n+1) = \rho$, i.e., the representations $(\rho-1,0)$, $(\rho-\frac{3}{2},\frac{1}{2}),\ldots,(0,\rho-1)$. But these two sets of sl(2,C) representations are disjoint, and moreover cannot be linked by the operators Δ, $\Sigma_{\mu\nu}$ and κ_μ: the operators Δ and $\Sigma_{\mu\nu}$ cannot link inequivalent representations of sl(2,C) since they commute with M and N; and the four-vector operator κ_μ can link[56] a representation (m,n) only with $(m+\frac{1}{2},n+\frac{1}{2})$, $(m+\frac{1}{2},n-\frac{1}{2})$, $(m-\frac{1}{2},n+\frac{1}{2})$ and $(m-\frac{1}{2},n-\frac{1}{2})$. Thus, Δ, $\Sigma_{\mu\nu}$ and κ_μ cannot link the subspaces $\mathscr{V}_{a\rho}$ and $\mathscr{V}_{b\rho}$ which are therefore separately invariant under the action of $\mathscr{W}$. Thus we see that the decomposition

$$\mathscr{V} = \sum_{\theta\in S_1}^{\oplus} \mathscr{V}_{a\theta} \oplus \sum_{\nu\in S_2}^{\oplus} \mathscr{V}_{b\nu} \tag{2.45}$$

is a decomposition of $\mathscr{V}$ into $\mathscr{W}$-invariant subspaces. It defines a decompositions of $\mathscr{T}$ into a direct sum of subrepresentations of $\mathscr{W}$.

It follows that if the given representation $\mathscr{T}$ is indecomposable, only one of the subspaces $\mathscr{V}_{a\theta},\mathscr{V}_{b\nu}$ is nontrivial.

Definition 2.2: A representation of $\mathscr{W}$ of Class $\mathscr{Q}$ will be called a $\langle\theta\rangle$-representation, where θ is a non-negative integer or semi-integer, if its basis operators Δ, κ_μ, $\Sigma_{\mu\nu}$ and the non-negative operators M,N defined by Eqs. (2.11), satisfy Eqs. (2.40). It will be called a $\{\nu\}$-representation, where $\nu (\nu \geqslant 1)$ is an integer or semi-integer, if Eqs. (2.42) are satisfied. □

Then we have proved the following:

Lemma 2.1: Any indecomposable representation of $\mathscr{W}$ of Class $\mathscr{Q}$ is either a $\langle\theta\rangle$-representation for some θ, or a $\{\nu\}$-representation for some ν. □

In the context of this work, we find that $\{\nu\}$-representations are not of interest. This is fortunate, because we shall see that in every $\langle\theta\rangle$-representation Δ is diagonalizable, while the same cannot be said of every $\{\nu\}$-representation, as the example of a $\{1\}$-representation defined by Eqs. (2.17) shows. The structure of $\langle\theta\rangle$-representations is comparatively simple. Let us look at this structure a little more closely, for an arbitrary $\langle\theta\rangle$-representation $\mathscr{T}$, with representation space $\mathscr{V}$. Noting that Eq. (2.40a) holds by definition, we define the projectors

$$P_+ = -\tfrac{1}{2}i[\Delta + i(M+N+1)][M+N+1]^{-1},$$
$$P_- = +\tfrac{1}{2}i[\Delta - i(M+N+1)][M+N+1]^{-1}, \tag{2.46}$$

which satisfy

$$P_+ + P_- = 1,\ P_+P_+ = P_+,\ P_-P_- = P_-,$$
$$P_+P_- = P_-P_+ = 0. \tag{2.47}$$

Then $\mathscr{V}$ is a direct sum of the corresponding subspaces $\mathscr{V}_+$ and $\mathscr{V}_-$,

$$\mathscr{V}_+ = P_+\mathscr{V},\quad \mathscr{V}_- = P_-\mathscr{V}. \tag{2.48}$$

On $\mathscr{V}_+$ we have

$$\Delta = +i(M+N+1), \tag{2.49}$$

and on $\mathscr{V}_-$ we have

$$\Delta = -i(M+N+1). \tag{2.50}$$

Because P_+ and P_- commute with $\Sigma_{\mu\nu}$, the subspaces $\mathscr{V}_+$ and $\mathscr{V}_-$ are separately sl(2,C)-invariant. It follows that we can choose bases in these subspaces such M and N, and hence Δ, are diagonal. This justifies our assertion above that Δ is always diagonalizable in a $\langle\theta\rangle$-representation. Now on $\mathscr{V}$ we also have, by definition of a $\langle\theta\rangle$-representation,

$$(M-N)^2 = \theta^2, \tag{2.51}$$

and it then follows from Eq. (2.49) that on $\mathscr{V}_+$, $-i\Delta$ has eigenvalues belonging to the series $(\theta+1)$, $(\theta+2)$, $(\theta+3),\ldots$, while on $\mathscr{V}_-$ it has eigenvalues belonging to the series $-(\theta+1)$, $-(\theta+2)$, $-(\theta+3),\cdots$. Consider the effect of Δ, $\Sigma_{\mu\nu}$, and κ_μ on a basis vector in $\mathscr{V}_-$. Since Δ and $\Sigma_{\mu\nu}$ commute with $-i\Delta$, and so cannot change its eigenvalue, they must carry such a vector back into $\mathscr{V}_-$. Now Eq. (2.3c) says that κ_μ converts an eigenvector of $-i\Delta$ with eigenvalue δ, into one with eigenvalue $(\delta+1)$. Since any eigenvalue from the first series above is greater by at least two units than any eigenvalue from the second series, it follows that κ_μ carries no basis vector from $\mathscr{V}_-$ into $\mathscr{V}_+$. In this way we see that $\mathscr{V}_-$ is invariant under the action of the operators of $\mathscr{W}$. By a similar argument we deduce that $\mathscr{V}_+$ is $\mathscr{W}$-invariant, and we conclude that the decomposition (2.48) defines a decomposition of $\mathscr{T}$ into a direct sum of subrepresentations. If $\mathscr{T}$ is indecomposable, one or the other of $\mathscr{V}_+,\mathscr{V}_-$ must be trivial.

Definition 2.3: A $\langle\theta\rangle$-representation of $\mathscr{W}$ will be called a $\langle\theta,+\rangle$-representation [respectively, a $\langle\theta,-\rangle$-representation] if, with the same notation as before,

$$\Delta = +i(M+N+1) \tag{2.52}$$

[respectively

$$\Delta = -i(M+N+1)]. \tag{2.53}$$

□

Then we have proved:

Lemma 2.2: Any indecomposable $\langle\theta\rangle$-representation is either a $\langle\theta,+\rangle$-representation or a $\langle\theta,-\rangle$-representation. □

Comment: A similar analysis cannot be performed for an arbitrary $\{\nu\}$-representation. In place of the operator $(M+N+1)$ in Eq. (2.46) above we would have $(M-N)$, which is not always invertible. [See again the example defined by Eqs. (2.17), for which $M=N=0$.] □

We can carry our investigation of $\langle \theta \rangle$-representations still further. Consider a $\langle \theta, + \rangle$-representation $\mathscr{T}$, with representation space $\mathscr{V}$, which has $\theta > 0$ but is otherwise arbitrary. As for the general case of a Class $\mathscr{Q}$ representation described above, introduce the projectors P_{mn}, $(m,n) \in S$. In the present case, $(m-n)^2 = \theta^2$ for every $(mn) \in S$. Let us split the set S of ordered pairs (m,n) into two subsets S_α and S_β according as $(m-n) = +\theta$ or $-\theta$, and define the corresponding projectors

$$P_\alpha = \sum_{(m,n)\in S_\alpha} P_{mn}, P_\beta = \sum_{(m,n)\in S_\beta} P_{mn}. \tag{2.54}$$

Then

$$P_\alpha + P_\beta = 1, \quad P_\alpha P_\alpha = P_\alpha, \quad P_\beta P_\beta = P_\beta,$$
$$P_\alpha P_\beta = P_\beta P_\alpha = 0, \tag{2.55}$$

and we can write $\mathscr{V}$ as a direct sum of the corresponding subspaces $\mathscr{V}_\alpha$ and $\mathscr{V}_\beta$,

$$\mathscr{V}_\alpha = P_\alpha \mathscr{V}, \quad \mathscr{V}_\beta = P_\beta \mathscr{V}. \tag{2.56}$$

Then, on $\mathscr{V}_\alpha$

$$M - N = +\theta, \tag{2.57}$$

while on $\mathscr{V}_\beta$,

$$M - N = -\theta. \tag{2.58}$$

It follows that vectors in $\mathscr{V}_\alpha$ belong to certain representations (m,n) of sl$(2,C)$ from the series $(\theta,0)$, $(\theta + \frac{1}{2},\frac{1}{2})$, $(\theta + 1,1),\cdots$, while those in $\mathscr{V}_\beta$ belong to certain representations (m,n) from the series $(0,\theta)$, $(\frac{1}{2},\theta + \frac{1}{2})$, $(1,\theta + 1)\cdots$. It is at once clear that Δ and $\Sigma_{\mu\nu}$, which commute with M and N, leave the two subspaces $\mathscr{V}_\alpha$ and $\mathscr{V}_\beta$ separately invariant. As we remarked before, κ_μ can only link the representation (m,n) with $(m \pm \frac{1}{2}, n + \frac{1}{2})$ and $(m \pm \frac{1}{2}, n - \frac{1}{2})$. Then it follows at once that, at least for $\theta > \frac{1}{2}$, κ_μ leaves $\mathscr{V}_\alpha$ and $\mathscr{V}_\beta$ separately invariant. In the case $\theta = \frac{1}{2}$, it is at first glance conceivable that κ_μ could link a vector in $\mathscr{V}_\alpha$ belonging to $(\frac{1}{2},0)$ with one in $\mathscr{V}_\beta$ belonging to $(0,\frac{1}{2})$, and one in $\mathscr{V}_\alpha$ belonging to $(1,\frac{1}{2})$ with one in $\mathscr{V}_\beta$ belonging to $(\frac{1}{2},1)$ etc. However, we recall that on $\mathscr{V}$, by definition of a $\langle \theta, + \rangle$-representation,

$$\Delta = i(M + N + 1) \tag{2.59}$$

so that Δ has the same value $3i/2$ on the first two vectors mentioned, and the same value $5i/2$ on the second two, etc. But Eq. (2.3e) shows that κ_μ cannot transform one eigenvector of Δ into another with the same eigenvalue. In this way we see that for every θ, $\theta > 0$, the two subspaces $\mathscr{V}_\alpha, \mathscr{V}_\beta$ are separately $\mathscr{W}$-invariant, and the decomposition of $\mathscr{V}$ defines a decomposition of $\mathscr{T}$ into a direct sum of subrepresentation. If $\mathscr{T}$ is indecomposable, one of $\mathscr{V}_\alpha, \mathscr{V}_\beta$ must be trivial. (The case $\theta = 0$ is special: there is just one subspace, on which $M = N$.) A completely analogous analysis can be given in the case of a $\langle \theta, - \rangle$-representation, with $\theta > 0$.

Definition 2.4: A $\langle \theta, + \rangle$-representation of $\mathscr{W}$ (with $\theta > 0$ or $\theta = 0$) will be called a $[+\theta, +]$-representation if, with the same notation as before, Eq. (2.57) holds:

$$M - N = +\theta.$$

It will be called a $[-\theta, +]$-representation if Eq. (2.58) holds:

$$M - N = -\theta.$$

Similarly, we define $[+\theta, -]$-representation as a $\langle \theta, - \rangle$-representation in which Eq. (2.57) holds; and a $[-\theta, -]$-representation as one in which Eq. (2.58) holds. □

Then we have proved

Lemma 2.3: Any indecomposable $\langle \theta, + \rangle$-representation is either a $[+\theta, +]$-representation, or a $[-\theta, +]$-representation. □

Rather than refer $[+\theta, +]$-, $[0, +]$-, and $[-\theta, +]$-representations, where 2θ is a positive integer, we can henceforth refer simply to $[\lambda, +]$-representations, with 2λ an integer, positive, negative or zero. Such a representation is characterized by Eqs. (2.18), and in addition[59]

$$M - N = \lambda, \tag{2.60a}$$
$$\Delta = +i(M + N + 1). \tag{2.60b}$$

Similarly, a $[\lambda, -]$-representation is characterized by Eqs. (2.18) and

$$M - N = \lambda, \tag{2.61a}$$
$$\Delta = -i(M + N + 1). \tag{2.61b}$$

We shall give one further result concerning the structure of such representations. Recall that Δ is diagonalizable in these cases.

Definition 2.5: A $[\lambda, +]$-representation will be called a $[\lambda, +; l,u]$-representation, where l and u are non-negative integers with $u \geqslant l$, if the eigenvalues of $(-i\Delta)$ are

$$|\lambda| + l + 1, |\lambda| + l + 2, \ldots, |\lambda| + u + 1. \tag{2.62}$$

Similarly, a $[\lambda, -]$-representation will be called a $[\lambda, -; l,u]$-representation if the eigenvalues of $(-i\Delta)$ are

$$-(|\lambda| + l + 1), -(|\lambda| + l + 2), \ldots,$$
$$-(|\lambda| + u + 1). \tag{2.63}$$

□

Lemma 2.4: An indecomposable $[\lambda, +]$-representation is a $[\lambda, +; l,u]$-representation for some l and u; and an indecomposable $[\lambda, -]$-representation is a $[\lambda, -; l,u]$-representation for some l and u.

Proof: Consider an indecomposable $[\lambda, +]$-representation, with representation space $\mathscr{V}$. Then Eqs. (2.60) hold, so that

$$-i\Delta = (2M + 1 - \lambda) = (2N + 1 + \lambda). \tag{2.64}$$

Because $2M$ and $2N$ have non-negative integral eigenvalues, we see that every eigenvalue δ of $(-i\Delta)$ in this representation is of the form

$$\delta = |\lambda| + t + 1 \tag{2.65}$$

with t a non-negative integer. If there is only one such t, we set $l = t = u$ and the proof is complete. If there are more than one, we order them thus:

$$0 \leqslant l = t_1 < t_2 < \cdots < t_n = u. \tag{2.66}$$

Then we have to show that $t_1, t_2 \cdots t_n$ comprise all the integers from l to u. Suppose this is not the case, so that for some integral value of i between 1 and $n-1$,

$$t_{i+1} > t_i + 1. \tag{2.67}$$

Since $(-i\Delta)$ is diagonalizable, $\mathscr{V}$ is the direct sum of the eigenspaces of $(-i\Delta)$. Let $\mathscr{V}_i$ be the direct sum of the eigenspaces corresponding to values of t not greater than t_i, and

$\mathscr{V}_i'$ the direct sum of those corresponding to values of t not less than t_{i+1}. Then

$$\mathscr{V} = \mathscr{V}_i \oplus \mathscr{V}_i'. \tag{2.68}$$

Since Δ and $\Sigma_{\mu\nu}$ commute with $(-i\Delta)$, they leave $\mathscr{V}_i$ and $\mathscr{V}_i'$ separately invariant. According to Eq. (2.3e),

$$\Delta\kappa_\mu = \kappa_\mu(\Delta + i),$$

so the action of κ_μ is to increase the eigenvalue of $(-i\Delta)$ by one unit. Because of the inequality (2.67), it follows then that κ_μ cannot carry a vector from $\mathscr{V}_i$ into $\mathscr{V}_i'$, nor from $\mathscr{V}_i'$ into $\mathscr{V}_i$: these spaces are also separately invariant under the action of κ_μ. In this way we see that Eq. (2.68) defines a decomposition of $\mathscr{V}$ into a direct sum of $\mathscr{W}$-invariant subspaces. Since the given representation is indecomposable, we have a contradication, and the inequality (2.67) cannot hold. An analogous proof applies in the case of an indecomposable $[\lambda, -]$-representation. □

Combining Lemmas 2.1, 2.2, 2.3, and 2.4 we have

Theorem 2.1: An indecomposable representation of $\mathscr{W}$ of Class $\mathscr{D}$ must be one of the following types:

(i) $[\lambda, +; l, u]$ or $[\lambda, -; l, u]$, for some integer or semi-integer λ (positive, negative or zero) and some non-negative integers l and u $(u \geqslant l)$.

