Parastatistics and the quark model

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An extension of ordinary parastatistics is considered which makes use of all the representations of the parastatistics algebra obtained from the usual ansatz. Govorkov's demonstration that such an extension, for parastatistics of order 2, implies a $U(2)$ symmetry, is generalized for parastatistics of order p. The parastatistics algebra, restricted to N dynamical states, is characterized by the irreducible representations of $U(N)$, $SO(2N)$, and $SO(2N+1)$ which it contains. It is shown that these representations have multiplicities equal to the dimensions of associated representations of $U(p)$, $O(p)$ and $C(p)$, respectively, where $C(p)$ is a subalgebra of the enveloping algebra of $O(p)$, but is not a Lie algebra. The symmetric group $S(p)$ also appears, as a subalgebra of the enveloping algebra of $C(p)$. It is shown how a nondegenerate vacuum state may be defined for the generalized parastatistics algebra of order p, and how to construct state vectors corresponding to arbitrary numbers of quarklike particles and antiparticles. Such states belong to irreducible representations of $U(N)$, and can be obtained by the application of one kind of creation and annihilation operators to certain basic states, here called reservoir states, which correspond to the different irreducible representations of $SO(2N+1)$. The specialization to parastatistics of order 3 is discussed in detail with the application to a quark model of the hadrons in view. It is shown how to define isospin and hypercharge in a significant way in this model, which, however, differs in some respects from Gell-Mann's well-known 3-fermion model, and also from Greenberg's 3-parafermion model. Some of the physical implications are examined.

1. INTRODUCTION

The idea that all particles appearing in nature should be formed from particles of spin half is an old one, which suggested, for instance, de Broglie's theory of fusion,[1] the neutrino theory of light,[2] and Yukawa's nonlocal model for composite particles.[3] All such theories have met with grave difficulties, but a more recent manifestation of the same idea, the quark model of the hadrons,[4] has been sufficiently successful to be taken seriously. Since quarks have not been positively identified in isolation,[5] however, there is ample room for speculation concerning their nature and properties.

In the original proposal of Gell–Mann,[4] a triplet of quark fermi fields is considered, the three types of quarks and antiquarks being assigned to the triplet and antitriplet representations of $U(3)$. However, Greenberg[6] has suggested that quarks may be parafermions, satisfying parastatistics of order 3. As Fritzsch and Gell–Mann[7] have pointed out, the introduction of a triplet of parafermi fields of order 3 is in a certain sense equivalent to the introduction of nine fermi fields, together with supplementary conditions which place a restriction on the allowed states. To see this, one need only recall that a parafermi field of order 3 may be thought of as constructed from three (commuting) fermi fields via the ansatz introduced by one of us[8] in the original formulation of parafermi field quantization. The supplementary conditions then reflect the fact that the ansatz yields a reducible representation of the parafermi field algebra, from which a suitable irreducible representation is to be selected.

These observations suggest a further possibility, that only one type of parafermi field of order 3 need be introduced in order to describe all the hadrons and the associated $U(3)$ multiplet structure. In the context of the usual formulation of the quantization of such a field, this suggestion proves unrewarding, because the states available do not form complete $U(3)$ multiplets. This deficiency may be traced to the fact that one has restricted one's attention to an irreducible representation of the parafermion algebra, and, following Govorkov,[9] one is led to consider more general representations previously rejected on the grounds that the "vacuum state" appeared to be degenerate.[10] From Govorkov's point of view, this degeneracy was only a consequence of the incorrect identification of the vacuum state, and the generalized parafermi statistics had real possibilities for the definition of such physical quantities as isospin and hypercharge. He considered generalized parafermi statistics of order $p = 2$, showed that n-particle states form $U(2)$ multiplets, and gave the expressions for the $U(2)$ generators. However, he was unable to complete the corresponding task for $p = 3$, although he did indicate that one-, two-, and three-particle states form $SU(3)$ multiplets, and, despite some effort,[11] his idea has not been properly realized.

In the meantime, progress has been made in the investigation of parastatistics algebras,[12] and the authors[13] have described the structure of those representations of the parafermion algebras usually adopted, with emphasis on the representations of subalgebras isomorphic to the Lie algebras of certain unitary groups. Here we undertake a similar task for the generalized parafermion algebras, with particular reference to the case $p = 3$.

Irreducible representations of the algebra of N pairs of parafermion creation and annihilation operators are known to correspond to irreducible representations of $SO(2N+1)$. We adopt a certain reducible representation of $SO(2N+1)$, defined by the well-known ansatz,[8] and are then concerned with labeling not only the various irreducible representations of $SO(2N+1)$ which occur, but also the irreducible representations of $U(N)$ contained within each irreducible representation of $SO(2N+1)$, since these correspond to collections of states with a fixed number of particles present, and a fixed symmetry type. So we are led to consider a rather difficult state labeling problem, essentially that of *completing* the set of commuting operators provided by the invariants of the chain,

$$SO(2N+1) \supset O(2N) \supset SO(2N) \supset U(N) [\supset U(N-1) \supset \cdots \supset U(1)]. \quad (1)$$

We find that there is a related chain of algebras also represented in the space of generalized parafermi statistics of order p, viz.

