Local conformal invariance of the wave equation for finite-component fields. II. Classification of relevant indecomposable fields

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It has been shown in Part I that the requirement of local conformal invariance of the wave equation for finite-component fields focuses attention on fields whose index spaces carry a certain type of finite-dimensional, indecomposable representation of the nonsemisimple Lie algebra $((k_4 \oplus d) \oplus \mathrm{sl}(2, C))$. All representations of this type are here described in complete detail, in each case in an sl(2, C) basis. Although indecomposable, these representations are in general not fully reducible.

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I. INTRODUCTION

In an earlier work[1] (henceforth referred to as BJ1), we have considered the conditions for local conformal invariance of the wave equation

$$\Box\psi(x) = 0, \quad x = (x^\mu), \quad \mu = 0, 1, 2, 3, \tag{1.1}$$

where ψ is a classical field with some fixed, finite number of complex-valued components. The index space of this field is assumed to carry a corresponding finite-dimensional representation of the Lie algebra

$$\mathscr{W} = ((k_4 \oplus d) \oplus \mathrm{sl}(2, C)), \tag{1.2}$$

with basis operators κ_μ, Δ, and $\Sigma_{\mu\nu} (= -\Sigma_{\nu\mu})$ satisfying the commutation relations

$$i[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = g_{\mu\rho}\Sigma_{\nu\sigma} + g_{\nu\sigma}\Sigma_{\mu\rho} - g_{\nu\rho}\Sigma_{\mu\sigma} - g_{\mu\sigma}\Sigma_{\nu\rho}, \tag{1.3a}$$

$$i[\kappa_\mu, \Sigma_{\nu\rho}] = g_{\mu\rho}\kappa_\nu - g_{\mu\nu}\kappa_\rho, \tag{1.3b}$$

$$[\kappa_\mu, \kappa_\nu] = 0, \tag{1.3c}$$

$$[\Sigma_{\mu\nu}, \Delta] = 0, \tag{1.3d}$$

$$i[\kappa_\mu, \Delta] = \kappa_\mu. \tag{1.3e}$$

Only if this assumption is made[2] can one define, for an arbitrary infinitesimal conformal transformation of space-time, an appropriate cotransformation law for the field ψ. The generators of infinitesimal conformal transformations of ψ then take the forms

homogeneous Lorentz transformations: $\quad i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}, \quad$ (1.4a)

space-time translations: $\quad i\partial_\mu, \quad$ (1.4b)

dilations: $\quad ix^\mu\partial_\mu + \Delta, \quad$ (1.4c)

special conformal transformations: $\quad 2ix_\mu x^\nu\partial_\nu + 2x_\mu\Delta - ix^\nu x_\nu\partial_\mu + 2\Sigma_{\mu\nu}x^\nu + \kappa_\mu, \quad$ (1.4d)

and satisfy on suitably smooth ψ the commutation relations appropriate to the Lie algebra of the conformal group. The significance of $\mathscr{W}$ in this connection stems from the fact that it is the Lie subalgebra associated with those conformal transformations which leave invariant the point $x = 0$, *viz.* those composed of homogeneous Lorentz transformations, dilations, and special conformal transformations. (An isomorphic subalgebra is associate with the dilation group, composed of homogeneous Lorentz transformations, dilations, and space-time translations, and is of independent interest. The dilation group, like the conformal group itself, has been discussed as a possible approximate space-time symmetry group in particle physics. In that context, however, the main interest is in infinite-dimensional representations[3] of $\mathscr{W}$.)

The problem of classifying all finite-component field types having inequivalent cotransformation laws for infinitesimal conformal transformations, is seen to correspond to the problem of classifying all inequivalent finite-dimensional representations of $\mathscr{W}$. Such representations have been called[2] of type I, as distinct from infinite-dimensional (type II) representations. More particularly, a finite-dimensional representation and corresponding field is called of type Ia if the associated basis operators κ_μ vanish identically, and of type Ib otherwise. The Lie algebra $\mathscr{W}$ is not semisimple, and its representations of type I or II are not in general fully reducible. The problem of classifying all inequivalent representations of type Ib in particular seems quite beyond our present powers.

In BJ1, we have defined the wave equation (1.1) to be locally conformal-invariant on a vector space $\mathscr{U}$ of smooth solutions, if $\mathscr{U}$ is invariant under the action of the conformal algebra (1.4). Then we have shown that the non-zero components of any ψ in such a $\mathscr{U}$ must belong to a representation of $\mathscr{W}$ from a certain class $\mathscr{Q}$, characterized by the property that the basis operators of any representation from this class satisfy the $\mathscr{W}$-invariant set of equations

$$\kappa_\mu\kappa^\mu = 0, \tag{1.5a}$$

$$\Sigma_{\mu\nu}\kappa^\nu = (\Delta + i)\kappa_\mu, \tag{1.5b}$$

$$\Delta^4 + (C_1 + 1)\Delta^2 + (C_2)^2 = 0, \tag{1.5c}$$

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where

$$C_1 = \tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}, \tag{1.6a}$$

$$C_2 = (1/8)i\epsilon^{\mu\nu\rho\sigma}\Sigma_{\mu\nu}\Sigma_{\rho\sigma}. \tag{1.6b}$$

(The metric tensor is diagonal with $g_{00} = -g_{11} = -g_{22} = -g_{33} = +1$, and the alternating tensor has $\epsilon^{0123} = +1$.) Accordingly, we have restricted our attention to indecomposable (but not necessarily irreducible) representations of $\mathscr{W}$ in Class $\mathscr{D}$, and associated indecomposable Class $\mathscr{D}$ fields.