(ii) $\{\nu\}$, for some integer or semi-integer $\nu (\nu > 1)$. □

Comments:

1. We are not concerned at this stage with proving the existence of any of these representation types. The only Class $\mathscr{D}$ representation we have exhibited so far is the $\{1\}$-representation defined by Eqs. (2.17).

2. It is, of course, not true that a given representation of any one of these types need be indecomposable. Moreover, we have not proved that any two given representations of the same type (for example, any two $[\lambda, +; l, u]$-representations having the same values of λ, l, and u) are necessarily equivalent, even if they are both indecomposable.

3. We shall refer to $\psi(x)$ as an (indecomposable) Class $\mathscr{D}$ field if its index space carries an (indecomposable) representation of $\mathscr{W}$ of Class $\mathscr{D}$. Similarly, we shall refer to (indecomposable) $[\lambda, +; l, u]$-fields, $\{\nu\}$-fields, etc. □

We complete this section by presenting some results valid for any representation of the Lie algebra sl(2,C) (whether or not finite-dimensional, and whether or not contained in a representation of $\mathscr{W}$). These results will be required below.

Lemma 2.5: Let $\Sigma_{\mu\nu}$ $(= -\Sigma_{\mu\nu})$ be linear operators defined everywhere on a vector space, and satisfying there the commutation relations (2.3a) of sl(2,C). Define the Casimir operators C_1 and C_2 as in Eqs. (2.7). Then the following identities hold on that vector space:

(i) $$\tilde{\Sigma}_{\mu\nu}\Sigma^{\nu\lambda} = \Sigma_{\mu\nu}\tilde{\Sigma}^{\nu\lambda} = iC_2\delta_\mu{}^\lambda + i\tilde{\Sigma}_\mu{}^\lambda, \tag{2.69}$$

where

$$\tilde{\Sigma}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Sigma^{\rho\sigma}; \tag{2.70}$$

(ii) $$\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}^{\nu\lambda} = C_1\delta_\mu{}^\lambda + \Sigma_{\mu\nu}\Sigma^{\nu\lambda} - 2i\Sigma_\mu{}^\lambda; \tag{2.71}$$

(iii) $$\Sigma_{\mu\nu}\Sigma^{\nu\rho}\Sigma_{\rho\sigma}\Sigma^{\sigma\tau} - 4i\Sigma_{\mu\nu}\Sigma^{\nu\rho}\Sigma_\rho{}^\tau + (C_1 - 5)\Sigma_{\mu\nu}\Sigma^{\nu\tau} - 2i(C_1 - 1)\Sigma_\mu{}^\tau - [C_1 - (C_2)^2]\delta_\mu{}^\tau = 0; \tag{2.72}$$

(iv) $$\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}^{\nu\rho}\tilde{\Sigma}_{\rho\sigma}\tilde{\Sigma}^{\sigma\tau} - (C_1 + 1)\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}^{\nu\tau} + (C_2)^2\delta_\mu{}^\tau = 0; \tag{2.73}$$

(v) $$\Sigma_{\mu\nu}\Sigma^{\nu\rho}\Sigma_\rho{}^\mu = -2iC_1. \tag{2.74}$$

If, in addition, the vector space is finite-dimensional, so that the operators M and N can be introduced as in Eqs. (2.11) and (2.13) above, then the following identities also hold:

(vi) $$[-i\Sigma_{\mu\nu} - (M - N + 1)g_{\mu\nu}] \times [-i\Sigma^{\nu\rho} + (M - N - 1)g^{\nu\rho}] \times [-i\Sigma_{\rho\sigma} - (M + N + 2)g_{\rho\sigma}] \times [-i\Sigma^{\sigma\tau} + (M + N)g^{\sigma\tau}] = 0; \tag{2.75}$$

(vii) $$[\tilde{\Sigma}_{\mu\nu} - (M - N)g_{\mu\nu}] \times [\tilde{\Sigma}^{\nu\rho} + (M - N)g^{\nu\rho}] \times [\tilde{\Sigma}_{\rho\sigma} - (M + N + 1)g_{\rho\sigma}] \times [\tilde{\Sigma}^{\sigma\tau} + (M + N + 1)g^{\sigma\tau}] = 0. \tag{2.76}$$

Proof. (i) This result is obtained by substitution of various values for μ and λ, and use of the commutation relations (2.3a). For example, with $\mu = 0$, $\lambda = 1$ we have

$$\begin{aligned}\tilde{\Sigma}_{\mu\nu}\Sigma^{\nu\lambda} &= \tilde{\Sigma}_{02}\Sigma^{21} + \tilde{\Sigma}_{03}\Sigma^{31},\\ &= -\Sigma^{31}\Sigma^{21} + \Sigma^{21}\Sigma^{31},\\ &= i\Sigma^{23},\\ &= i\tilde{\Sigma}_0{}^1,\end{aligned} \tag{2.77}$$

as required.

(ii) We note that

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\nu\lambda\alpha\beta} = \begin{pmatrix}\lambda\alpha\beta\\ \mu\rho\sigma\end{pmatrix} - \begin{pmatrix}\lambda\alpha\beta\\ \mu\sigma\rho\end{pmatrix} + \begin{pmatrix}\lambda\alpha\beta\\ \rho\sigma\mu\end{pmatrix} - \begin{pmatrix}\lambda\alpha\beta\\ \rho\mu\sigma\end{pmatrix} + \begin{pmatrix}\lambda\alpha\beta\\ \sigma\mu\rho\end{pmatrix} - \begin{pmatrix}\lambda\alpha\beta\\ \sigma\rho\mu\end{pmatrix}, \tag{2.78}$$

where, for example,

$$\begin{pmatrix}\lambda\alpha\beta\\ \mu\rho\sigma\end{pmatrix} = \delta_\mu{}^\lambda\delta_\rho{}^\alpha\delta_\sigma{}^\beta. \tag{2.79}$$

Then we have

$$\begin{aligned}4\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}^{\nu\lambda} &= \epsilon_{\mu\nu\rho\sigma}\epsilon^{\nu\lambda\alpha\beta}\Sigma^{\rho\sigma}\Sigma_{\alpha\beta},\\ &= \delta_\mu{}^\lambda(\Sigma^{\alpha\beta}\Sigma_{\alpha\beta} - \Sigma^{\beta\alpha}\Sigma_{\alpha\beta})\\ &\quad + \Sigma^{\lambda\alpha}\Sigma_{\alpha\mu} - \Sigma^{\lambda\beta}\Sigma_{\mu\beta}\\ &\quad + \Sigma^{\beta\lambda}\Sigma_{\mu\beta} - \Sigma^{\alpha\lambda}\Sigma_{\alpha\mu},\end{aligned} \tag{2.80}$$

which yields the result (2.71), with the help of the commutation relations (2.3a).

(iii) Define

$$\begin{aligned}A_\mu{}^\nu &= -i\Sigma_\mu{}^\nu - \delta_\mu{}^\nu,\\ B_\mu{}^\nu &= \tilde{\Sigma}_\mu{}^\nu,\end{aligned} \tag{2.81}$$

and, suppressing tensor indices for the moment, write

$$\begin{aligned}&A \text{ for } A_\mu{}^\nu,\ 1 \text{ for } \delta_\mu{}^\nu,\ B \text{ for } B_\mu{}^\nu,\\ &A\circ A \text{ for } A_\mu{}^\lambda A_\lambda{}^\nu,\ A\circ B \text{ for } A_\mu{}^\lambda B_\lambda{}^\nu,\end{aligned} \tag{2.82}$$

and so forth. Then Eqs. (2.69) and (2.71), respectively, read as

$$A\circ B = B\circ A = C_2, \tag{2.83}$$

$$B\circ B = -A\circ A + C_1 + 1. \tag{2.84}$$

It follows from the second of these, multiplying on the left or right by $A\circ A$, that

$$A\circ A\circ A\circ A = -A\circ A\circ B\circ B + (C_1 + 1)A\circ A. \tag{2.85}$$

Using Eq. (2.83) twice in succession we see that Eq. (2.85) reduces to

$$A\circ A\circ A\circ A - (C_1+1)A\circ A + (C_2)^2 = 0, \tag{2.86}$$

which is equivalent to Eq. (2.72).

(vi) On substituting for C_1 and C_2 in Eq. (2.72), in terms of M and N from Eqs. (2.13), we get

$$[A-(M-N)]\circ[A+(M-N)]\circ$$
$$[A-(M+N+1)]\circ[A+(M+N+1)] = 0, \tag{2.87}$$

which is equivalent to Eq. (2.75).

(iv) Multiplying Eq. (2.84) on the left or right by $B\circ B$, we get

$$B\circ B\circ B\circ B = -B\circ B\circ A\circ A + (C_1+1)B\circ B. \tag{2.88}$$

Again using Eq. (2.83) twice, we get

$$B\circ B\circ B\circ B - (C_1+1)B\circ B + (C_2)^2 = 0, \tag{2.89}$$

which is equivalent to Eq. (2.73).

(vii) On substituting for C_1 and C_2 in Eq. (2.73) in terms of M and N from Eqs. (2.13), we get

$$[B-(M-N)]\circ[B+(M-N)]\circ$$
$$[B-(M+N+1)]\circ[B+(M+N+1)] = 0, \tag{2.90}$$

which is equivalent to Eq. (2.76).

(v) Using the commutation relations (2.3a), it is straightforward to show that if

$$\Gamma_{\mu\nu} = \Sigma_{\mu\alpha}\Sigma^{\alpha\beta}\Sigma_{\beta\nu} - 3i\Sigma_{\mu}{}^{\alpha}\Sigma_{\alpha\nu} - iC_1 g_{\mu\nu}, \tag{2.91}$$

then

$$\Gamma_{\mu\nu} = -\Gamma_{\nu\mu}. \tag{2.92}$$

It follows that $\Gamma_{\mu}{}^{\mu} = 0$, whence

$$\Sigma_{\mu\alpha}\Sigma^{\alpha\beta}\Sigma_{\beta}{}^{\mu} - 3i\Sigma_{\mu\alpha}\Sigma^{\alpha\mu} - 4iC_1 = 0, \tag{2.93}$$

which is equivalent to Eq. (2.74). □

Comment:

1. Some of the identities given here were presented earlier by Bracken and Green[60] in the general context of identities for the generators of representations of SO(n). □

3. CONDITIONS TO BE SATISFIED FOR LOCAL CONFORMAL-INVARIANCE OF THE WAVE EQUATION

We are concerned with massless fields, and we shall take that to mean that they satisfy[61] the wave equation

$$\Box\psi = -P^{\mu}P_{\mu}\psi = 0. \tag{3.1}$$

Definition 3.1: This equation will be said to be locally conformal-invariant on a vector space $\mathscr{U}$ ($\subseteq \mathscr{D}$) consisting of solutions, if $\mathscr{U}$ is $\mathscr{A}$-invariant; that is to say, if $\psi\in\mathscr{U}$ implies $X\psi\in\mathscr{U}$, where X is any element of the Lie algebra $\mathscr{A}$ spanned by D, P_μ, K_μ, and $M_{\mu\nu}$. □

Comments:

1. We do not require that $\mathscr{U}$ must consist of all the solutions of Eq. (3.1) which lie in $\mathscr{D}$. Nor do we require that if $\psi\in\mathscr{D}$ is a solution, then so is $X\psi$, where X is any element of $\mathscr{A}$. As we shall see, such requirements would rule out of further consideration such interesting cases as the free electromagnetic field $F_{\mu\nu}(x)$, where conformal invariance of the wave equation holds *not* on the space of all smooth solutions of that equation, but only on the subspace of fields satisfying certain extra equations, viz., Maxwell's equations.

2. If ψ is to be a *potential* for a massless field χ of a different type [e.g., with index space carrying a different finite-dimensional representation of sl(2,C)], then it may not be appropriate to require that ψ satisfy the wave equation; nor, when it does, to require local conformal-invariance of this equation in the manner defined. One might only expect these requirements to be met, roughly speaking, "up to a change of gauge" of ψ. Our results are relevant to a potential ψ only in the restricted situation where one chooses a gauge such that ψ satisfies the wave equation, and asks if this equation is locally conformal-invariant when ψ transforms as in Eqs. (2.4) and (2.5), supplementary gauge transformations being suppressed. It is known that in the case of the four-vector potential of the free electromagnetic field, the equation $\Box A_\mu = 0$ is not conformal-invariant in this sense.[22,32,38] We shall see that this result generalizes to all potentials. The only fields for which the wave equation is locally conformal-invariant are fields "having invariant helicity." □

In order to prove our first result, we exploit the isomorphism of $\mathscr{A}$ and the Lie algebra so (4,2). Following Mack *et al.*,[43] we define $J_{AB}(=-J_{BA})$, $A,B = 0,1,2,3,5,6$ by

$$J_{\mu\nu} = M_{\mu\nu}, \qquad J_{65} = D,$$
$$J_{5\mu} = \tfrac{1}{2}(P_\mu - K_\mu), \quad J_{6\mu} = \tfrac{1}{2}(P_\mu + K_\mu). \tag{3.2}$$

Then the commutation relations (2.6) can be written as

$$i[J_{AB}, J_{CD}] = g_{AC}J_{BD} + g_{BD}J_{AC} - g_{BC}J_{AD} - g_{AD}J_{BC}, \tag{3.3}$$

where the extended metric tensor is diagonal, with $g_{55} = -1$, $g_{66} = +1$.

Theorem 3.1: (1) The wave equation (3.1) is locally conformal invariant on a vector space $\mathscr{U} \subseteq \mathscr{D}$ if and only if $\mathscr{U}$ is $\mathscr{A}$ invariant and every field ψ in $\mathscr{U}$ satisfies

$$W_{AB}\psi = 0, \quad A,B = 0,1,2,3,5,6, \tag{3.4}$$

where

$$W_{AB} = J_{AC}J^{C}{}_{B} + J_{BC}J^{C}{}_{A} + \tfrac{1}{3}g_{AB}J_{CD}J^{CD}. \tag{3.5}$$

(2) Any one solution in $\mathscr{D}$ of Eqs. (3.4) generates under the action of $\mathscr{A}$ an $\mathscr{A}$-invariant space of such solutions, on which the wave equation is locally conformal invariant.

Proof: (1) Suppose that the wave equation is locally conformal invariant on $\mathscr{U}$, and $\psi\in\mathscr{U}$. It follows from Definition 3.1 that

$$[\cdots[[P^{\mu}P_{\mu}, X_1], X_2], \ldots, X_n]\psi = 0 \tag{3.6}$$

for any finite set of operators $X_1, X_2, \ldots, X_n$ in $\mathscr{A}$. Now from Eqs. (3.2),

$$\begin{aligned} P^{\mu}P_{\mu} &= (J_{5\mu} + J_{6\mu})(J_5{}^{\mu} + J_6{}^{\mu}) \\ &= -J_{5A}J^{A}{}_{5} - J_{6A}J^{A}{}_{6} - J_{5A}J^{A}{}_{6} - J_{6A}J^{A}{}_{5} \\ &= -\tfrac{1}{2}W_{55} - \tfrac{1}{2}W_{66} - W_{56}. \end{aligned} \tag{3.7}$$

Since W_{AB} by construction is an so (4.2)-tensor operator, we have (on $\mathscr{D}$)

$$i[W_{AB}, J_{CD}] = g_{AD}W_{CB} - g_{AC}W_{DB} + g_{BD}W_{AC} - g_{BC}W_{AD}, \tag{3.8}$$

and so

$$i[P_\mu P^\mu, J_{5\nu}] = -W_{5\nu} - W_{6\nu}$$
$$[[P_\mu P^\mu, J_{5\nu}], J_{5\rho}] = W_{\rho\nu} + g_{\nu\rho}W_{55} + g_{\nu\rho}W_{65}. \tag{3.9}$$

It then follows from Eq. (3.6) that

$$W_{\rho\nu}\psi = 0, \quad \rho\neq\nu \tag{3.10}$$

and

$$\begin{aligned}(W_{00}+W_{55}+W_{65})\psi &= 0,\\ (W_{11}-W_{55}-W_{65})\psi &= 0,\\ (W_{22}-W_{55}-W_{65})\psi &= 0,\\ (W_{33}-W_{55}-W_{65})\psi &= 0.\end{aligned} \tag{3.11}$$

Similarly, from the commutator

$$[[P_\mu P^\mu, J_{50}], J_{60}] = W_{66} - W_{00} + W_{65} \tag{3.12}$$

we deduce that

$$(W_{66} - W_{00} + W_{65})\psi = 0. \tag{3.13}$$

From Eq. (3.9) we also have, provided $\rho\neq\nu$,

$$i[[[P_\mu P^\mu J_{5\nu}], J_{5\rho}], J_{5\sigma}] = g_{\rho\sigma}W_{5\nu} + g_{\nu\sigma}W_{5\rho}, \tag{3.14}$$

from which we deduce (taking $\sigma=\rho\neq\nu$) that

$$W_{5\nu}\psi = 0. \tag{3.15}$$

Similarly, from the commutator ($\rho\neq\nu$ here)

$$i[[[P_\mu P^\mu, J_{5\nu}], J_{5\rho}], J_{6\sigma}] = g_{\rho\sigma}W_{6\nu} + g_{\nu\sigma}W_{6\rho}, \tag{3.16}$$

we deduce that

$$W_{6\nu}\psi = 0. \tag{3.17}$$

Finally, from the commutator

$$[[[[P_\mu P^\mu, J_{50}], J_{51}], J_{51}], J_{60}] = W_{65}$$

we have

$$W_{65}\psi = 0. \tag{3.18}$$

Noting from the definition (3.5) that

$$W_A{}^A = W_{00} - W_{11} - W_{22} - W_{33} - W_{55} + W_{66} = 0, \tag{3.19}$$

we can readily see from Eqs. (3.10–3.13, 3.15, and 3.17–3.19) that all of Eqs. (3.4) hold.

Conversely, suppose that every ψ in a vector space $\mathscr{U}(\subseteq\mathscr{D})$ satisfies Eqs. (3.4). Then by Eq. (3.7) every ψ in $\mathscr{U}$ satisfies the wave equation. If in addition $\mathscr{U}$ is $\mathscr{A}$-invariant, then the wave equation is by definition locally conformal-invariant on $\mathscr{U}$.

(2) Suppose $\psi(\in\mathscr{D})$ satisfies Eqs. (3.4). Then it is obvious from the relations (3.8) that $\psi' = X_1X_2\cdots X_n\psi$ also satisfies these equations, where $X_1X_2,\ldots,X_n$ is any finite set of elements of $\mathscr{A}$. Let $\mathscr{U}_\psi$ be the vector subspace of $\mathscr{D}$ consisting of all finite linear combinations of all such ψ'. Then $\mathscr{U}_\psi$ is an $\mathscr{A}$-invariant space of solutions in $\mathscr{D}$ of Eqs. (3.4), and so by the first part of this theorem, is a space on which the wave equation is locally conformal-invariant. □

Comments:

1. This theorem enables us to replace the problem of finding for which field types there exist $\mathscr{A}$-invariant spaces of solutions of the wave equation with the simpler problem of finding for which field types there exist *any* solutions of the Equations (3.4). This is the advantage of having found an irreducible $\mathscr{A}$-tensor set of equations.

2. There is an obvious generalization to any situation where one has a representation $\mathscr{A}'$, on a vector space $\mathscr{D}'$, of the so(4,2) Lie algebra, with basis P'_μ, K'_μ, D', and $M'_{\mu\nu}$. The equation

$$P'_\mu P'^\mu\psi' = 0, \quad \psi'\in\mathscr{D}' \tag{3.20}$$

will hold on an $\mathscr{A}'$-invariant subspace $\mathscr{U}'$ of $\mathscr{D}'$ if and only if every vector ψ' in $\mathscr{U}'$ satisfies

$$W_{AB}{}'\psi' = 0, \tag{3.21}$$

where $W_{AB}{}'$ and $J_{AB}{}'$ are defined in terms of P'_μ, etc., as in Eqs. (3.2) and (3.5). And any one vector in $\mathscr{D}'$ satisfying Eqs. (3.21) will generate under the action of $\mathscr{A}'$ an $\mathscr{A}'$-invariant space of such vectors.

3. Barut and Böhm[62] have shown that the self-adjoint generators $j_{AB}(=-j_{BA})$ of any irreducible unitary representation of SU(2,2), in the ladder series, satisfy (on a suitable domain)

$$j_{AC}j^C{}_B + j_{BC}j^C{}_A + \tfrac{1}{3}g_{AB}j_{CD}J^{CD} = 0. \tag{3.22}$$

These representations are associated with the mass-zero representations of ISL(2,C), as remarked in the Introduction, and this result can be seen to be a corollary to Theorem 3.1—or rather, to its generalization described in Comment 2. However, we emphasize that we do not assume the representation (unitary or otherwise) of any *group* on the fields ψ and we are not concerned with any Hilbert space structure for such fields. □

We proceed to investigate the content of the (20 linearly independent) equations (3.4), writing them out in SO(3,1)-tensor form. We have:

$$\begin{aligned}A=\mu,\ B=\nu:&(M_{\mu\rho}M^\rho{}_\nu + M_{\nu\rho}M^\rho{}_\mu - \tfrac12 K_\mu P_\nu\\ &- \tfrac12 K_\nu P_\mu - \tfrac12 P_\mu K_\nu - \tfrac12 P_\nu K_\mu)\psi\\ &= -\tfrac13 g_{\mu\nu}(J_{CD}J^{CD})\psi.\end{aligned} \tag{3.23a}$$

$$\begin{aligned}A=\mu,\ B=5:&[-M_{\mu\nu}(P^\nu - K^\nu) + (P^\nu - K^\nu)M_{\nu\mu}\\ &-(P_\mu + K_\mu)D - D(P_\mu + K_\mu)]\psi = 0.\end{aligned} \tag{3.23b}$$

$$\begin{aligned}A=\mu,\ B=6:&[-M_{\mu\nu}(P^\nu + K^\nu) + (P^\nu + K^\nu)M_{\nu\mu}\\ &-(P_\mu - K_\mu)D - D(P_\mu - K_\mu)]\psi = 0.\end{aligned} \tag{3.23c}$$

$$\begin{aligned}A=5,\ B=5:&[D^2 + \tfrac14(P^\mu P_\mu - K^\mu P_\mu - P^\mu K_\mu + K^\mu K_\mu)]\psi\\ &= -\tfrac16(J_{CD}J^{CD})\psi.\end{aligned} \tag{3.23d}$$

$$A=5, B=6: P^\mu P_\mu\psi = K^\mu K_\mu\psi. \tag{3.23e}$$

$$\begin{aligned}A=6, B=6:&[D^2 - \tfrac14(P^\mu P_\mu + K^\mu P_\mu + P^\mu K_\mu + K^\mu K_\mu)]\psi\\ &= -\tfrac16(J_{CD}J^{CD})\psi.\end{aligned} \tag{3.23f}$$

Also, we note that

$$\begin{aligned}J_{CD}J^{CD} &= M_{\mu\nu}M^{\mu\nu} + K^\mu P_\mu + K^\mu P_\mu - 2D^2,\\ &= M_{\mu\nu}M^{\mu\nu} + 2K^\mu P_\mu + 8iD - 2D^2,\end{aligned} \tag{3.24}$$

using Eqs. (2.6). A set of equations equivalent to Eqs. (3.23) and more convenient than them is obtained by taking certain linear combinations and using the commutation relations (2.6) to reorder factors in some products. We get

$$P^\mu P_\mu\psi = 0, \tag{3.25a}$$

$$K^\mu K_\mu\psi = 0, \tag{3.25b}$$

$$M_{\mu\nu}P^\nu\psi = (i - D)P_\mu\psi, \tag{3.25c}$$

$$M_{\mu\nu}K^\nu\psi = (i + D)K_\mu\psi, \tag{3.25d}$$

$$\begin{aligned}&(M_{\mu\rho}M^\rho{}_\nu + M_{\nu\rho}M^\rho{}_\mu - K_\mu P_\nu - K_\nu P_\mu)\psi\\ &\quad = -g_{\mu\nu}(M_{\rho\sigma}M^{\rho\sigma} - 2iD + 2D^2)\psi,\end{aligned} \tag{3.25e}$$

and

$$K_\mu P^\mu \psi = (M_{\mu\nu} M^{\mu\nu} - 4iD + 4D^2)\psi. \quad (3.26)$$

We note also that when Eq. (3.26) holds, we have from Eq. (3.24)

$$J_{CD} J^{CD} \psi = (3M_{\mu\nu} M^{\mu\nu} + 6D^2)\psi. \quad (3.27)$$

Finally we note that Eq. (3.26) is redundant, as it follows from Eq. (3.25e) by contraction. We therefore drop it from the set, leaving again $(1 + 1 + 4 + 4 + 10 =)20$ equations to be satisfied by ψ.

We now obtain an equivalent set of 20 equations involving the generators $\Sigma_{\mu\nu}, \Delta$, and κ_μ, by substituting into Eqs. (3.25) the expressions (2.5) for $M_{\mu\nu}, D$ and K_μ. At first sight it appears that the resulting equations will be very complicated, but great simplifications occur. For example, consider the third equation. We have from Eqs. (2.5)

$$M_{\mu\nu} P^\nu = x_\mu (P_\nu P^\nu) - (x_\nu P^\nu) P_\mu + \Sigma_{\mu\nu} P^\nu \quad (3.28)$$

and

$$(i - D) P_\mu = (i - x_\nu P^\nu - \Delta) P_\mu, \quad (3.29)$$

and so

$$M_{\mu\nu} P^\nu \psi = (i - D) P_\mu \psi \Rightarrow [x_\mu (P_\nu P^\nu) + \Sigma_{\mu\nu} P^\nu] \psi = (i - \Delta) P_\mu \psi. \quad (3.30)$$

Since we shall retain $P_\nu P^\nu \psi = 0$ as one equation in our set, Eq. (3.30) reduces to

$$\Sigma_{\mu\nu} P^\nu \psi = (i - \Delta) P_\mu \Psi. \quad (3.31)$$

It is no surprise that all x-dependent terms disappear in the transition from Eq. (3.25c) to Eq. (3.31): as Eqs. (3.25) are locally conformal-invariant, they are locally translation-invariant. This can be exploited in the reduction of the remaining equations in the set (3.25). We obtain

Theorem 3.2: Equations (3.4) are equivalent to Eqs. (3.25). For fields on which the generators of infinitesimal conformal transformations have the form (2.5), they are also equivalent to the following:

$$P_\mu P^\mu \psi = 0, \quad (3.32a)$$

$$\kappa_\mu \kappa^\mu \psi = 0 \quad (3.32b)$$

$$\Sigma_{\mu\nu} P^\nu \psi = (i - \Delta) P_\mu \psi, \quad (3.32c)$$

$$\Sigma_{\mu\nu} \kappa^\nu \psi = (i + \Delta) \kappa_\mu \psi, \quad (3.32d)$$

$$(\Sigma_{\mu\rho} \Sigma^\rho{}_\nu + \Sigma_{\nu\rho} \Sigma^\rho{}_\mu - \kappa_\mu P_\nu - \kappa_\nu P_\mu)\psi = -g_{\mu\nu}(\Sigma_{\rho\sigma} \Sigma^{\rho\sigma} + 2\Delta^2 - 2i\Delta)\psi. \quad (3.32e)$$

Proof: Suppose Eqs. (3.25) hold, and consider Eq. (3.25d)

$$M_{\mu\nu} K^\nu \psi = (i + D) K_\mu \psi, = K_\mu (2i + D)\psi, \quad (3.33)$$

using Eq. (2.6b). Noting the forms (2.5) of $M_{\mu\nu}$ and K_μ, we proceed to simplfy the left-hand side. We have

$$\begin{aligned}(x_\mu P_\nu - x_\nu P_\mu) K^\nu \psi &= [2x_\mu (x_\nu P^\nu)^2 + 8ix_\mu (x_\nu P^\nu) \\ &- 2ix_\mu (x_\nu P^\nu) + 2x_\mu (x_\nu P^\nu)\Delta + 8ix_\mu \Delta \\ &+ 2x_\mu x_\nu (\Delta - i) P^\nu + x_\mu (P_\nu \kappa^\nu) - 2(x_\nu x^\nu) P_\mu (x_\rho P^\rho) \\ &- 2ix_\mu (x_\nu P^\nu) + (x_\nu x^\nu)(x_\rho P^\rho) P_\mu + 2ix_\mu (x_\nu P^\nu) \\ &- 2(x_\nu x^\nu) P_\mu \Delta - 2ix_\mu \Delta + 2x_\nu P_\mu x^\rho \Sigma_\rho{}^\nu - (x_\nu \kappa^\nu) P_\mu]\psi,\end{aligned} \quad (3.34)$$

and

$$\Sigma_{\mu\nu} K^\nu \psi = [2x^\nu (x_\rho P^\rho) \Sigma_{\mu\nu} - (x_\nu x^\nu)(i - \Delta) P_\mu + 2x^\nu \Sigma_{\mu\nu} \Delta + 2x_\rho \Sigma_{\mu\nu} \Sigma^{\nu\rho} + \Sigma_{\mu\nu} \kappa^\nu]\psi, \quad (3.35)$$

noting Eqs. (3.32a) and (3.32c), already seen to follow from Eqs. (3.25). The right-hand side of Eq. (3.33) is

$$\begin{aligned}&[2x_\mu (x_\nu P^\nu)^2 + 2x_\mu (x_\nu P^\nu)\Delta + 4ix_\mu (x_\nu P^\nu) - (x_\nu x^\nu) P_\mu (x_\rho P^\rho) \\ &- (x_\nu x^\nu) P_\mu \Delta - 2i(x_\nu x^\nu) P_\mu + 2x_\mu (x_\nu P^\nu)\Delta \\ &+ 2x_\mu \Delta^2 + 4ix_\mu \Delta \\ &+ 2x^\rho (x_\nu P^\nu) \Sigma_{\mu\rho} + 2x^\rho \Sigma_{\mu\rho} \Delta + 4ix^\rho \Sigma_{\mu\rho} \\ &+ (x_\nu P^\nu) \kappa_\mu + \kappa_\mu \Delta + 2i\kappa_\mu]\psi.\end{aligned}$$

Combining these results, we see that Eq. (3.33) is

$$\begin{aligned}&[x_\mu (P_\nu \kappa^\nu) + 2ix_\mu \Delta - (x_\nu \kappa^\nu) P_\mu + 2x_\rho \Sigma_{\mu\nu} \Sigma^{\nu\rho} \\ &+ \Sigma_{\mu\nu} \kappa^\nu - 2x_\mu \Delta^2 - 2ix^\rho \Sigma_{\mu\rho} \\ &- (x_\nu P^\nu) \kappa_\mu - \kappa_\mu \Delta - 2i\kappa_\mu]\psi = 0,\end{aligned} \quad (3.36)$$

which we write as $A\psi = 0$.