We have shown furthermore that if (a) ψ is an indecomposable Class $\mathscr{D}$ field, (b) a locally conformal-invariant vector space $\mathscr{U}$ of solutions of the wave equation (1.1) does exist, and (c) at least one of the solutions in $\mathscr{U}$ is a plane wave, then the associated indecomposable Class $\mathscr{D}$ representation of $\mathscr{W}$ must be, for some integer 2λ and non-negative integer u, a representation of the type we have labeled $[\lambda, +; 0, u]$. Since we are interested primarily in the possibility of using locally conformal-invariant spaces of solutions of Eq. (1.1) in the description of free massless particles, the condition (c) is important, and our attention has therefore been limited further, to indecomposable $[\lambda, +; 0, u]$-representations and fields.

The basis operators of such a representation satisfy, by definition, certain conditions additional to (1.5). In order to be able to describe these conditions, we must first recall that every finite-dimensional representation of $\mathscr{W}$ must be fully reducible when regarded as a representation of the sl(2, C) subalgebra associated with the basis operators $\Sigma_{\mu\nu}$. Let (m, n) label the $(2m+1)(2n+1)$-dimensional irreducible representation[4] of sl(2, C), where $2m$ and $2n$ are non-negative integers, associated with eigenvalues $2[m(m+1) + n(n+1)]$ and $[m(m+1) - n(n+1)]$ of the sl(2, C)-invariants C_1 and C_2, respectively, of Eqs. (1.6). An arbitrary finite-dimensional representation of $\mathscr{W}$ must decompose into a direct sum of such representations (m, n), with various values of m and n, and various multiplicities. The operators C_1 and C_2 in such a representation of $\mathscr{W}$ will therefore have the form

$$C_1 = 2M(M+1) + 2N(N+1), \tag{1.7a}$$

$$C_2 = M(M+1) - N(N+1), \tag{1.7b}$$

where M and N are non-negative, simultaneously diagonalizable, sl(2, C) scalar operators whose eigenvalues are non-negative integers or semi-integers. On that subspace of the representation space for $\mathscr{W}$ which is associated with the totality of irreducible representations (m, n) of sl(2, C) for fixed m and n, M and N have the eigenvalues m and n, respectively.

The additional defining properties of a $[\lambda, +; 0, u]$-representation of $\mathscr{W}$ are then

(i) $\Delta = i(M + N + 1)$, (1.8a)

implying in particular that Δ is diagonalizable,

(ii) $M - N = \lambda$, (1.8b)

(iii) the eigenvalues of $-i\Delta$ are exactly the set of numbers

$$\{|\lambda| + 1, \quad |\lambda| + 2, \cdots, \quad |\lambda| + u + 1\}. \tag{1.8c}$$

The conditions (i) and (ii) taken together are stronger than, and imply condition (1.5c), as can be seen with the help of Eqs. (1.7). Thus the independent conditions characterizing a $[\lambda, +; 0, u]$-representation are Eqs. (1.5a), (1.5b) and conditions (i)–(iii) above.

In BJ1 we have shown that a $[\lambda, +; 0, u]$-representation is of type Ia if and only if $u = 0$. We have shown also that for each integer 2λ there exists, up to equivalence, exactly one indecomposable $[\lambda, +; 0, 0]$-representation. It is in fact irreducible, and remains so when restricted to sl(2, C), being then labeled $(\lambda, 0)$ if $\lambda \geqslant 0$ and $(0, -\lambda)$ if $\lambda < 0$. The basis operator Δ is constant, having the value $i(|\lambda| + 1)$, and of course $\kappa_\mu = 0$.

We have shown also that if ψ is an indecomposable $[\lambda, +; 0, 0]$ field, and lies in a locally conformal-invariant vector space $\mathscr{U}$ of solutions of Eq. (1.1), then ψ actually satisfies a set of equations including (1.1). These equations are equivalent to the scalar wave equation if $\lambda = 0$; to two-component neutrino equations if $|\lambda| = \frac{1}{2}$; to Maxwell's free field equations if $|\lambda| = 1$; and in general to the Bargmann–Wigner equations for massless fields of helicity λ. The conformal invariance of these sets of equations is well known.[5] In order to find new conformal-invariant free massless field theories, possibly of interest to physics, it is therefore necessary to look at what are in effect, the only remaining possibilities, indecomposable $[\lambda, +; 0, u]$ fields with $u > 0$. These are fields of type Ib, and the corresponding representations, although indecomposable, are not irreducible.

We have not attempted a complete description of these representations in BJ1. Indeed, we have not even proved their existence for arbitrary integers 2λ and $u > 0$. It is the purpose of this work to fill these gaps. That we are able to achieve this object completely, as the ensuing Theorem 2.1 shows, is remarkable, given the apparent intractability of the corresponding task for the totality of representations of type Ib, or even those of Class $\mathscr{D}$. Our success depends upon the diagonalizability of Δ in $[\lambda, +; 0, u]$-representations, and the availability of Gabriel's theorem,[6,7] whose substance should not be underestimated. We were able to derive our own proof of the latter from "first principles" for the special case of interest to us (i.e., for the quiver corresponding to the Dynkin diagram for A_{u+1} - see Sec. 2) but this proof runs to several pages.

In subsequent work we shall describe the structure of the new sets of locally conformal-invariant massless field equations obtained for indecomposable $[\lambda, +; 0, u]$ fields with $u > 0$.