Now we note that since Eqs. (3.25) form an $\mathscr{A}$-invariant set, $P_\lambda \psi$ satisfies those equations whenever ψ does. Therefore it is also true that

$$AP_\lambda \psi = 0, \quad (3.37)$$

and hence that

$$[A, P_\lambda]\psi = 0. \quad (3.38)$$

Evaluating the commutator appearing here, we then get from Eq. (3.36)

$$[g_{\lambda\mu}(\kappa_\nu P^\nu) + 2ig_{\lambda\mu}\Delta - \kappa_\lambda P_\mu + 2\Sigma_{\mu\nu}\Sigma^\nu{}_\lambda - 2g_{\lambda\mu}\Delta^2 - 2i\Sigma_{\mu\lambda} - \kappa_\mu P_\lambda]\psi = 0, \quad (3.39)$$

and Eq. (3.36) then implies further that

$$[\Sigma_{\mu\nu}\kappa^\nu - \kappa_\mu (2i + \Delta)]\psi = 0, \quad (3.40)$$

which is Eq. (3.32d). Contracting Eq. (3.39) with $g^{\lambda\mu}$ we get

$$\kappa_\nu P^\nu \psi = (\Sigma_{\nu\rho}\Sigma^{\nu\rho} + 4\Delta^2 - 4i\Delta)\psi, \quad (3.41)$$

and combining this with Eq. (3.39) we get

$$(2\Sigma_{\mu\nu}\Sigma^\nu{}_\lambda - 2i\Sigma_{\mu\lambda} - \kappa_\mu P_\lambda - \kappa_\lambda P_\mu)\psi = -g_{\lambda\mu}(\Sigma_{\nu\rho}\Sigma^{\nu\rho} + 2\Delta^2 - 2i\Delta)\psi, \quad (3.42)$$

which we see is equivalent to Eq. (3.32e), using the commutation relations (2.3a). Note that Eq. (3.42) is equivalent to Eq. (3.39), as it also implies Eq. (3.41) on contraction with $g^{\lambda\mu}$.

In a similar way we first reduce Eq. (3.25b) to

$$[4x^\rho x_\sigma \Sigma_{\rho\nu}\Sigma^{\nu\sigma} + 4i(x_\nu x^\nu)\Delta - 4(x_\nu x^\nu)\Delta^2 + 2(x_\nu x^\nu)(P_\rho \kappa^\rho) - 4(x_\nu P^\nu)(x_\rho \kappa^\rho)]\psi = \kappa_\nu \kappa^\nu \psi. \quad (3.43)$$

But the left-hand side of this vanishes, as is seen by contracting Eq. (3.39) with $x^\lambda x^\mu$. Therefore Eq. (3.32b) holds. Equation (3.25e) yields no equations not included in Eqs. (3.32).

To complete the proof, we need to show that Eqs. (3.32) imply Eqs. (3.25). It is easy to see that this is so for Eqs.

(3.25a)–(3.25d), essentially by reversing the arguments above. In order to prove it so for Eq. (3.25e), we can proceed in the same way, or, more simply, as follows:

If ψ satisfies Eqs. (3.32), then so does $P_\lambda \psi$. But Eqs. (3.32) imply Eq. (3.25d), and hence

$$[M_{\mu\nu}K^\nu - (i+D)K_\mu]P_\lambda\psi = 0 \tag{3.44}$$

as well. But then it follows that

$$[M_{\mu\nu}K^\nu - (i+D)K_\mu, P_\lambda]\psi = 0, \tag{3.45}$$

or, using Eqs. (2.6),

$$[2M_{\mu\nu}M^\nu{}_\lambda - 2iM_{\mu\lambda} + g_{\lambda\mu}(K_\nu P^\nu) + 2ig_{\lambda\mu}D - 2g_{\lambda\mu}D^2 - K_\lambda P_\mu - K_\mu P_\lambda]\psi = 0. \tag{3.46}$$

Contracting with $g^{\lambda\mu}$ we get Eq. (3.26), and substituting this back in Eq. (3.46), and noting the relations (2.6), we get Eq. (3.25e) as required. □

Note: We also find that for fields satisfying Eqs. (3.32), Eq. (3.24) reduces to

$$J_{CD}J^{CD}\psi = 6(C_1 + \Delta^2)\psi. \tag{3.47}$$

Comments:

1. In view of Theorem 3.1 (2), any one (smooth) solution of Eqs. (3.32) generates an $\mathscr{A}$-invariant vector space of such solutions. Our main problem is to find for which field types, i.e., for which finite-dimensional representations of $\mathscr{W}$ with basis operators $\Sigma_{\mu\nu}, \kappa_\mu$, and Δ, there exist *any* solutions of Eqs. (3.32).

2. Any finite-dimensional representation of $\mathscr{W}$ can be reduced to a direct sum of indecomposable representations, not necessarily irreducible, and correspondingly, any field ψ can be written as a direct sum of $\mathscr{W}$-indecomposable fields. Now as far as the index space of the field ψ is concerned, Eqs. (3.32) involve only the $\mathscr{W}$ operators. It follows that when these equations hold, they hold separately on each $\mathscr{W}$-indecomposable component field in the direct sum decomposition of ψ. In addition to this, consider the above-mentioned $\mathscr{A}$-invariant space $\mathscr{U}_\psi$ of solutions of Eqs. (3.32), generated by one solution ψ in the manner described in the proof of Theorem 3.1. The operators in $\mathscr{A}$, as far as their action on the index space of ψ is concerned, only involve the $\mathscr{W}$-operators, according to their definitions (2.5). Therefore $\mathscr{U}_\psi$ is the direct sum of the $\mathscr{A}$-invariant spaces generated by the $\mathscr{W}$-indecomposable components of ψ. For these reasons it is sufficient at the outset to consider fields ψ which are $\mathscr{W}$-indecomposable, i.e., whose index space carries an indecomposable representation of $\mathscr{W}$. □

In examining the implications of Eqs. (3.32), we begin with (3.32e), which we write in the form

$$\tau_{\mu\nu}\psi = r_{\mu\nu}\psi, \tag{3.48}$$

with

$$\tau_{\mu\nu} = \Sigma_{\mu\rho}\Sigma^\rho{}_\nu + \Sigma_{\nu\rho}\Sigma^\rho{}_\mu + g_{\mu\nu}C_1, = 2\Sigma_{\mu\rho}\Sigma^\rho{}_\nu - 2i\Sigma_{\mu\nu} + g_{\mu\nu}C_1, \tag{3.49}$$

$$r_{\mu\nu} = \kappa_\mu P_\nu + \kappa_\nu P_\mu - g_{\mu\nu}G, \tag{3.50}$$

$$G = C_1 + 2\Delta^2 - 2i\Delta, \tag{3.51}$$

and C_1 as in Eqs. (2.7).

We note that

$$\tau_{\mu\nu} = \tau_{\nu\mu}, \quad \tau_\mu{}^\mu = 0, \quad r_{\mu\nu} = r_{\nu\mu}. \tag{3.52}$$

Then Eq. (3.48) implies that

$$r_\mu{}^\mu\psi = 0, \tag{3.53}$$

or equivalently,

$$\kappa_\mu P^\mu\psi = 2G\psi. \tag{3.54}$$

Equation (3.48) also implies that

$$\tau^{\mu\nu}\tau_{\mu\nu}\psi = \tau^{\mu\nu}r_{\mu\nu}\psi. \tag{3.55}$$

Using Eqs. (3.52) we see that

$$\begin{aligned}\tau^{\mu\nu}\tau_{\mu\nu} &= 2\Sigma^{\mu\alpha}\Sigma_\alpha{}^\nu(2\Sigma_{\nu\rho}\Sigma^\rho{}_\mu - 2i\Sigma_{\nu\mu} + g_{\nu\mu}C_1)\\ &= 4\Sigma^{\mu\alpha}\Sigma_{\alpha\nu}\Sigma^{\nu\rho}\Sigma_{\rho\mu} - 4i\Sigma^{\mu\alpha}\Sigma_{\alpha\nu}\Sigma^\nu{}_\mu - 4(C_1)^2\\ &= 4(C_1)^2 - 16(C_2)^2,\end{aligned} \tag{3.56}$$

using Eqs. (2.72) and (2.74) of Lemma 2.5. Now consider

$$\begin{aligned}\Sigma_\mu{}^\nu r_{\nu\rho}\psi &= (\Sigma_{\mu\nu}\kappa^\nu P_\rho + \Sigma_{\mu\nu}\kappa_\rho P^\nu - \Sigma_{\mu\rho}G)\psi\\ &= (P_\rho\Sigma_{\mu\nu}\kappa^\nu + \kappa_\rho\Sigma_{\mu\nu}P^\nu - ig_{\mu\rho}\kappa_\nu P^\nu + i\kappa_\mu P_\rho - \Sigma_{\mu\rho}G)\psi\end{aligned}$$

[using Eqs. (2.3b)]

$$= [P_\rho(i+\Delta)\kappa_\mu + \kappa_\rho(i-\Delta)P_\mu + i\kappa_\mu P_\rho - (\Sigma_{\mu\rho} + 2ig_{\mu\rho})G]\psi$$

[using Eqs. (3.32) and (3.54)]

$$= [\Delta(\kappa_\mu P_\rho - \kappa_\rho P_\mu) + 2i(\kappa_\mu P_\rho + \kappa_\rho P_\mu) - (\Sigma_{\mu\rho} + 2ig_{\mu\rho})G]\psi \tag{3.57}$$

[using Eq. (2.3e)]. Then

$$\begin{aligned}\Sigma^{\rho\mu}\Sigma_\mu{}^\nu r_{\nu\rho}\psi &= [\Delta\Sigma^{\rho\mu}(\kappa_\mu P_\rho - \kappa_\rho P_\mu) - \Sigma^{\rho\mu}\Sigma_{\mu\rho}G]\psi\\ &= [2\Delta(i+\Delta)\kappa_\rho P^\rho + 2C_1G]\psi\end{aligned}$$

[using Eq. (3.32d)]

$$\begin{aligned}&= 2(2\Delta^2 + 2i\Delta + C_1)G\psi\\ &= 2[4\Delta^4 + 4(C_1+1)\Delta^2 + (C_1)^2]\psi.\end{aligned} \tag{3.58}$$

Now

$$\begin{aligned}\tau^{\nu\rho}r_{\nu\rho}\psi &= \tau^{\rho\nu}r_{\nu\rho}\psi\\ &= 2\Sigma^{\rho\mu}\Sigma_\mu{}^\nu r_{\nu\rho}\psi,\end{aligned} \tag{3.59}$$

using the definition (3.49) and noting Eqs. (3.52) and (3.53). Combining Eqs. (3.55), (3.56), (3.58), and (3.59) we get

$$[4(C_1)^2 - 16(C_2)^2]\psi = 4[4\Delta^4 + 4(C_1+1)\Delta^2 + (C_1)^2]\psi,$$

i.e.,

$$[\Delta^4 + (C_1+1)\Delta^2 + (C_2)^2]\psi = 0. \tag{3.60}$$

Now consider this equation, together with Eqs. (3.32b) and (3.32d). Any field satisfying Eqs. (3.32) must satisfy these three equations in particular. In Sec. 2 we have shown that this set of equations is $\mathscr{W}$-invariant, and in fact characterizes what we have called a representation of $\mathscr{W}$ of Class $\mathscr{Q}$. Therefore we have

Theorem 3.3: The nonzero components of any field ψ satisfying Eqs. (3.32), belong to a representations of $\mathscr{W}$ of Class $\mathscr{Q}$. □

Comment:

1. In the context of free, massless fields satisfying locally conformal-invariant equations it follows that we can, without significant loss of generality, limit ourselves at the outset to fields whose index spaces carry indecomposable representations of $\mathscr{W}$ of Class $\mathscr{D}$. Then Eqs. (3.32b), (3.32d), and (3.60) hold identically. However, we must bear in mind that such an indecomposable Class $\mathscr{D}$ field *may* represent only some of the components of a given field, whose index space carries a larger indecomposable representation of $\mathscr{W}$; and whose extra components, though set to zero by Eqs. (3.32b), (3.32d), and (3.60) when the field is free and massless, could become operative when the field is "in interaction." Such a possibility exists because the algebra $\mathscr{W}$ has representations which are not fully reducible. A classification of all such possibilities would require a classification of all indecomposable representations of $\mathscr{W}$ which "contain" a representation of Class $\mathscr{D}$. Such a classification will not be attempted here, and we restrict our attention henceforth to indecomposable Class $\mathscr{D}$ fields. □

We know that an indecomposable representation of $\mathscr{W}$ of Class $\mathscr{D}$ is of one of the types listed in Theorem 2.1. We shall show that if Eqs. (3.32) are required to admit plane wave solutions, then representations of all types except $[\lambda, +;0,u]$ are eliminated. The existence of plane wave solutions is essential if the associated fields are to be able to describe free, massless particles (at the many-particle or one-particle level, according as the fields are quantized or not).

Definition 3.2: A massless plane wave is a field $\psi(x)$ of the form

$$\psi(x) = \psi_0 \exp(-ik^\mu x_\mu), \tag{3.61}$$

where ψ_0 is a constant nonzero field and the k^μ are real constants, not all zero, satisfying

$$k^\mu k_\mu = 0. \tag{3.62}$$

□

Lemma 3.1: Let $\psi(x)$ be a field whose index space carries the irreducible representation (m,n) of sl(2,C), with basis operators $\Sigma_{\mu\nu}$. If the equations

$$\Sigma_{\mu\nu}\partial^\nu\psi = i(\alpha+1)\partial_\mu\psi, \tag{3.63}$$

where α is a constant, admit a massless plane wave solution, then

$$\alpha = -(m+n+1). \tag{3.64}$$

Proof: In the notation used in the proof of Lemma 2.5, Eq. (3.63) reads as

$$A\circ\partial\psi = \alpha\partial\psi. \tag{3.65}$$

Suppose that these equations admit a solution in the form of a massless plane wave (3.61). Then it follows that

$$A\circ k\psi_0 = \alpha k\psi_0. \tag{3.66}$$

Now in the representation (m,n), according to Lemma 2.5, Eq. (2.87),

$$[A-(m-n)]\circ[A+(m-n)]\circ[A-(m+n+1)]\circ$$
$$[A+(m+n+1)] = 0. \tag{3.67}$$

Applying the operator on the left-hand side of this identity to $k\psi_0$, we get from Eq. (3.66)

$$[\alpha-(m-n)][\alpha+(m-n)][\alpha-(m+n+1)]$$
$$[\alpha+(m+n+1)]k_\mu\psi_0 = 0 \tag{3.68}$$

Since $k_\mu\psi_0$ by assumption does not vanish for all μ, it follows that

$$\alpha\epsilon\{m-n, n-m, m+n+1, -(m+n+1)\}. \tag{3.69}$$

Case (1): $\alpha\neq 0$.