II. STRUCTURE OF THE RELEVANT REPRESENTATIONS OF $\mathscr{W}$

Theorem 2.1: Up to equivalence, there is exactly one indecomposable $[\lambda, +; 0, u]$-representation of $\mathscr{W}$ for each integer or semi-integer λ and each non-negative integer u. When regarded as a representation of sl(2, C), this representation of $\mathscr{W}$ has the decomposition

$$(\lambda, 0) \oplus (\lambda + \tfrac{1}{2}, \tfrac{1}{2}) \oplus \ldots \oplus (\lambda + \tfrac{1}{2}u, \tfrac{1}{2}u) \tag{2.1}$$

if $\lambda \geqslant 0$, and

$$(0, -\lambda) \oplus (\tfrac{1}{2}, \tfrac{1}{2} - \lambda) \oplus \ldots \oplus (\tfrac{1}{2}u, \tfrac{1}{2}u - \lambda) \tag{2.2}$$

if $\lambda < 0$. The dimension of the representation is

$$d = \tfrac{1}{6}(u+1)(u+2)(2u+3+6|\lambda|). \tag{2.3}$$

A basis consisting of vectors

$$\begin{aligned}
&|\delta, s, s_3\rangle,\\
&\delta \in \{|\lambda|+1, |\lambda|+2, \ldots, |\lambda|+1+u\},\\
&s \in \{|\lambda|, |\lambda|+1, \ldots, \delta-1\},\\
&s_3 \in \{s, s-1, \ldots, -s\},
\end{aligned} \tag{2.4}$$

can be introduced, on which the operators Δ, $\Sigma_{\mu\nu}$, κ_μ, and the related operators M, N, C_1, and C_2 of Eqs. (1.6) and (1.7) act as follows [we write

$$(\Sigma_{23}, \Sigma_{31}, \Sigma_{12}) = \mathbf{S}, \tag{2.5a}$$

$$(\Sigma_{01}, \Sigma_{02}, \Sigma_{03}) = \mathbf{T}]: \tag{2.5b}$$

$$\Delta\,|\delta, s, s_3\rangle = i\delta\,|\delta, s, s_3\rangle, \tag{2.6a}$$

$$\mathbf{S}\cdot\mathbf{S}|\delta, s, s_3\rangle = s(s+1)|\delta, s, s_3\rangle, \tag{2.6b}$$

$$S_3|\delta, s, s_3\rangle = s_3|\delta, s, s_3\rangle, \tag{2.6c}$$

$$C_1|\delta, s, s_3\rangle = (\lambda^2 + \delta^2 - 1)|\delta, s, s_3\rangle, \tag{2.6d}$$

$$C_2|\delta, s, s_3\rangle = \lambda\delta\,|\delta, s, s_3\rangle, \tag{2.6e}$$

$$M\,|\delta, s, s_3\rangle = \tfrac{1}{2}(\delta + \lambda - 1)|\delta, s, s_3\rangle, \tag{2.6f}$$

$$N\,|\delta, s, s_3\rangle = \tfrac{1}{2}(\delta - \lambda - 1)|\delta, s, s_3\rangle, \tag{2.6g}$$

$$\begin{aligned}
&(S_1 \pm iS_2)|\delta, s, s_3\rangle\\
&\quad = [(s \pm s_3 + 1)(s \mp s_3)]^{1/2}|\delta, s, s_3 \pm 1\rangle,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&T_3|\delta, s, s_3\rangle\\
&\quad = D(s)[(\delta - s)(\delta + s)(s - s_3)(s + s_3)]^{1/2}\\
&\qquad \times |\delta, s-1, s_3\rangle + s_3\delta E(s)|\delta, s, s_3\rangle\\
&\qquad - D(s+1)[(\delta - s - 1)(\delta + s + 1)\\
&\qquad \times (s - s_3 + 1)(s + s_3 + 1)]^{1/2}|\delta, s+1, s_3\rangle,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&(T_1 \pm iT_2)|\delta, s, s_3\rangle\\
&\quad = \pm D(s)[(\delta - s)(\delta + s)(s \mp s_3)(s \mp s_3 - 1)]^{1/2}\\
&\qquad \times |\delta, s-1, s_3 \pm 1\rangle\\
&\qquad + \delta E(s)[(s \mp s_3)(s \pm s_3 + 1)]^{1/2}|\delta, s, s_3 \pm 1\rangle\\
&\qquad \pm D(s+1)[(\delta - s - 1)(\delta + s + 1)\\
&\qquad \times (s \pm s_3 + 1)(s \pm s_3 + 2)]^{1/2}\\
&\qquad \times |\delta, s+1, s_3 \pm 1\rangle,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&\kappa_0|\delta, s, s_3\rangle\\
&\quad = \kappa[(\delta - s)(\delta + s + 1)]^{1/2}|\delta + 1, s, s_3\rangle,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&\kappa_3|\delta, s, s_3\rangle\\
&\quad = i\kappa D(s)[(\delta - s)(\delta - s + 1)(s - s_3)(s + s_3)]^{1/2}\\
&\qquad \times |\delta + 1, s - 1, s_3\rangle\\
&\qquad + i\kappa s_3 E(s)[(\delta - s)(\delta + s + 1)]^{1/2}|\delta + 1, s, s_3\rangle\\
&\qquad - i\kappa D(s+1)[(\delta + s + 1)(\delta + s + 2)(s - s_3 + 1)\\
&\qquad \times (s + s_3 + 1)]^{1/2}|\delta + 1, s + 1, s_3\rangle,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&(\kappa_1 \pm i\kappa_2)|\delta, s, s_3\rangle\\
&\quad = \pm i\kappa D(s)[(\delta - s)(\delta - s + 1)(s \mp s_3)\\
&\qquad \times (s \mp s_3 - 1)]^{1/2}|\delta + 1, s - 1, s_3 \pm 1\rangle\\
&\qquad + i\kappa E(s)[(\delta - s)(\delta + s + 1)(s \mp s_3)\\
&\qquad \times (s \pm s_3 + 1)]^{1/2}|\delta + 1, s, s_3 \pm 1\rangle\\
&\qquad \pm i\kappa D(s+1)[(\delta + s + 1)(\delta + s + 2)\\
&\qquad \times (s \pm s_3 + 1)(s \pm s_3 + 2)]^{1/2}\\
&\qquad \times |\delta + 1, s + 1, s_3 \pm 1\rangle.
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
&D(s) = (s^2 - \lambda^2)^{1/2}/s(4s^2 - 1)^{1/2},\\
&E(s) = -i\lambda/s(s+1),
\end{aligned} \tag{2.13}$$