Multiply Eq. (3.66) on the left by B (again in the notation of Lemma 2.5). Then we get

$$B\circ A\circ k\psi_0 = \alpha B\circ k\psi_0 \tag{3.70}$$

whence, with the help of Lemma 2.5, Eq. (2.83) we have

$$B\circ k\psi_0 = \alpha^{-1}C_2 k\psi_0 \tag{3.71}$$

or, in view of Eqs. (2.8),

$$\widetilde{\Sigma}_{\mu\nu}k^\nu\psi_0 = \alpha^{-1}(m-n)(m+n+1)k_\mu\psi_0,$$

i.e.,

$$\widetilde{\Sigma}_{\mu\nu}k^\nu\psi_0 = \beta k_\mu\psi_0, \tag{3.72}$$

where

$$\beta = \alpha^{-1}(m-n)(m+n+1). \tag{3.73}$$

In view of Eq. (2.90), we then have in addition

$$\beta\in\{m-n, n-m, m+n+1, -(m+n+1)\}. \tag{3.74}$$

Consider the $\mu=0$ component of Eq. (3.72):

$$\widetilde{\Sigma}_{0i}k^i\psi_0 = \beta k_0\psi_0$$

i.e.,

$$\mathbf{S}\cdot\mathbf{k}\psi_0 = -\beta k_0\psi_0 \tag{3.75}$$

where

$$\mathbf{S} = (\widetilde{\Sigma}_{10},\widetilde{\Sigma}_{20},\widetilde{\Sigma}_{30}) = (\Sigma_{23},\Sigma_{31},\Sigma_{12})$$
$$\mathbf{k} = (k^1,k^2,k^3). \tag{3.76}$$

Let (s) denote the $(2s+1)$-dimensional irreducible representation of su(2). It is known that the representation (m,n) of sl(2,C), when regarded as a representation of su(2) with basis operators $\mathbf{S}$, is a direct sum of those irreducible representations (s) with

$$s\in\{m+n, m+n-1,\ldots,|m-n|\}, \tag{3.77}$$

each such representation occurring once. It is also known that if $\mathbf{n}$ is a real unit vector, then in the representation (s), the operator $\mathbf{S}\cdot\mathbf{n}$ has eigenvalues $s, s-1,\ldots,-s$. It follows that in the representation (m,n) of sl(2,C), $\mathbf{S}\cdot\mathbf{n}$ has eigenvalues $m+n, m+n-1,\ldots,-(m+n)$; in particular, the largest eigenvalue of $(\mathbf{S}\cdot\mathbf{n})^2$ equals $(m+n)^2$. Now Eq. (3.75) implies

$$(\mathbf{S}\cdot\mathbf{k})^2\psi_0 = \beta^2\mathbf{k}\cdot\mathbf{k}\psi_0, \tag{3.78}$$

since, by assumption, $(k_0)^2 = \mathbf{k}\cdot\mathbf{k}$. Thus on ψ_0, $(\mathbf{S}\cdot\mathbf{n})^2$ has the eigenvalue β^2, where

$$\mathbf{n} = \mathbf{k}/|\mathbf{k}|. \tag{3.79}$$

It then follows that

$$\beta^2 \leqslant (m+n)^2. \tag{3.80}$$

Next consider the $\mu=0$ component of Eq. (3.63),

$$\Sigma_{0i}k^i\psi_0 = i(\alpha+1)k_0\psi_0$$

i.e.,

$$\mathbf{T}\cdot\mathbf{k}\psi_0 = i(\alpha+1)k_0\psi_0, \tag{3.81}$$

where

$$\mathbf{T} = (\Sigma_{01},\Sigma_{02},\Sigma_{03}). \tag{3.82}$$

Let us define the operators

$$\mathbf{S}_\pm = \tfrac{1}{2}(\mathbf{S} \pm i\mathbf{T}). \tag{3.83}$$

Then it is easily checked from Eqs. (2.3a) that the S_{+i} (and likewise the S_{-i}) satisfy the su(2) commutation relations. Moreover, the S_{+i} commute with the S_{-i}, and

$$\begin{aligned}\mathbf{S}_+\cdot\mathbf{S}_+ &= \tfrac{1}{4}(\mathbf{S}\cdot\mathbf{S} - \mathbf{T}\cdot\mathbf{T} + 2i\mathbf{S}\cdot\mathbf{T})\\ &= \tfrac{1}{8}\Sigma_{\mu\nu}\Sigma^{\mu\nu} + \tfrac{1}{8}i\tilde{\Sigma}_{\mu\nu}\Sigma^{\mu\nu}\\ &= \tfrac{1}{4}(C_1 + 2C_2)\\ &= m(m+1)\end{aligned} \tag{3.84}$$

in the representation (m,n). Similarly,

$$\mathbf{S}_-\cdot\mathbf{S}_- = n(n+1) \tag{3.85}$$

in this representation. We can regard $\mathbf{S}_+$ as the basis operators of a representation (m) of su(2), and $\mathbf{S}_-$ as the basis operators of a representation (n) of su(2). Then, by the argument employed above for the operators $\mathbf{S}$, we can deduce that if $\mathbf{n}$ is any real unit vector, the maximum eigenvalue of $(\mathbf{S}_+\cdot\mathbf{n})^2$ is m^2 in the representation (m,n); and the maximum eigenvalue of $(\mathbf{S}_-\cdot\mathbf{n})^2$ is n^2. But according to Eqs. (3.75) and (3.81) we have

$$\mathbf{S}_\pm\cdot\mathbf{k}\psi_0 = -\tfrac{1}{2}[\beta \pm (\alpha+1)]k_0\psi_0, \tag{3.86}$$

whence

$$(\mathbf{S}_\pm\cdot\mathbf{n})^2\psi_0 = \tfrac{1}{4}[\beta \pm (\alpha+1)]^2\psi_0, \tag{3.87}$$

with $\mathbf{n}$ as in Eqs. (3.79). From this we can conclude that

$$\begin{aligned}&\tfrac{1}{4}(\beta+\alpha+1)^2 \leqslant m^2\\ &\tfrac{1}{4}(\beta-\alpha-1)^2 \leqslant n^2.\end{aligned} \tag{3.88}$$

The only pair of numbers α,β satisfying the conditions (3.69), (3.74), (3.80), and (3.88) is

$$\alpha = -(m+n+1),\ \beta = n-m \tag{3.89}$$

Case (2): $\alpha = 0$

According to the Lemma to be proved, there should be no massless plane wave solutions of Eq. (3.63) in this case, since $(m+n+1)$ is never zero. Suppose on the contrary that such a solution does exist. From Eq. (3.81) we have

$$\mathbf{T}\cdot\mathbf{k}\psi_0 = ik_0\psi_0, \tag{3.90}$$

while from the $\mu = (1,2,3)$ components of Eq. (3.63) we get

$$(\Sigma_{ij}k^j + \Sigma_{i0}k^0)\psi_0 = ik_i\psi_0,$$

i.e.,

$$(\mathbf{k}\wedge\mathbf{S} - k_0\mathbf{T})\psi_0 = -i\mathbf{k}\psi_0. \tag{3.91}$$

Now take the dot product of Eq. (3.91) on the left with $\mathbf{T}$, noting Eq. (3.90), to obtain

$$(\mathbf{T}\cdot\mathbf{k}\wedge\mathbf{s} - k_0\mathbf{T}\cdot\mathbf{T})\psi_0 = k_0\psi_0. \tag{3.92}$$

Next take the cross product of Eq. (3.91) on the left with $\mathbf{k}$ to obtain

$$[(\mathbf{S}\cdot\mathbf{k})\mathbf{k} - (\mathbf{k}\cdot\mathbf{k})\mathbf{S} - k_0(\mathbf{k}\wedge\mathbf{T})]\psi_0 = 0. \tag{3.93}$$

Noting that $(\mathbf{k}\cdot\mathbf{k}) = (k_0)^2$, and that

$$\begin{aligned}\mathbf{S}\cdot(\mathbf{k}\wedge\mathbf{T})\psi_0 &= [-\mathbf{T}\cdot(\mathbf{k}\wedge\mathbf{S}) - 2i\mathbf{T}\cdot\mathbf{k}]\psi_0\\ &= -\mathbf{T}\cdot(\mathbf{k}\wedge\mathbf{S})\psi_0 + 2k_0\psi_0,\end{aligned} \tag{3.94}$$

we take the dot product of Eq. (3.93) on the left with $\mathbf{S}$ to obtain

$$[(\mathbf{S}\cdot\mathbf{k})^2 - (k_0)^2\mathbf{S}\cdot\mathbf{S} + k_0\mathbf{T}\cdot(\mathbf{k}\wedge\mathbf{S}) - 2(k_0)^2]\psi_0 = 0. \tag{3.95}$$

Combining this equation with Eq. (3.92), we get

$$(\mathbf{S}\cdot\mathbf{k})^2\psi_0 = (k_0)^2[1 + \mathbf{S}\cdot\mathbf{S} - \mathbf{T}\cdot\mathbf{T}]\psi_0. \tag{3.96}$$

But if $\alpha = 0$, it follows from Eq. (3.69) that $m = n = r$, say. In the representation (r,r) of sl(2,C),

$$C_1 = \tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu} = \mathbf{S}\cdot\mathbf{S} - \mathbf{T}\cdot\mathbf{T} = 4r(r+1), \tag{3.97}$$

so that we have from Eq. (3.96)

$$(\mathbf{S}\cdot\mathbf{n})^2\psi_0 = (2r+1)^2\psi_0, \tag{3.98}$$

with $\mathbf{n}$ again as in Eq. (3.79). However, as argued above, the maximum eigenvalue of $(\mathbf{S}\cdot\mathbf{n})^2$ in the representation (r,r) is $(2r)^2$. Thus we have a contradiction, and there is no massless plane wave solution if $\alpha = 0$. □

A closely related result is

Lemma 3.2 (Weinberg's Lemma): Let $\psi(x)$, $\Sigma_{\mu\nu}$ be as in Lemma 3.1. If the equations

$$\tilde{\Sigma}_{\mu\nu}\partial^\nu\psi = \beta\partial_\mu\psi, \tag{3.99}$$

where β is a constant, admit a massless plane wave solution, then

$$\beta = n - m. \tag{3.100}$$

Proof: On a massless plane wave solution, Eqs. (3.99) reduce to

$$B\circ k\psi_0 = \beta k\psi_0 \tag{3.101}$$

(again in the notation used in the proof of Lemma 3.1). Multiplying on the left with A and using Eq. (2.83) we have

$$\begin{aligned}\beta A\circ k\psi_0 &= C_2 k\psi_0\\ &= (m-n)(m+n+1)k\psi_0.\end{aligned} \tag{3.102}$$

Suppose $\beta = 0$. Then Eq. (3.102) implies $m = n$, so that $\beta = n - m$ as required. Suppose $\beta \neq 0$. Then Eq. (3.102) becomes

$$A\circ k\psi_0 = \beta^{-1}(m-n)(m+n+1)k\psi_0 \tag{3.103}$$

and by Lemma 3.1,

$$\beta^{-1}(m-n)(m+n+1) = -(m+n+1) \tag{3.104}$$

whence $\beta = n - m$ as required. □

Comment:

1. Weinberg[42] considered free, *quantized*, positive-energy, massless fields, belonging to the irreducible representation (m,n) of sl(2,C). He showed that if such a field has (Lorentz-invariant) helicity h then, in our notation, $h = m - n$. Now the covariant statement that the field has invariant helicity h is Eq. (3.99), with $\beta = -h$ [as Eq. (3.75) shows when $k_0 = |\mathbf{k}| > 0$]. Furthermore, the possibility of quantizing a field ψ which satisfies Eqs. (3.99) is, in the usual formulations, dependent upon the existence of plane-wave solutions of those equations. For these reasons it seems appropriate to call Lemma 3.2 "Weinberg's Lemma", as we have done here. □

Theorem 3.4: If ψ is an indecomposable Class $\mathscr{D}$ field, and Eqs. (3.32) admit a massless plane wave solution, then ψ is an indecomposable $[\lambda, + ;0,u]$ field, for some integer or semi-integer λ, and some non-negative integer u.

Proof: In view of Theorem 2.1, it suffices to show that ψ cannot be (1) a $[\lambda, - ;l,u]$ field, (2) a $[\lambda, + ;l,u]$ field, where $l>0$, or (3) a $\{\nu\}$ field.

1. Suppose that ψ is both a $[\lambda, - ;l,u]$ field and a solution of Eqs. (3.32). Then Eqs. (2.61) hold, and Eq. (3.32c) yields

$$\Sigma_{\lambda\mu} P^\mu \psi = i(M+N+2)P_\lambda \psi. \tag{3.105}$$

Let P_{mni} denote the projector onto the ith one (in some ordering) of the r_{mn} multiples of the irreducible representation (m,n) of sl$(2,C)$ carried by the index space of ψ. (cf. Sec. 2). Then P_{mni} commutes with $\Sigma_{\lambda\mu}$, M, and N, so that

$$\Sigma_{\lambda\mu} P^\mu \psi_{mni} = i(m+n+2)P_\lambda \psi_{mni}, \tag{3.106}$$

where

$$\psi_{mni} = P_{mni}\psi. \tag{3.107}$$

Now if ψ is a massless plane wave, so is ψ_{mni}, if it does not vanish. But Lemma 3.1 shows that there are no massless plane wave solutions of Eqs. (3.106). Thus ψ_{mni} vanishes, for every i and every possible (m,n). But then ψ vanishes, and we have a contradiction. Thus ψ cannot be a massless plane wave.

2. Suppose that ψ is both a $[\lambda, + ;l,u]$ field (with $l>0$), and a solution of Eqs. (3.32). Then by Definition 2.5 the smallest eigenvalue of $-i\Delta$ is $(|\lambda|+1+l)$, which is greater than $(|\lambda|+1)$. Since Eqs. (2.60) hold here, it follows that the least eigenvalue of $(M+N)$ is greater than $|\lambda|$, and that the least eigenvalues of M and N are both greater than 0. Hence the operator (MN) is invertible. Now let P_t be the projector onto that subspace of the index space associated with the eigenvalue $(|\lambda|+1+t)$ of $-i\Delta$ $(l<t<u)$. Since ψ satisfies Eq. (3.32e), it satisfies (by contraction)

$$\kappa_\mu P^\mu \psi = (\Sigma_{\mu\nu}\Sigma^{\mu\nu} + 4\Delta^2 - 4i\Delta)\psi, \tag{3.108a}$$
$$= -8MN\psi. \tag{3.108b}$$

If ψ is nontrivial, then not every P_t can annihilate ψ. Of those P_t satisfying

$$P_t\psi \neq 0 \tag{3.109}$$

let P be the one having the smallest value of t. Since Eq. (2.3e) implies that κ_μ shifts the eigenvalue of $-i\Delta$ (and hence the value of t) upward by one unit, it then follows that

$$P\kappa_\mu P^\mu \psi = 0. \tag{3.110}$$

Eqs. (3.108) and (3.110) together imply

$$P(MN)\psi = 0. \tag{3.111}$$

But the projectors P_t evidently all commute with M and N, so that

$$(MN)P\psi = 0, \tag{3.112}$$

and since (MN) is invertible, we have

$$P\psi = 0, \tag{3.113}$$

contradicting the definition of P. Thus ψ cannot be a nontrivial solution of Eqs. (3.32). [Note that we did not need to assume ψ to be a massless plane wave solution. There are *no* nontrivial solutions to Eqs. (3.32) if ψ is a $[\lambda, + ;l,u]$ field with $l>0$.]

3. Suppose that ψ is both a $\{\nu\}$ field and a massless plane wave solution (3.61) of Eqs. (3.32). Contracting Eq. (3.32c) on the left with $\tilde{\Sigma}^{\rho\mu}$, using Lemma 2.5, Eq. (2.69), and noting Eqs. (2.13), we obtain

$$\Delta \tilde{\Sigma}_{\rho\nu} k^\nu \psi_0 = (M-N)(M+N+1)k_\rho \psi_0. \tag{3.114}$$

The $\rho = 0$ component of this equation is

$$\Delta\, \mathbf{S}\cdot\mathbf{k}\psi_0 = (M-N)(M+N+1)k_0\psi_0 \tag{3.115}$$

where $\mathbf{S}$ and $\mathbf{k}$ are defined as in Eqs. (3.76). Since the operators Δ, $\mathbf{S}\cdot\mathbf{k}$, M and N all commute, we get from Eq. (3.115)

$$\Delta^2(\mathbf{S}\cdot\mathbf{k})^2\psi_0 = (M-N)^2(M+N+1)^2(k_0)^2\psi_0. \tag{3.116}$$

Since Eq. (2.42a) holds in a $\{\nu\}$ representation, we then have [introducing $\mathbf{n}$ as in Eq. (3.79)]

$$(M-N)^2(\mathbf{S}\cdot\mathbf{n})^2\psi_0 = (M-N)^2(M+N+1)^2\psi_0. \tag{3.117}$$

Now introduce, as in Sec. 2, the projector P_{mn} onto that subspace of the index space associated with the totality of representations (m,n) of sl$(2,C)$ that are contained in the given $\{\nu\}$ representation of $\mathscr{W}$. Recalling that, for each chosen m and n, this projector commutes with Δ, M, N and $\Sigma_{\mu\nu}$, we get from Eq. (3.117)

$$(m-n)^2(\mathbf{S}\cdot\mathbf{n})^2\psi_{mn} = (m-n)^2(m+n+1)^2\psi_{mn}, \tag{3.118}$$

where

$$\psi_{mn} = P_{mn}\psi_0. \tag{3.119}$$

If $m\neq n$ we have then

$$(\mathbf{S}\cdot\mathbf{n})^2\psi_{mn} = (m+n+1)^2\psi_{mn}. \tag{3.120}$$

But, as remarked in the proof of Lemma 3.1, the largest eigenvalue of $(\mathbf{S}\cdot\mathbf{n})^2$ in the representation (m,n) of sl$(2,C)$ is equal to $(m+n)^2$. It follows that

$$\psi_{mn} = 0, \quad m\neq n. \tag{3.121}$$

Now in a $\{\nu\}$ representation of $\mathscr{W}$, Eq. (2.42b) holds, and we see that the only representation (m,m) of sl$(2,C)$ which can occur have

$$m = n = \tfrac{1}{2}(\nu-1). \tag{3.122}$$

Thus we have

$$P_{rr}\psi_0 = \psi_0, \quad r = \tfrac{1}{2}(\nu-1), \tag{3.123}$$

whence

$$M\psi_0 = N\psi_0 = r\psi_0. \tag{3.124}$$

We recall again that the four-vector operator κ_μ can link a representation (m,n) of sl$(2,C)$ only with $(m\pm\tfrac{1}{2}, n+\tfrac{1}{2})$ and $(m\pm\tfrac{1}{2}, n-\tfrac{1}{2})$. It follows that

$$P_{rr}\kappa_\mu P_{rr} = 0. \tag{3.125}$$

Now ψ satisfies Eq. (3.108a), so that

$$\kappa_\mu k^\mu\psi_0 = 4[M(M+1) + N(N+1) + \Delta^2 - i\Delta]\psi_0,$$
$$= 4[2MN + M + N - i\Delta]\psi_0, \tag{3.126}$$

using Eq. (2.42a). Multiplying on the left by P_{rr}, using Eqs. (3.123) and (3.125), and noting that P_{rr} commutes with M, N, and Δ, we get

$$(2MN + M + N - i\Delta)\psi_0 = 0. \tag{3.127}$$

Then Eqs. (3.124) imply

$$\Delta\psi_0 = -2ir(r+1)\psi_0. \tag{3.128}$$

Now, since Eqs. (2.42a) and (3.124) hold, we have

$$\Delta^2\psi_0 = 0. \tag{3.129}$$

Consistency of Eqs. (3.128) and (3.129) requires

$$r = 0 (\Rightarrow \nu = 1) \tag{3.130}$$

and

$$\Delta\psi_0 = 0. \tag{3.131}$$

Now consider Eq. (3.32c), which is supposed to be satisfied by ψ. On a plane wave solution, we have

$$\Sigma_{\mu\nu} k^\nu \psi_0 = (i - \Delta) k_\mu \psi_0, \tag{3.132}$$

so that Eq. (3.131) implies

$$\Sigma_{\mu\nu} k^\nu \psi_0 = i k_\mu \psi_0. \tag{3.133}$$

But Eqs. (3.124) and (3.130) imply

$$P_{00}\psi_0 = \psi_0, \tag{3.134}$$

so that we have

$$\Sigma_{\mu\nu} k^\nu P_{00}\psi_0 = i k_\mu \psi_0. \tag{3.135}$$

Since (0,0) is the trivial representation of sl(2,C),

$$\Sigma_{\mu\nu} P_{00} = 0, \tag{3.136}$$

and Eq. (3.135) yields

$$k_\mu \psi_0 = 0, \tag{3.137}$$

providing a contradiction. Thus ψ cannot be both a $\{\nu\}$ field and a massless plane wave solution of Eqs. (3.32). □

Comment:

1. We have yet to show that indecomposable $[\lambda, +; 0, u]$ representations exist, and that plane wave solutions of Eqs. (3.32) exist if ψ is a $[\lambda, +; 0, u]$ field. These questions will be examined in full in subsequent papers. In the next section we shall see that well-known sets of conformal-invariant free-field equations do provide illustrative examples, but all corresponding to cases with $u = 0$. □

Now if ψ is a $[\lambda, +; 0, u]$ field, then in particular,

$$\Delta = i(M + N + 1), \tag{3.138a}$$

$$M - N = \lambda, \tag{3.138b}$$

and Eq. (3.32c) becomes

$$\Sigma_{\mu\nu} P^\nu \psi = i(M+N) P_\mu \psi. \tag{3.139}$$

Contracting on the left with $\tilde{\Sigma}_\rho{}^\mu$ and using Lemma 2.5, Eq. (2.69), we get

$$iC_2 P_\rho \psi = -i(M+N+1)\tilde{\Sigma}_{\rho\mu} P^\mu \psi. \tag{3.140}$$

Using Eqs. (2.13) and (3.138b), and noting that $(M+N+1)$ has a well-defined inverse, we then obtain

$$\tilde{\Sigma}_{\mu\nu} P^\nu \psi = -\lambda P_\mu \psi. \tag{3.141}$$

If this ψ is a positive-energy (resp., negative-energy) plane wave, Eq. (3.141) is a covariant statement that ψ has helicity λ (resp., $-\lambda$.) (cf. Comment 1 following Lemma 3.2). Noting Theorems 3.1, 3.2, 3.3, 2.1, and 3.4, we therefore have

Theorem 3.5: If the wave equation (1.1) is locally conformal-invariant on a vector space $\mathscr{U} \subseteq \mathscr{D}$, then the nonzero components of any plane-wave solution $\psi \in \mathscr{U}$ belong to a direct sum of indecomposable $[\lambda, +; 0, u]$ representations of $\mathscr{W}$, for various values of λ and u. Moreover, if ψ_λ is a direct-summand of such a plane wave solution, corresponding to the representation $[\lambda, +; 0, u]$ for some u, then ψ_λ has Lorentz-invariant helicity λ or $-\lambda$ according as the plane wave has positive or negative energy. □

Comment:

1. In this sense we justify our assertion in the Introduction that Eq. (1.1) is not locally conformal-invariant when ψ is a potential, since such finite-component fields do not have (manifestly) Lorentz-invariant helicity,[31] i.e., they do not satisfy equations of the general form of Eq. (3.141). □

4. CONNECTION WITH EARLIER WORK

Most earlier works on the conformal-invariance of massless field equations have been concerned with fields corresponding to representations of $\mathscr{W}$ of Type Ia, in the notation of Mack *et al.*[43], i.e., representations in which the $\kappa_\mu = 0$. In the light of Theorem 3.5, the following result is significant for such fields:

Theorem 4.1: An indecomposable $[\lambda, +; 0, u]$-representation of $\mathscr{W}$ is of Type Ia if and only if $u = 0$. For each integral and semi-integral λ, there exists exactly one (up to equivalence) indecomposable $[\lambda, +; 0, u]$-representation. It is in fact irreducible, and remains so when restricted to sl(2,C), the sl(2,C) content being $(\lambda, 0)$ when $\lambda \geqslant 0$, and $(0, -\lambda)$ when $\lambda < 0$. In either case, the basis operator Δ satisfies

$$\Delta = i(|\lambda| + 1). \tag{4.1}$$

Proof: In an indecomposable $[\lambda, +; 0, u]$ representation, the eigenvalues of $-i\Delta$ are, according to (2.62),

$$|\lambda| + 1, |\lambda| + 2, \cdots |\lambda| + u + 1.$$

Since $-i\Delta$ is diagonalizable, the representation space is a direct sum of the corresponding eigenspaces. But if $\kappa_\mu = 0$, Eqs. (2.3) show that these eigenspaces are separately invariant under the action of the $\mathscr{W}$ algebra, contradicting the assumed indecomposability unless $u = 0$.

Conversely, when $u = 0$ the representation space consists of the single eigenspace corresponding to the eigenvalue $(|\lambda| + 1)$ of $-i\Delta$. Since the action of κ_μ is to increase the eigenvalue of $-i\Delta$ by one unit, it follows that in such a representation

$$\kappa_\mu = 0, \tag{4.2a}$$

$$\Delta = i(|\lambda| + 1). \tag{4.2b}$$

In view of the defining relations (3.138) of such a representation, we have then

$$M + N = |\lambda|, \quad M - N = \lambda \tag{4.3}$$

so that if $\lambda \geqslant 0$,

$$M = \lambda, \quad N = 0, \tag{4.4a}$$

and if $\lambda < 0$,

$$M = 0, \quad N = -\lambda. \tag{4.4b}$$

It follows from the meaning of M and N that if $\lambda \geqslant 0$, the representation $[\lambda, +; 0, 0]$, regarded as a representation of sl(2,C), is a direct sum of replicas of $(\lambda, 0)$; while if $\lambda < 0$, it is a direct sum of replicas of $(0, -\lambda)$. But when Eqs. (4.2) hold, the corresponding irreducible sl(2,C) subspaces are also $\mathscr{W}$-invariant, so that if the given representation of $\mathscr{W}$ is indecomposable, it must consist of a single irreducible representation $(\lambda,0)$ or $(0,-\lambda)$ of sl(2,C).

It can now be seen that there exists exactly one (up to equivalence) indecomposable representation of $\mathscr{W}$ satisfying all these conditions for a given value of λ. It consists of the representation $(\lambda,0)$ of sl(2,C) [or $(0,-\lambda)$, if $\lambda<0$], extended to a representation of $\mathscr{W}$ by defining κ_μ and Δ via Eqs. (4.2). It is evidently irreducible. □

For an irreducible $[\lambda,+;0,0]$ field then, Eq. (4.2b) holds and it can be seen from the cotransformation law (2.4) and (2.5) for the field under changes of scale in particular, that such a field has the length dimension $-(|\lambda|+1)$. This is the "canonical" dimension of a field corresponding to a representation $(|\lambda|,0)$ or $(0,|\lambda|)$ of sl(2,C).

Combining Theorems 3.5 and 4.1, we have (cf. Bracken[41]):

Theorem 4.2: If ψ is a field of Type Ia, and the wave equation (1.1) is locally conformal-invariant on a vector space $\mathscr{U}\subseteq\mathscr{D}$, then the non zero components of any positive-energy (respectively, negative-energy) plane wave solution in $\mathscr{U}$ belong to a direct sum of irreducible representations of sl(2,C) of the type $(m,0)$ or $(0,n)$, with the corresponding length dimensions $(-m-1)$ and $(-n-1)$, and corresponding Lorentz-invariant helicities m and $-n$ [respectively, $-m$ and n]. □

What is the content of the critical Eqs. (3.32) for irreducible $[\lambda,+;0,0]$ fields, or direct sums of such fields for various values of λ? Equations (3.32b) and (3.32d) are satisfied identically. We note that since the only representations (m,n) of sl(2,C) involved here have $mn=0$, then

$$MN=0, \tag{4.5}$$

and Eq. (3.32e) can be written with the help of Eqs. (4.2a), (3.138a), (4.5), and (2.13) as

$$\tau_{\mu\nu}\psi=0, \tag{4.6}$$

with $\tau_{\mu\nu}$ as in Eq. (3.49). But in a representation of the type under consideration, $\tau_{\mu\nu}$ vanishes identically because of the following:

Lemma 4.1: Let $\Sigma_{\mu\nu}$ be basis operator of a finite-dimensional representation $(m,0)$ or $(0,n)$ of sl(2,C). Then the tensor $\tau_{\mu\nu}$, defined as in Eq. (3.49), vanishes identically.

Proof: In the representation $(m,0)$ we have [cf. Eqs. (3.83) and (3.85)]

$$\tilde{\Sigma}_{\mu\nu}=-i\Sigma_{\mu\nu} \tag{4.7}$$

and Eq. (2.69) of Lemma 2.5 becomes

$$\begin{aligned}-i\Sigma_{\mu\nu}\Sigma^{\nu}{}_{\lambda}-\Sigma_{\mu\lambda}&=iC_2g_{\mu\lambda}\\&=im(m+1)g_{\mu\lambda}\\&=\tfrac{1}{2}iC_1g_{\mu\lambda},\end{aligned}$$

i.e.,

$$\tau_{\mu\nu}=0.$$

The argument is similar for the representation $(0,n)$. □

It follws that for fields which correspond to a direct sum of irreducible $[\lambda,+;0,0]$ representations of $\mathscr{W}$, Eqs. (3.32) reduce to (3.32a) and (3.32c), i.e.,

$$P_\mu P^\mu\psi=0, \tag{4.8a}$$

$$\Sigma_{\mu\nu}P^\nu\psi=(i-\Delta)P_\mu\psi\equiv-i(M+N)P_\mu\psi. \tag{4.8b}$$

And furthermore, if the direct sum of fields contains no summand ψ_λ with $\lambda=0$, then Eq. (4.8a) is implied by Eq. (4.8b), since

$$(M+N)\psi_\lambda=(-i\Delta-1)\psi_\lambda=|\lambda|\psi_\lambda, \tag{4.9}$$

and contracting Eq. (4.8b) on the left with P^μ gives

$$-i(M+N)P^\mu P_\mu\psi_\lambda=0. \tag{4.10}$$

We now consider the results of earlier investigations in relation to ours.