and κ is a nonzero constant. The nonzero value of κ is immaterial, representations which differ only in this value being equivalent.[8] The formulas (2.10, 2.11) for κ_μ applied to $|\delta, s, s_3\rangle$ are only valid for $\delta < (|\lambda| + 1 + u)$, and

$$\kappa_\mu|(|\lambda| + 1 + u), s, s_3\rangle = 0. \tag{2.14}$$

The operators κ_μ are nilpotent, and the product $(\kappa_\mu\kappa_\nu\ldots\kappa_\rho)$ is not identically zero only if it does not contain more than u factors. (In particular, if $u = 0$ then $\kappa_\mu = 0$.)

Proof: We know that in such an indecomposable representation of $\mathscr{W}$, $(-i\Delta)$ has the eigenvalues (1.8c)

$$\delta = |\lambda| + 1, |\lambda| + 2, \ldots, |\lambda| + 1 + u.$$

Since Eqs. (1.8a, b) hold, it follows that, if $\lambda \geqslant 0$, the pair (M, N) has eigenvalue pairs

$$(m, n) = (\lambda, 0), (\lambda + \tfrac{1}{2}, \tfrac{1}{2}), \ldots, (\lambda + \tfrac{1}{2}u, \tfrac{1}{2}u), \tag{2.15}$$

while if $\lambda < 0$, it has eigenvalue pairs

$$(m, n) = (0, -\lambda), (\tfrac{1}{2}, \tfrac{1}{2} - \lambda), \ldots, (\tfrac{1}{2}u, \tfrac{1}{2}u - \lambda). \tag{2.16}$$

Accordingly, this representation of $\mathscr{W}$, when regarded as a representation of sl(2, C), has the general form

$$r_0(\lambda, 0) \oplus r_1(\lambda + \tfrac{1}{2}, \tfrac{1}{2}) \oplus \cdots \oplus r_u(\lambda + \tfrac{1}{2}u, \tfrac{1}{2}u), \tag{2.17}$$

for $\lambda \geqslant 0$, or

$$r_0(0, -\lambda) \oplus r_1(\tfrac{1}{2}, \tfrac{1}{2} - \lambda) \oplus \cdots \oplus r_u(\tfrac{1}{2}u, \tfrac{1}{2}u - \lambda), \tag{2.18}$$

for $\lambda < 0$, where $r_0, r_1, \ldots, r_u$ are certain *positive* integers. It is convenient at this stage to go from the (m, n) to the $[k_0, c]$ labeling scheme[4] for the finite-dimensional irreducible representations of sl(2, C), where

$$\begin{aligned}
&k_0 = m - n,\\
&c = m + n + 1.
\end{aligned} \tag{2.19}$$

In the case at hand, because Eq. (1.8b) holds, we get only representations with $k_0 = \lambda$, and the decompositions (2.15) and (2.16) have the common form

$$r_0[\lambda, |\lambda| + 1] \oplus r_1[\lambda, |\lambda| + 2] \oplus \cdots \oplus r_u[\lambda, |\lambda| + u]. \tag{2.20}$$

We see that the eigenspace $\mathscr{V}_\delta$, associated with the eigenvalue δ of $(-i\Delta)$, carries the direct sum of r_τ copies of the representation $[\lambda, \delta]$ of sl(2, C), where $\tau = \delta - |\lambda| - 1$. We imagine these copies ordered in some definite way, and labeled by an index α taking values 1, 2, ..., r_τ. Now each representation $[\lambda, \delta]$ of sl(2, C), when regarded as a representation of its su(2) subalgebra spanned by the operators $\mathbf{S}$, is a direct sum of $(2s+1)$-dimensional irreducible representations (s) of su(2), for $s = |\lambda|, |\lambda| + 1, \ldots, \delta - 1$ (each occurring once). And each representation (s) of su(2), when regarded as a representation of its u(1) subalgebra spanned by the operator S_3, is a direct sum of one-dimensional representations $[s_3]$ of u(1), for $s_3 = s, s-1, \ldots, -s$ (each occurring once.) Accordingly, we can introduce a set of basis vectors for the whole carrier space of the given representation of $\mathscr{W}$, labeled