A. The scalar field

The index space is one-dimensional in this case, and carries the trivial representation $(0,0)$ of sl(2,C). This can be extended to the nontrivial representation $[0,+;0,0]$ of $\mathscr{W}$, by taking $\kappa_\mu=0$ and $\Delta=+i$. The dimension of the field is then (-1). Eq. (4.8b) is trivial in this case as $\Sigma_{\mu\nu}=0=M=N$. We are left with the single Eq. (4.8a), i.e., the wave equation, in our locally conformal-invariant set.

B. The two- and four-component neutrino equations

Consider the two-component neutrino field χ, with index space carrying the representation $(\frac{1}{2},0)$ of sl(2,C) with basis operators

$$\mathbf{S}=\tfrac{1}{2}\boldsymbol{\sigma},\ \mathbf{T}=-\tfrac{1}{2}i\boldsymbol{\sigma} \tag{4.11}$$

in the notation of Lemma 3.1. Here $\boldsymbol{\sigma}$ are the Pauli matrices. This representation can be extended to the representation $[\frac{1}{2},+;0,0]$ of $\mathscr{W}$, by taking $\kappa_\mu=0$ and $\Delta=3i/2$. Then χ has dimension $(-3/2)$. A locally conformal-invariant set of equations (implying $\Box\chi=0$) is then Eq. (4.8b), which is (since $M=\frac{1}{2}$, $N=0$ here)

$$\Sigma_{\mu\nu}\partial^\nu\chi=-\tfrac{1}{2}i\partial_\mu\chi, \tag{4.12}$$

or equivalently

$$\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\chi=-\partial_0\chi, \tag{4.13a}$$

$$(\boldsymbol{\sigma}\wedge\boldsymbol{\nabla}+i\boldsymbol{\sigma}\partial_0)\chi=-i\boldsymbol{\nabla}\chi, \tag{4.13b}$$

where

$$\boldsymbol{\nabla}=(\partial_1,\partial_2,\partial_3). \tag{4.14}$$

Eq. (4.13b) is implied by Eq. (4.13a), so we can consider Eq. (4.13a) alone, the Weyl equation, as a locally conformal-invariant equation. It implies that a positive energy field has helicity $(+\frac{1}{2})$.

The case of a two-component field corresponding to the representation $(0,\frac{1}{2})$ of sl(2,C), and $[-\frac{1}{2},+;0,0]$ of $\mathscr{W}$, is similar. Again the field has dimension $(-\frac{3}{2})$. The four-component (Dirac bispinor) neutrino field ψ is the direct sum of these two two-component fields. The appropriate representation of $\mathscr{W}$ is $[\frac{1}{2},+;0,0]\oplus[-\frac{1}{2},+;0,0]$, with basis operators

$$\Sigma_{\mu\nu}=\tfrac{1}{4}i[\gamma_\mu,\gamma_\nu],\quad\kappa_\mu=0,\quad\Delta=(\tfrac{3}{2})i, \tag{4.15}$$

where γ_μ are the Dirac matrices, satisfying

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}. \tag{4.16}$$

Equation (4.8b) reads

$$\tfrac{1}{4}i[\gamma_\mu,\gamma_\nu]\partial^\nu\psi = -\tfrac{1}{2}i\partial_\mu\psi, \qquad (4.17)$$

which, with the help of Eq. (4.16) can be reduced to

$$\gamma_\mu(\gamma_\nu\partial^\nu)\psi = 0,$$

or equivalently,

$$\gamma^\nu\partial^\nu\psi = 0. \qquad (4.18)$$

The two-component fields are recovered with the use of the projectors

$$P_\pm = \tfrac{1}{2}(1 \pm \gamma_5), \qquad (4.19)$$

where

$$\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3. \qquad (4.20)$$

Thus if

$$\psi_\pm = P_\pm\psi, \qquad (4.21)$$

then $\psi_\pm$ satisfies

$$\gamma_5\psi_\pm = \pm\psi_\pm. \qquad (4.22)$$

and corresponds to the representation $[\pm\frac{1}{2},+;0,0]$ of $\mathscr{W}$. This can be seen by evaluating

$$C_1 = \tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu} = \tfrac{3}{2} \qquad (4.23a)$$

$$C_2 = \tfrac{1}{4}i\widetilde{\Sigma}_{\mu\nu}\Sigma^{\mu\nu} = \tfrac{1}{4}i(-i\gamma_5\Sigma_{\mu\nu})\Sigma^{\mu\nu} = (\tfrac{3}{4})\gamma_5. \qquad (4.23b)$$

A comparison with Eq. (2.13) shows that on $\psi_\pm$, $M=\frac{1}{2}$ and $N=0$, while on ψ_-, $N=\frac{1}{2}$ and $M=0$. The fields $\psi_\pm$ have helicity $\pm\frac{1}{2}$ (for positive energy) in accordance with Theorems 3.5 and 4.2.

C. Maxwell's equations for the free electromagnetic field

The index space of the electromagnetic field $F_{\mu\nu}(x)[=-F_{\nu\mu}(x)]$ carries the representation $(1,0)\oplus(0,1)$ of sl(2,C). We can extend this to a representation $[1,+;0,0]\oplus[-1,+;0,0]$ of $\mathscr{W}$, by taking $\kappa_\mu=0$ and $\Delta=2i$. Then $F_{\mu\nu}$ has dimension (-2). Since $(M+N)=1$ here, a locally conformal-invariant set of equations (implying $\Box F=0$) is, from Eq. (4.8b),

$$\Sigma_{\mu\nu}\partial^\nu F = -i\partial_\mu F. \qquad (4.24)$$

The sl(2,C) operators act on $F_{\alpha\beta}$ as

$$(\Sigma_{\mu\nu}F)_{\alpha\beta} \equiv (\Sigma_{\mu\nu})_{\alpha\beta}{}^{\rho\sigma}F_{\rho\sigma}, \qquad (4.25)$$

where

$$\begin{aligned}-2i(\Sigma_{\mu\nu})_{\alpha\beta}{}^{\rho\sigma} = {} & (g_{\mu\alpha}\delta_\nu{}^\rho - g_{\nu\alpha}\delta_\mu{}^\rho)\delta_\beta{}^\sigma \\ & + \delta_\alpha{}^\rho(g_{\mu\beta}\delta_\nu{}^\sigma - g_{\nu\beta}\delta_\mu{}^\sigma) \\ & - (g_{\mu\beta}\delta_\nu{}^\rho - g_{\nu\beta}\delta_\mu{}^\rho)\delta_\alpha{}^\sigma \\ & - \delta_\beta{}^\rho(g_{\mu\alpha}\delta_\nu{}^\sigma - g_{\nu\alpha}\delta_\mu{}^\sigma), \end{aligned} \qquad (4.26)$$

and on substituting this expression in Eq. (4.24) and using the antisymmetry of $F_{\mu\nu}$, we get

$$g_{\mu\alpha}\partial^\rho F_{\rho\beta} - g_{\mu\beta}\partial^\rho F_{\rho\alpha} = -\partial_\mu F_{\alpha\beta} - \partial_\alpha F_{\beta\mu} - \partial_\beta F_{\mu\alpha}. \qquad (4.27)$$

Contracting both sides with $g_{\mu\alpha}$ we find

$$\partial^\rho F_{\rho\alpha} = 0, \qquad (4.28)$$

and Eq. (4.27) then implies also

$$\partial_\mu F_{\alpha\beta} + \partial_\beta F_{\mu\alpha} + \partial_\alpha F_{\beta\mu} = 0. \qquad (4.29)$$

Eqs. (4.28) and (4.29) are the free-field Maxwell's equations. They are written in compact form in Eq. (4.24) [or Eq. (4.27)]. Note that by Theorems 3.5 and 4.2, the $[\pm 1,+;0,0]$ component $F_{\mu\nu}{}^{(\pm)}$ of $F_{\mu\nu}$ satisfies also

$$\widetilde{\Sigma}_{\mu\nu}\partial^\nu F^{(\pm)} = \mp\partial_\mu F^{(\pm)}, \qquad (4.30)$$

an equation which is also locally conformal-invariant, and which states that the invariant helicity of (positive energy) fields $F_{\mu\nu}{}^{(\pm)}$ is ± 1. It is easily checked that

$$F_{\mu\nu}{}^{(\pm)} = \tfrac{1}{2}(F_{\mu\nu} \mp i\widetilde{F}_{\mu\nu}), \qquad (4.31)$$

where

$$\widetilde{F}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \qquad (4.32)$$

Thus $F^{(\pm)}$ (respectively, $F^{(-)}$) is the right (respectively, left) circularly polarized component of F.

D. The Bargmann–Wigner equations

The index space of the fields ξ used by Bargmann *et al.*[18] to describe massive and massless particles with spin $s(>0)$ is the *symmetrized* tensor product of $2s$ identical, four-dimensional Dirac bispinor spaces, which we may label with $\alpha = 1,2,\ldots,2s$. Let $\gamma_\mu{}^{(\alpha)}$ be the Dirac matrices acting on the αth four-dimensional space. Then for each α, the relations (4.16) are satisfied, and $\gamma_\mu{}^{(\alpha)}$ commutes with $\gamma_\mu{}^{(\beta)}$ if $\alpha\neq\beta$. Introduce also $\gamma_5{}^{(\alpha)}$, $\alpha=1,2,\ldots,2s$, by analogy with Eq. (4.20).

For massless particles with helicity $+s$, Bargmann *et al.* further required that $\xi\,(=\xi_+$ now) satisfies

$$\gamma_5{}^{(\alpha)}\xi_+ = \xi_+, \quad \alpha = 1,2,\ldots,2s. \qquad (4.33)$$

Since the eigenvalues $+1$ and -1, respectively, of $\gamma_5{}^{(\alpha)}$ label the representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ of sl(2,C) carried by the αth factor space, it follows that the index space of ξ_+ carries the symmetrized tensor product of the representation $(\frac{1}{2},0)$ with itself ($2s$) times. This is the representation $(s,0)$. Similarly we may introduce ξ_- satisfying

$$\gamma_5{}^{(\alpha)}\xi_-, \quad \alpha = 1,2,\ldots,2s, \qquad (4.34)$$

and associated with the representation $(0,s)$ of sl(2,C).

The sl(2,C) basis operators in both cases are (restrictions of)

$$\begin{aligned}\Sigma_{\mu\nu} &= \frac{1}{4}i\sum_{\alpha=1}^{2s}[\gamma_\mu{}^{(\alpha)},\gamma_\nu{}^{(\alpha)}] \\ &= -isg_{\mu\nu} + \frac{1}{2}i\sum_{\alpha=1}^{2s}\gamma_\mu{}^{(\alpha)}\gamma_\nu{}^{(\alpha)}, \end{aligned} \qquad (4.35)$$

so that

$$C_1 = \tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu} = 2s(s+1) - \frac{1}{4}\sum_{\alpha<\beta=1}^{2s}(\gamma_\mu{}^{(\alpha)}\gamma^{\mu(\beta)})^2. \qquad (4.36)$$

According to Eqs. (2.8), on the representation $(s,0)$ or $(0,s)$,

$$C_1 = 2s(s+1). \qquad (4.37)$$

It follows that

$$\sum_{\alpha<\beta=1}^{2s}(\gamma_\mu{}^{(\alpha)}\gamma^{\mu(\beta)})^2\xi_\pm = 0. \qquad (4.38)$$

Now $\gamma_0^{(\alpha)}$ and $i\gamma_j^{(\alpha)}$ $(j = 1,2,3)$ can be taken to be Hermitian, without loss of generality, for each value of α. Thus $(\gamma_\mu^{(\alpha)}\gamma^{\mu(\beta)})$ is Hermitian. It then follows from Eq. (4.38) that

$$(\gamma_\mu^{(\alpha)}\gamma^{\mu(\beta)})\xi_\pm = 0 \ , \ \alpha \neq \beta. \tag{4.39}$$

Conversely, it can easily be seen that if ξ satisfies Eqs. (4.39), then it belongs to that part of the tensor-product space assoicated with the representation $(s,0) \oplus (0,s)$ of sl$(2,C)$. Eqs. (4.39), and equivalently the symmetrization conditions and Eqs. (4.33) and (4.34) of Bargmann *et al*, are not to be thought of as dynamical conditions, but rather as statements defining the index space of the fields to be used to describe massless particles. One could, of course, start with $2(2s+1)$-component fields corresponding to this representation of sl$(2,C)$, but the advantage of the approach used by Bargmann *et al.*—introducing redundant components and then imposing conditions which set them to zero—is simply that one can employ the familiar algebra of the Dirac matrices.

The representation $(s,0) \oplus (0,s)$ can be extended to the representation $[s, + ;0,0] \oplus [-s, + ;0,0]$ of $\mathscr{W}$, by setting $\kappa_\mu = 0$ and $\Delta = +i(s+1)$. Then ξ has the canonical dimension $(-s-1)$. Since $(M+N) = s$ here, a locally conformal-invariant set of equations for ξ (implying $\Box\xi = 0$) is then, from Eq. (4.8b),

$$\Sigma_{\mu\nu}\partial^\nu\xi = -is\xi. \tag{4.40}$$

Substituting for $\Sigma_{\mu\nu}$ from Eq. (4.35), we get

$$\sum_{\alpha=1}^{2s} \gamma_\mu^{(\alpha)}(\gamma_\nu^{(\alpha)}\partial^\nu)\xi = 0. \tag{4.41}$$

Contracting on the left with $\gamma^{\mu(\beta)}$, using the commutation and anticommutation relations between the $\gamma_\rho^{(\delta)}$, and noting Eqs. (4.39), we get

$$(\gamma_\mu^{(\beta)}\partial^\mu)\xi = 0, \quad \beta = 1,2,...,2s. \tag{4.42}$$

Conversely, if Eqs. (4.42) hold, then so do Eqs. (4.41) and hence Eqs. (4.40). Thus the locally conformal-invariant Eq. (4.8b) is in this case equivalent to Eqs. (4.42), which are the Bargmann–Wigner equations.[18] The component $\xi_\pm$ corresponding to the representation $[\pm s, + ;0,0]$ of $\mathscr{W}$ can be obtained as

$$\xi_\pm = \prod_{\alpha=1}^{2s} [\tfrac{1}{2}(1 \pm \gamma_5^{(\alpha)})]\xi. \tag{4.43}$$

It satisfies the locally conformal-invariant equation (3.141) with $\lambda = \pm s$ [and also Eqs. (4.33) or (4.34)], and so (for positive energy) has helicity $\pm s$ as expected.