$$|\delta, \alpha, s, s_3\rangle, \tag{2.21}$$

where δ runs over the eigenvalues of $(-i\Delta)$ as in Eqs. (2.4); for each δ $(=\tau + |\lambda| + 1)$, α runs over the values $1, 2, ..., r_\tau$, and independently s runs over the values $|\lambda|, |\lambda| + 1, ..., \delta - 1$; and for each s, s_3 runs over the values $s, s-1, ..., -s$. On the basis vector (2.21), the operators Δ, $\mathbf{S}\cdot\mathbf{S}$ and S_3 will have the eigenvalues $i\delta$, $s(s+1)$, and s_3, respectively. Moreover, in view of Eqs. (1.6) and (1.7), the operators M, N, C_1, and C_2 will have the eigenvalues $\frac{1}{2}(\delta + \lambda - 1)$, $\frac{1}{2}(\delta - \lambda - 1)$, $(\lambda^2 + \delta^2 - 1)$, and $\lambda\delta$, respectively. The action of the sl(2, C) operators in an su(2)$\geqslant$u(1) basis of an irreducible representation $[k_0, c]$ is well known.[4] We get Eqs. (2.7), (2.8), and (2.9) with $|\delta, s, s_3\rangle$ replaced by $|\delta, \alpha, s, s_3\rangle$ throughout. (These operators do not "see" the label α.)

We now turn to the action of the operators κ_μ. In view of the commutation relation (1.3e) and the fact that κ_0 commutes with $\mathbf{S}$, we must have

$$\kappa_0|\delta, \alpha, s, s_3\rangle = \sum_\beta A_{\beta\alpha}|\delta + 1, \beta, s, s_3\rangle, \tag{2.22}$$

for some complex numbers $A_{\beta\alpha}$, which *á priori* could depend on δ and s (but not on s_3). The sum is over the $r_{\tau+1}$ values of β (with $\tau = \delta - |\lambda| - 1$). Equation (2.22) can only hold for $\delta < \delta_{\max} = (|\lambda| + 1 + u)$, and we must have also

$$\kappa_0|(|\lambda| + 1 + u), \alpha, s, s_3\rangle = 0. \tag{2.23}$$

According to Eq. (1.3b), κ_μ is a four-vector operator. The most general structure possible for such operators within a finite-dimensional representation of sl(2, C) is well known.[4] We can apply these known general results to the particular situation at hand, or determine the structure directly, noting that a necessary and sufficient condition for κ_0 as in Eqs. (2.22) and (2.23) to be the fourth component of a four-vector is that

$$[[\kappa_0, T_3], T_3] = -\kappa_0. \tag{2.24}$$

[The remaining components of κ_μ can then be defined by

$$i\kappa_i = [\kappa_0, T_i], \tag{2.25}$$

and the commutation relations (1.3b) will then be satisfied.] We get, in place of Eq. (2.22),

$$\kappa_0|\delta, \alpha, s, s_3\rangle = \sum_\beta B^{(\tau)}_{\beta\alpha}[(\delta - s)(\delta + s + 1)]^{1/2}|\delta + 1, \beta, s, s_3\rangle, \tag{2.26}$$

where the $B^{(\tau)}_{\beta\alpha}$, $\tau \equiv (\delta - |\lambda| - 1) = 0, 1, ..., u-1$, are complex numbers which do not depend on s or s_3, but are otherwise not restricted by Eq. (2.24). For each value of τ, we may regard them as the elements of an $(r_{\tau+1} \times r_\tau)$ matrix $B^{(\tau)}$. We might expect these matrices to be restricted in form by the relations (1.5a), (1.5b) and (1.3c) which are required of a $[\lambda, +; 0, u]$ representation, but in fact this is not the case. These relations place no restrictions whatsoever on the $B^{(\tau)}$ but are satisfied once κ_0 and κ_i have the forms determined by Eqs. (2.26), (2.23), and (2.25). We see this most simply as follows.

The operators κ_μ as defined so far are shift operators for $(-i\Delta)$, M, and N, and in fact we have

$$M\kappa_\mu = \kappa_\mu(M + \tfrac{1}{2}), \quad N\kappa_\mu = \kappa_\mu(N + \tfrac{1}{2}). \tag{2.27}$$

It follows that

$$\begin{aligned} M[\kappa_\mu, \kappa_\nu] &= [\kappa_\mu, \kappa_\nu](M+1), \\ N[\kappa_\mu, \kappa_\nu] &= [\kappa_\mu, \kappa_\nu](N+1). \end{aligned} \tag{2.28}$$

Thus $[\kappa_\mu, \kappa_\nu]$ shifts any vector from a repesentation subspace of sl(2, C) labeled (m, n) to one labeled $(m+1, n+1)$. But, just as a four-vector operator [transforming according to the representation $(\frac{1}{2}, \frac{1}{2})$ itself] can only link[4] (m, n) with $(m + \frac{1}{2}, n \pm \frac{1}{2})$ and $(m - \frac{1}{2}, n \pm \frac{1}{2})$, so any antisymmetric tensor operator like $[\kappa_\mu, \kappa_\nu]$ [transforming according to the representation $(1,0) \oplus (0,1)$] can only link (m, n) with $(m \pm 1, n)$, (m, n), and $(m, n \pm 1)$. It cannot link (m, n) with $(m+1, n+1)$—and to avoid a contradiction it must be true that

$$[\kappa_\mu, \kappa_\nu] = 0. \tag{2.29}$$

Similarly, we have

$$\begin{aligned} M(\kappa_\mu\kappa^\mu) &= (\kappa_\mu\kappa^\mu)(M+1), \\ N(\kappa_\mu\kappa^\mu) &= (\kappa_\mu\kappa^\mu)(N+1). \end{aligned} \tag{2.30}$$

But a scalar operator like $(\kappa_\mu\kappa^\mu)$ cannot link (m, n) with $(m+1, n+1)$, and so

$$\kappa_\mu\kappa^\mu = 0. \tag{2.31}$$

Consider the commutator

$$\begin{aligned} [\kappa_\mu, C_1] &= [\kappa_\mu, \tfrac{1}{2}\Sigma_{\nu\rho}\Sigma^{\nu\rho}] \\ &= 2i\Sigma_{\mu\rho}\kappa^\rho + 3\kappa_\mu \end{aligned} \tag{2.32}$$

using the relations (1.3b), already established. In view of Eqs. (1.7) and (2.27) we then have

$$\begin{aligned} i\Sigma_{\mu\rho}\kappa^\rho + \tfrac{3}{2}\kappa_\mu &= [\kappa_\mu, M(M+1)] + [\kappa_\mu, N(N+1)] \\ &= \kappa_\mu M(M+1) - M(M+1)\kappa_\mu \\ &\quad + \kappa_\mu N(N+1) - N(N+1)\kappa_\mu \\ &= \kappa_\mu M(M+1) - \kappa_\mu(M+\tfrac{1}{2})(M+\tfrac{3}{2}) \\ &\quad + \kappa_\mu N(N+1) - \kappa_\mu(N+\tfrac{1}{2})(N+\tfrac{3}{2}) \\ &= -\kappa_\mu(M+N+\tfrac{3}{2}), \end{aligned} \tag{2.33}$$

so that

$$\begin{aligned} \Sigma_{\mu\rho}\kappa^\rho &= i\kappa_\mu(M+N+3) \\ &= i(M+N+2)\kappa_\mu = (i+\Delta)\kappa_\mu, \end{aligned} \tag{2.34}$$

as required. Thus we see that Eqs. (1.3c), (1.5a), and (1.5b) are all satisfied.

How then are the matrices $B^{(\tau)}$ restricted? It is easy to see that for no τ can $B^{(\tau)}$ be identically zero; otherwise the representation space splits into the direct sum of nontrivial $\mathscr{W}$-invariant subspaces, contradicting the assumed indecomposability of the given representation. But the indecomposability restricts them much more than this. Consider the effect of a change of basis, of the special form

$$|\delta, \alpha, s, s_3\rangle' = \sum_\beta S^{(\tau)}_{\beta\alpha}|\delta, \beta, s, s_3\rangle, \tag{2.35}$$

where, for each δ as in Eqs. (2.4) and corresponding $\tau = \delta - |\lambda| - 1$, the $(r_\tau \times r_\tau)$ matrix $S^{(\tau)}$ with complex elements $S^{(\tau)}_{\beta\alpha}$ is nonsingular. Then

$$|\delta,\alpha,s,s_3\rangle = \sum_\beta S^{(\tau)-1}_{\beta\alpha}|\delta,\beta,s,s_3\rangle', \qquad (2.36)$$

where $S^{(\tau)-1}$ is the inverse of $S^{(\tau)}$, and so, from Eq. (2.26),

$$\kappa_0\sum_\beta S^{(\tau)-1}_{\beta\alpha}|\delta,\beta,s,s_3\rangle' = \sum_\gamma\sum_\beta B^{(\tau)}_{\beta\alpha}[(\delta-s)(\delta+s+1)]^{1/2} \times S^{(\tau+1)-1}_{\gamma\beta}|\delta+1,\gamma,s,s_3\rangle',$$

i.e.,

$$\kappa_0|\delta,\alpha,s,s_3\rangle' = \sum_\beta B^{(\tau)'}_{\beta\alpha}[(\delta-s)(\delta+s+1)]^{1/2}|\delta,\beta,s,s_3\rangle', \qquad (2.37)$$

where

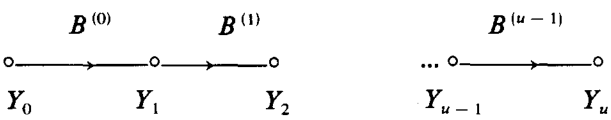
$$B^{(\tau)'}_{\beta\alpha} = \sum_\gamma\sum_\sigma S^{(\tau+1)-1}_{\beta\gamma}B^{(\tau)}_{\gamma\sigma}S^{(\tau)}_{\sigma\alpha}. \qquad (2.38)$$

In short,

$$B^{(\tau)'} = S^{(\tau+1)-1}B^{(\tau)}S^{(\tau)}, \quad \tau = 0, 1, ..., u-1. \qquad (2.39)$$

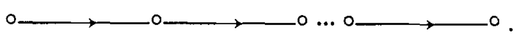
Since we are only interested in the structure of the representation $[\lambda, +; 0, u]$ up to equivalence, we may look for a canonical form of the matrices $B^{(\tau)}$ with respect to transformations of the form (2.39).

Consider a sequence of $(u+1)$ complex vector spaces Y_τ, $\tau = 0, 1, ..., u$ of dimension $r_0, r_1, ..., r_u$, respectively. The matrices $B^{(\tau)}$ define a sequence of linear mappings between the spaces Y_τ, shown diagrammatically thus:



(2.40)

Now consider in abstraction the oriented, connected graph appearing in that diagram,



(2.41)

Such a graph, and more generally, any finite, oriented, connected graph, is called a *quiver*. If with each vertex of the quiver (2.41) is associated a finite-dimensional vector space, and with each directed edge a linear mapping in the appropriate direction, as in the diagram (2.40), then one has a *representation* (Y, B) of the quiver. The *direct sum* of two such representations (Y, B), (Y', B') is the representation (Y'', B''), where for each τ,

$$Y_\tau'' = Y_\tau \oplus Y_\tau', \quad B^{(\tau)''} = B^{(\tau)} \oplus B^{(\tau)'}. \qquad (2.42)$$