The Eqs. (4.40), where the index space of ξ carries the representation $(s,0)$, $(0,s)$, or $(s,0) \oplus (0,s)$ of sl$(2,C)$, were consiered before the work of Bargmann *et al.* by Dirac,[15] Fierz,[16] and Gårding,[14] using the dotted–undotted spinor formalism. The complete equivalence of these different ways of writing *the same equations* must be emphasized. Those sections in other works,[19–23,26] concerned only with showing the conformal invariance of these equations, were repeating in different formalisms part of the work of Gross.[17] McLennan[13] had previously shown the local invariance of these same equations.[63] Note also that for $s = \frac{1}{2}$ and $s = 1$ the Bargmann–Wigner equations are completely equivalent to the neutrino equations, and Maxwell's equations, respectively, as can be seen from our discussion above.

When we write all these equations in the forms (4.12), (4.17), (4.24), and (4.40), we see most clearly that they belong to one family—*the* family of conformal-invariant equations for Type Ia fields.

E. Errors in the work of McLennan and Post

McLennan[13] claimed to prove the conformal invariance of certain sets of field equations described by Gårding.[14] In these papers the dotted-undotted spinor formalism is used. The index space of a field with p undotted indices and q dotted ones (p and q are non-negative integers), separately symmetric in each set, carries the irreducible representation $(\frac{1}{2}p,\frac{1}{2}q)$ of sl$(2,C)$ in our notation. In particular, fields ψ corresponding to the representation $(\frac{1}{2}p,\frac{1}{2}q) \oplus (\frac{1}{2}q,\frac{1}{2}p)$ [with $p \neq q$] are considered, together with first-order field equations [McLennan's Eqs. (3.19)] which imply that the wave equation (1.1) is satisfied. According to our results above, these equations can not be locally conformal invariant unless $pq = 0$. This contradicts a claim made by McLennan, but it is easy to find an error in his analysis. He supposes [see his Eq. (6.4)] that under a special conformal transformation, a component of the field corresponding to the representation $(\frac{1}{2}p, \frac{1}{2}q)$ transforms in such a way that its p undotted indices are not affected. Similarly, for a component corresponding to $(\frac{1}{2}q, \frac{1}{2}p)$, the p dotted indices are not affected. But such transformation laws are not consistent with the structure of the Lie algebra of the conformal group, for an infinitesimal translation does not affect spinor indices, but the commutator of our infinitesimal special conformal transformation along one spatial axis, and an infinitesimal translation along another, is an infinitesimal rotation about the third [cf. Eq. (2.6i)], and so affects all dotted and undotted indices. Therefore, an infinitesimal special conformal transformation must in general also affect all dotted and undotted indices. McLennan's proposed transformation law is not consistent if $p \neq 0$.

In claiming to deduce the conformal invariance of equations satisfied by fields with $p = q$ and zero helicity (such fields can also be thought of as symmetric, traceless, tensor fields $\varphi_{\mu\nu...\rho}$ with p indices), McLennan merely remarked that such sets of equations "are equivalent to the scalar or pseudo-scalar wave equation" (1.1), which is conformal invariant. In fact one can show that[42]

$$\varphi_{\mu\nu\cdots\rho} = \partial_\mu\partial_\nu\cdots\partial_\rho\varphi. \tag{4.44}$$

where φ satisfies Eq. (1.1). However, the conformal invariance of Eq. (1.1) for φ does not ensure the invariance of the equations satisfied by $\varphi_{\mu\nu\cdots\rho}$ defined as in Eq. (4.44), and in fact our results imply that these equations are not invariant. The index-space representation of sl$(2,C)$ associated with this tensor field is $(\frac{1}{2}p, \frac{1}{2}p)$. This can be extended to a representation of $\mathscr{W}$ only by setting $\kappa_\mu = 0$ and Δ equal to a constant, so that the field is in particular of Type Ia. But then Theorem 4.2 shows that the wave equation is not locally invariant on such a field, if $p \neq 0$. The reason for this breakdown of conformal invariance in the passage from φ to $\varphi_{\mu\nu\cdots\rho}$ is easily seen—the operators $\partial_\mu, \partial_\nu \cdots$ in Eq. (4.44) are Lorentz-covariant but not conformal-covariant objects.

More recently Post[26] considered free, massless, positive-energy fields $\psi^{(m,n)}(x)$ whose index space carries an irreducible representation (m,n) of sl$(2,C)$, and which have Lorentz-invariant helicity $\lambda = (m-n)$ [cf. Lemma 3.2]. He claimed to prove that the equations satisfied by such fields, including the wave equation (1.1), are conformal-invariant, even if $mn \neq 0$. This contradicts our results, and indeed, the result given by one of us[41] before Post's work appeared. His proof is incorrect, and depends crucially on a result attributed to Mack *et al.*[22] [See the paragraph following Post's Eq. (5.11).] This result, which is in fact invalid, was not proved in Ref. 22, though its validity was implied there. The result in question can be described as follows.

A Hilbert space of the fields $\psi^{(m,n)}$ can be defined, carrying the unitary, irreducible representation of ISL $(2,C)$ appropriate to a massless "particle" with positive energy and helicity $(m-n)$. This representation extends to a unitary irreducible representation of SU(2,2), with self-adjoint generators $P'_\lambda, K'_\lambda, D'$, and $M'_{\mu\nu}$ satisfying, on a suitable domain, the commutation relations (2.6). Then these operators can be identified on the Hilbert space with the generators (2.5) of conformal transformations for these fields, after appropriate choices for κ_μ and Δ are made.

Mack and Todorov showed that this is so if $mn = 0$, but they did not consider directly the cases with $mn \neq 0$. Instead they quoted a result of Weinberg,[42] who showed that if a free, massless positive-energy field χ corresponds to an irreducible index space representation (m,n) of sl$(2,C)$ with $m - n = \lambda$, and has Lorentz-invariant helicity λ, then χ is a linear combination of the rth partial derivatives with respect to the variables x^μ, of a field ξ which also has invariant helicity λ. If $\lambda \geqslant 0$, then ξ corresponds to an index-space representation $(\lambda,0)$, and $r = 2n$. If $\lambda < 0$, then ξ corresponds to $(0, -\lambda)$, and $r = 2m$. On this basis, Mack and Todorov concluded that they could restrict their attention to the cases with $mn = 0$, in order to prove the desired result for the operators $P'_\lambda, K'_\lambda, D'$, and $M'_{\mu\nu}$. However, as remarked in the Introduction, and as implied by Theorem 4.2, the result in question is not valid if $mn \neq 0$. In fact one finds that the operators K'_λ in these cases, unlike the K_λ of Eqs. (2.5), are nonlocal. The reason for this breakdown of conformal invariance, in the context of Weinberg's result, is again that the operator ∂_μ relating massless fields with $mn = 0$ to ones with $mn \neq 0$ [cf. Eq. (4.44)] is not conformal covariant. Essentially the same misunderstanding of this point led McLennan into error, as noted above.

F. Other related works

Several authors[36–40,43] have considered the conditions to be satisfied if classical field equations derivable from an action principle are to be conformal invariant. However, they have not been concerned with the specific situation where the wave equation (1.1) is required to be one of the field equations obtained. The conditions obtained are accordingly much less specific than ours. (In another sense, they are more specific, since it is not clear which of the sets of field equations we have described are derivable from an action principle.) Furthermore, these works have concentrated on fields of Type Ia.

The conformal invariance (in a weaker sense) of wave equations for massive particles has been considered by other authors.[5,11,24,54,64] Because the taking of the zero mass limit is a nontrivial matter, particularly in the context of conformal invariance,[65] it is not clear how the results obtained in these works relate to ours.

The conditions under which Lorentz-invariant equations of the form (1.2) are also conformal invariant have been analyzed by Kotecky *et al.*[30] But again, because they did not specifically require that Eq. (1.1) should follow from Eq. (1.2), their results are not easily related to ours. They did relate their results to some extent with those of McLennan,[13] but did not detect any errors in that work. Only fields of Type Ia appear in the results of Kotecky *et al.* One reason for this is easily seen. If fields of Type Ib are involved, then one has a four-vector operator κ_μ acting on the index space, and having scale dimension $(+1)$. Then as well as equations of the forms (1.2), field equations of the form

$$L_\mu \partial^\mu \psi = A\psi \tag{4.45}$$

must also be considered, where A is a dimensionless matrix. Equation (3.54) provides an example. Massless wave equations of the general form (4.45) have appeared in a more general context in the work of Wightman.[66] Let us remark also that for field equations of the form (1.2), (4.45) where L_μ is rectangular, an important and nontrivial constraint [cf. Theorem 3.4], not considered by Kotecky *et al.*, is that the equation should admit plane wave solutions.

Fields of Type Ib have received comparatively little attention in the literature. Ciccariello and Sartori[52] (see also Ferrara *et al.*,[53] and Lopuszanski and Oziewicz[25]) considered fields of Type Ib and associated conformal-invariant wave equations, but once again, their aims were different from ours, and their results and ours are not easily related. Lopuszanski *et al.* did note the appearance of conformal-invariant equations of the form (4.40) for fields of Type Ia, as one of us had done earlier.[41] (See also Seetharaman.[46])

Since the Lie algebra $\mathscr{W}$ is a subalgebra of su(2,2), any finite-dimensional representation of the latter defines a representation of the former. Mack *et al.*[43] have considered fields of Type Ib generated in this way. But it must be emphasized that only a limited class of representations of $\mathscr{W}$, and consequently, only a limited class of field types, can be obtained in this way. There is a countable number of inequivalent, finite-dimensional representations for su(2,2), but an uncountable number for $\mathscr{W}$ and representations of $\mathscr{W}$ in which Δ is not diagonalizable [cf. Eqs. (2.17)] are not contained in representations of su(2,2).[67]

Dirac[9] and Hepner[51] (see also Mack *et al.*[43] and Budini[29]) have considered the particular case of Dirac spinors $\psi(x)$ and the associated four-dimensional representations of su(2,2) with [cf. Eqs. (2.5) and (2.6)]

$$p_\mu = \tfrac{1}{2}\kappa^{-1}(1 \pm \gamma_5)\gamma_\mu, \quad m_{\mu\nu} = \tfrac{1}{4}i[\gamma_\mu, \gamma_\nu],$$
$$d = \mp \tfrac{1}{2}i\gamma_5, \quad k_\mu = \tfrac{1}{2}\kappa(1 \mp \gamma_5)\gamma_\mu. \tag{4.46}$$

Here the Dirac matrices are as in Sec. 4.2, and κ is a nonzero constant with dimensions of length. (Representations with different values of κ are equivalent, so this value has no physical significance.) Then one may take for the generators of conformal transformations of $\psi(x)$

$$P_\mu = i\partial_\mu + p_\mu, \quad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + m_{\mu\nu},$$
$$D = x^\mu P_\mu + in + d, \quad K_\mu = 2x_\mu(x^\nu P_\nu + in) - x^\nu x_\nu P_\mu + k_\mu, \qquad (4.47)$$

where n is a constant. These operators satisfy the relations (2.6), but are not of the form (2.5). However, by a similarity transformation[43]

$$\psi(x) \rightarrow \exp(-ix^\nu p_\nu)\psi(x),$$
$$P_\mu \rightarrow \exp(-ix^\nu p_\nu)\cdot P_\mu \cdot \exp(ix^\nu p_\nu), \qquad (4.48)$$

etc., one can bring them to the form (2.5), with

$$\kappa_\mu = \tfrac{1}{2}\kappa(1 \mp \gamma_5), \quad \Delta = in \mp \tfrac{1}{2}i\gamma_5$$
$$\Sigma_{\mu\nu} = \tfrac{1}{4}i[\gamma_\mu, \gamma_\nu]. \qquad (4.49)$$

These operators (4.49) span a representation, $D_{n\mp}$ say, of $\mathscr{W}$, which is *not* a $[\lambda, +;0,u]$ representation for any λ, u. However, the representation D_{1+}, for example, is indecomposable but not irreducible, and contains the representation $[\frac{1}{2}, +;0,0]$ as an invariant subrepresentation, associated with the subspace of spinors on which $\gamma_5 = +1$. Accordingly, the equation (1.1) is then locally conformal invariant provided Eqs. (3.32) hold, and here they reduce to

$$(\gamma_\mu P^\mu)\psi = 0, \qquad (4.50a)$$
$$\gamma_5 \psi = \psi. \qquad (4.50b)$$

This is an example of the type of behavior whose possibility was indicated in Comment 1, following Theorem 3.3. In the present example, so long as we are concerned only with free massless fields, there is no real loss of generality if we restrict our attention to spinors for which Eq. (4.50b) is satisfied identically—i.e., essentially two-component spinors corresponding to the representation $[\frac{1}{2}, +;0,0]$ of $\mathscr{W}$ [cf. Sec. 4.2].

On the other hand, the equations

$$\gamma^\mu P_\mu \psi = 0 \qquad (4.51a)$$

and

$$\gamma_5 \psi = -\psi \qquad (4.51b)$$

are *not* conformal invariant if we adopt the representation D_{1+} for ψ, since they are not consistent with Eqs. (3.32). [The roles of the equations (4.50b) and (4.51b) are interchanged if we consider instead the representation D_{1-} for ψ.] The situation here is to be contrasted with that in Sec. 4.2, where the representation $[\frac{1}{2}, +;0,0] \oplus [-\frac{1}{2}, +;0,0]$ of $\mathscr{W}$ was adopted for ψ, and both sets of equations, (4.50) *and* (4.51) are conformal invariant. When we vary the relevant representation of $\mathscr{W}$ on Dirac spinors, we are really changing the field type, and when we talk about conformal invariance or noninvariance of equations like (4.50) of (4.51) we must be clear as to what type of fields we are considering. Failure to do so seems to have led to some confusion in the literature.[68,69] In particular, we should not confuse the results described here for spinors corresponding to the representations D_{1+}, D_{1-}, or $([\frac{1}{2}, +;0,0] \oplus [-\frac{1}{2}, +;0,0])$ of $\mathscr{W}$ with the result implied by Dirac[9] (see also Budini[29] and Castell[68]) that the equation

$$(1 \pm \gamma_5)\gamma^\mu P_\mu \psi = 0 \qquad (4.52)$$

is conformal invariant if ψ corresponds to the representation $D_{2\pm}$ of $\mathscr{W}$. Eq. (4.52) does not imply Eq. (1.1), so $\psi(x)$ is not a massless field according to our definition, and our general results are not directly relevant to this case.

5. CONCLUDING REMARKS

We have derived the conditions under which the wave equation (1.1) is locally conformal invariant, and have seen as a result that although some well-known sets of massless wave equations for fields of Type Ia are invariant, many others are not. Indeed, it is fair to say that most massless wave equations for fields of this type are not conformal invariant. In particular,[41] Eq. (1.1) is not invariant if the index space of ψ carries an irreducible representation (m,n) of sl$(2,C)$ with $mn \neq 0$.

Most generally, we have shown that only $[\lambda, +;0,u]$ fields are of direct interest in the discussion of locally-invariant wave equations, and that these always carry Lorentz-invariant helicity λ (for positive-energy plane waves). For $u > 0$, these fields are of Type Ib. In subsequent papers, we shall describe the representations $[\lambda, +;0,u]$ of $\mathscr{W}$ completely, and also examine in detail the consequences of Eqs. (3.32) for such fields, thus completing our analysis.

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