A representation (Y, B) is *indecomposable* if it cannot be represented as a direct sum of two nontrivial representations. Two representations (Y, B), (Y', B') are *equivalent* if there exist invertible mappings $S^{(\tau)}$

$$S^{(\tau)}: Y_\tau' \longrightarrow Y_\tau \qquad (2.43)$$

such that

$$B^{(\tau)'} = S^{(\tau+1)-1}B^{(\tau)}S^{(\tau)} \qquad (2.44)$$

for $\tau = 0, 1, ..., u-1$. It can be seen that an indecomposable $[\lambda, +; 0, u]$ representation of $\mathscr{W}$ defines an indecomposable representation of the quiver (2.41), and that any indecomposable representation of the quiver in which none of the $B^{(\tau)}$ is identically zero, defines a $[\lambda, +; 0, u]$ representation of $\mathscr{W}$. Moreover, equivalent representations of the quiver define equivalent representations of $\mathscr{W}$. The problem now arises of classifying the equivalence classes of indecomposable representations of the quiver (2.41). The notion of a representation, and of the indecomposability and equivalence thereof, can be defined for any quiver. Gabriel[6] (see also Bernstein *et al.*[7]) has posed and answered the following question: for which quivers are there finitely many equivalence classes of indecomposable representations? He found that a necessary and sufficient condition is that the graph, when unoriented (i.e., with the arrows removed from the edges) must coincide with the Dynkin diagram for one of the simple Lie algebras[7] $A_2, A_3, ..., D_4, D_5, ..., E_6, E_7$, or E_8. What is more remarkable is that in every such case there is a one-to-one correspondence between the equivalence classes and the positive (integral[7]) roots associated with the corresponding Lie algebra. In the case at hand, we have the Dynkin diagram of A_{u+1}, and the result is that, if the positive root is $(r_0, r_1, ..., r_u)$, then the dimension of Y_τ is r_τ in any representation (Y, B) from the corresponding class. There are $\frac{1}{2}(u+1)(u+2)$ positive roots of A_{u+1}, *viz.*[7]

$$\begin{array}{l}(1,0,0,...,0), (0,1,0,...,0), ..., (0,0,0,...,1)\\ (1,1,0,...,0), (0,1,1,0,...,0), ..., (0,0,0,...,0,1,1)\\ \vdots\\ (1,1,1,...,1)\end{array} \qquad (2.45)$$

But we are only interested in the situation where all $B^{(\tau)}$ are nontrivial, as already remarked, so only the last root is of relevance. (The others correspond to representations $[\lambda, +; l, v]$ of $\mathscr{W}$ with $l > 0$ or $v < u$.) Accordingly, each of the spaces Y_τ is one-dimensional, and

$$r_0 = r_1 = \cdots = r_u = 1. \qquad (2.46)$$

We may now drop the unnecessary label α from the basis vectors (2.21). Each matrix $B^{(\tau)}$ reduces to a nonzero constant—and furthermore, since there is just one equivalence class corresponding to the last of the roots (2.45), we can without loss of generality take all these constants equal, to κ say. Thus we arrive at the form (2.10) for the action of κ_0 on the basis vector $|\delta, s, s_3\rangle$, and the forms (2.11) for the remaining components are simply obtained from Eq. (2.25). The dimension d of the representation of $\mathscr{W}$ is now obtained by adding the dimensions of the irreducible representations $[\lambda, |\lambda|+1]$, $[\lambda, |\lambda|+2]$, ..., $[\lambda, |\lambda|+u]$ of sl(2, C), as

$$d = \sum_{\tau=0}^{u}(\tau+1)(2|\lambda|+\tau+1), \qquad (2.47)$$

yielding the result (2.3). That the product $(\kappa_\mu\kappa_\nu\cdots\kappa_\rho)$ is not identically zero only if it does not contain more than u factors, follows at once from the action of κ_μ as defined by Eqs. (2.10), (2.11), (2.12), and (2.14). □

III. AN ILLUSTRATIVE EXAMPLE

Consider a $[0, +; 0, 1]$ field. It has five components, and the sl(2, C) content of the index space representation is

$$(0,0)\oplus(\tfrac{1}{2},\tfrac{1}{2}). \tag{3.1}$$

The basis vectors $|\delta, s, s_3\rangle$ of Theorem 2.1 run over $|1,0,0\rangle$, $|2,0,0\rangle$, $|2,1,1\rangle$, $|2,1,0\rangle$, and $|2,1,-1\rangle$. Represent them by column vectors $(1\,0\,0\,0\,0)^T$, $(0\,1\,0\,0\,0)^T$, etc. Let E_{RS} $(R, S\in\{1,2,3,4,5\})$ denote the 5×5 matrix with a 1 in the R th row and S th column, and zeros elsewhere. Then according to Eqs. (2.6)–(2.14), the matrix representations of the $\mathscr{W}$ operators are

$$\begin{aligned}
&S_3 = E_{33} - E_{55}, \quad T_3 = E_{24} - E_{42},\\
&S_1 + iS_2 = (\sqrt{2})(E_{34} + E_{45}),\\
&S_1 - iS_2 = (\sqrt{2})(E_{43} + E_{54}),\\
&T_1 + iT_2 = (\sqrt{2})(E_{25} + E_{32}),\\
&T_1 - iT_2 = -(\sqrt{2})(E_{52} + E_{23}),\\
&\kappa_0 = \kappa(\sqrt{2})E_{21}, \quad \kappa_3 = -i\kappa(\sqrt{2})E_{41},\\
&\kappa_1 + i\kappa_2 = 2i\kappa E_{31}, \quad \kappa_1 - i\kappa_2 = -2i\kappa E_{51},\\
&\Delta = iE_{11} + 2i(E_{22} + E_{33} + E_{44} + E_{55}).
\end{aligned} \tag{3.2}$$

Now make a unitary transformation

$$A \rightarrow UAU^{\dagger} \tag{3.3}$$

of each of the $\mathscr{W}$ operators A, where

$$\begin{aligned}U = {}& E_{11} + E_{22} - iE_{54}\\ &+ (1/\sqrt{2})(iE_{33} - iE_{35} - E_{43} - E_{45}).\end{aligned} \tag{3.4}$$

This corresponds to a change from the su(2)$\supset$u(1) basis to a tensor basis. An arbitrary $[0, +; 0, 1]$ field then takes the form

$$\psi(x) = \begin{pmatrix}\varphi(x)\\ A_\mu(x)\end{pmatrix}, \tag{3.5}$$

where φ is an sl(2, C) scalar field, and A_μ a four-vector field. The action of the $\mathscr{W}$ operators is then found to be

$$\Sigma_{\mu\nu}\begin{pmatrix}\varphi\\ A_\rho\end{pmatrix} = i\begin{pmatrix}0\\ g_{\mu\rho}A_\nu - g_{\nu\rho}A_\mu\end{pmatrix}, \tag{3.6a}$$

$$\Delta\begin{pmatrix}\varphi\\ A_\mu\end{pmatrix} = i\begin{pmatrix}\varphi\\ 2A_\mu\end{pmatrix}, \tag{3.6b}$$

$$\kappa_\mu\begin{pmatrix}\varphi\\ A_\nu\end{pmatrix} = \kappa'\begin{pmatrix}0\\ g_{\mu\nu}\varphi\end{pmatrix}, \quad \kappa' = \kappa\sqrt{2}. \tag{3.6c}$$

Consider an infinitesimal scale transformation

$$x'^\mu = (1+\epsilon)x^\mu \tag{3.7}$$

and the corresponding transformation of ψ, as generated by the operators (1.4c),

$$\begin{aligned}\psi'(x) &= \psi(x) + i\epsilon(ix^\mu\partial_\mu + \Delta)\psi(x)\\ &= (1 + i\epsilon\Delta)\psi((1-\epsilon)x),\end{aligned}$$

i.e.,

$$\psi'(x') = (1 + i\epsilon\Delta)\psi(x). \tag{3.8}$$

Then

$$\varphi'(x') = (1-\epsilon)\varphi(x), \tag{3.9a}$$

$$A_\mu{}'(x') = (1-2\epsilon)A_\mu(x), \tag{3.9b}$$

so $\varphi(x)$ has length dimension (-1) and $A_\mu(x)$ has dimension (-2). [Note that the four-vector potential of the electromagnetic field has dimension (-1).] Now consider an infinitesimal special conformal transformation

$$x'^\mu = x^\mu + 2\theta^\nu x_\nu x^\mu - \theta^\mu x_\nu x^\nu \tag{3.10}$$

and the corresponding field transformation

$$\begin{aligned}\psi'(x) &= \psi(x) + i\theta^\mu(2ix_\mu x^\nu\partial_\nu + 2x_\mu\Delta\\ &\quad - ix^\nu x_\nu\partial_\mu + 2\Sigma_{\mu\nu}x^\nu + \kappa_\mu)\psi(x)\\ &= (1 + 2i\theta^\mu x_\mu\Delta + 2i\theta^\mu\Sigma_{\mu\nu}x^\nu + i\theta^\mu\kappa_\mu)\\ &\quad\times\psi(x^\mu - 2\theta^\nu x_\nu x^\mu + \theta^\mu x_\nu x^\nu),\end{aligned}$$

i.e.,

$$\begin{aligned}\psi'(x') = (1 &+ 2i\theta^\mu x_\mu\Delta\\ &+ 2i\theta^\mu\Sigma_{\mu\nu}x^\nu + i\theta^\mu\kappa_\mu)\psi(x).\end{aligned} \tag{3.11}$$

Then

$$\varphi'(x') = (1 - 2\theta^\mu x_\mu)\varphi(x), \tag{3.12}$$

the usual transformation law for a scalar field, while

$$\begin{aligned}A'_\mu(x') = {}& A_\mu(x) - 4\theta^\nu x_\nu A_\mu(x) - 2\theta_\mu x^\nu A_\nu(x)\\ &+ 2x_\mu\theta^\nu A_\nu(x) + i\kappa'\theta_\mu\varphi(x).\end{aligned} \tag{3.13}$$

Here we see the novel feature of Type Ib fields—under the action of the conformal group, fields belonging to different index-space irreducible representations of sl(2, C) are mixed together.

Note that the subspace of fields having $\varphi(x) = 0$ is invariant under this action. This corresponds to the fact that although the representation $[0, +; 0, 1]$ of $\mathscr{W}$ is indecomposable, it is not irreducible, and it "contains" the indecomposable (and irreducible) representation $[0, +; 1, 1]$ as an invariant subrepresentation. More generally, we can see that $[\lambda, +; 0, u]$ contains $[\lambda, +; 1, u]$, which contains $[\lambda, +; 2, u]$, etc. Of the representations $[\lambda, +; 0, u]$, only those with $u = 0$ are irreducible.

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