$$U(p) \supset O(p) \supset SO(p) \to C(p), \tag{2}$$

where $SO(p) \to C(p)$ means that the algebra $C(p)$, which is not a Lie algebra, is a subalgebra of the enveloping algebra of $SO(p)$. Recently Drühl, Haag, and Roberts, and subsequently Ohnuki and Kamefuchi,[14] have considered the generalized parafermion algebra from a quite different point of view, and in their work, the chain $U(p) \supset O(p) \supset SO(p)$ is identified and discussed as a chain of "non-Abelian gauge groups." We find that in fact the algebras $U(p) \otimes U(N)$, $O(p) \otimes SO(2N)$, $SO(p) \otimes O(2N)$, and $C(p) \otimes SO(2N+1)$ are all represented, and the Casimir invariants of associated algebras, such as $U(p)$ and $U(N)$, are so related that the problem of completing the commuting set of operators defined by the chain (1) is precisely that of completing the set defined by the chain (2), in a certain class of representations of $U(p)$. For $p \geq 3$, this is well-known to be a very difficult problem, and we do not find an explicit solution. However, for $p = 3$ we do find an operational way of establishing a satisfactory basis in the representation space, thus implicitly defining a solution to the problem. Stated briefly, the picture which emerges in that case is as follows.

One begins with an irreducible representation of three commuting fermi fields, the ansatz components. This contains a reducible representation of the parafermion algebra of order 3, whose irreducible components may be labeled completely by the eigenvalues of two operators I and I_3, which we identify with the total isospin and third component of the isospin: Thus for each $I = 0, \frac{1}{2}, 1, \ldots$, one has $I_3 = I, I-1, \ldots, -I$. Representations labeled by different values of I are inequivalent, while those labeled by the same I and different I_3 are equivalent. The representation with $I = I_3 = 0$ is the one usually adopted for the descriptions of parafermions of order 3, and it contains the nondegenerate vacuum state. Each of the other representations contains degenerate "reservoir states", which may be thought of as containing a number $2I$ of isospin-carrying particles and antiparticles, but on which all the parafermion annihilation operators vanish. Other states within each representation are obtained by applying polynomials in the parafermion creation operators to such reservoir states. Every state in a particular basis for the Fock space of the system can be characterized by the $U(3)$ multiplet to which it belongs and its associated eigenvalues of I, I_3 and the hypercharge Y [together with the $U(N-1) \supset U(N-2) \supset \cdots \supset U(1)$ labels associated with the space–time degrees of freedom]. The $U(3)$ multiplet structure is precisely the same as that obtained in the usual model with three anticommuting fermi fields. Indeed one can assert that within the space of the generalized parafermion algebra, there acts irreducibly the algebra of three anticommuting fermi fields, in terms of which the physical $U(3)$ generators may be defined in the usual way. However, such fields have no important role in the physical interpretation of the model. For example, they carry definite quanta of Y and I_3, while the parafermion creation and annihilation operators, which are more appropriately associated with the "quarks" in this model, are isoscalars and do not carry definite quanta of hypercharge. The hypercharge and other $U(3)$ labels only become diagonal in this picture when an n-particle state is appropriately symmetrized. So we have here a model which reproduces all the multiplet structure of the usual quark model, but has a quite different interpretation at the level of the constituent subparticles.

Section 2 is concerned with analysis of the structure of the representation space for quantized parafermi statistics of order p. Emphasis is given to the representations of the algebras $SO(2N+1) \supset O(2N) \supset SO(2N) \supset U(N)$ and $U(p) \supset O(p) \supset SO(p) \to C(p)$. The labeling problem is precisely defined. In Section 3, the structure and multiplicity of "reservoir states" are examined, together with the way in which other states can be constructed from these by application of parafermion creation operators. Section 4 is devoted to a more detailed treatment of the case $p = 3$, with implicit solution for the state-labeling problem, and definition of the $U(3)$ generators. A brief description of the physical interpretation of some $U(3)$ multiplets is given.

2. GENERALIZED PARASTATISTICS

The absorption and emission operators of a kind of parafermions will be represented by a_r and $a^r = a_r^*$, and for convenience we shall suppose that the affix r takes only a finite number of values $1, 2, \ldots, N$, though in the applications N is, of course, unlimited. As in our earlier paper,[13] we define a_ρ as equal to a_r when $\rho = r$, and equal to a^r when $\rho = r + N$, so that the Greek subscript takes values from 1 to $2N$. Then, if the nonvanishing elements of $g_{\rho\sigma}$ are

$$g_{\rho\sigma} = 1, \quad |\rho - \sigma| = N,$$

the commutation relations

$$[a_\rho, a_\sigma] = 2\alpha_{\rho\sigma},$$
$$[a_\rho, \alpha_{\sigma\tau}] = g_{\rho\sigma}a_\tau - g_{\rho\tau}a_\sigma$$

imply that the a_ρ and $\alpha_{\sigma\tau}$ may be regarded as generators of a representation of $SO(2N+1)$. We also define $g^{\rho\sigma} = g_{\rho\sigma}$, and $a^\rho = g^{\rho\sigma}a_\sigma$, etc.

If an arbitrary irreducible representation of $SO(2N+1)$ is labeled in the usual way by its highest weight $(L_1, L_2, \ldots, L_N)$, the representation $([\frac{1}{2}p]^N) = (\frac{1}{2}p, \frac{1}{2}p, \ldots, \frac{1}{2}p)$ corresponds[13] to ordinary parastatistics of order p. In the generalization which we wish to consider, the reducible representation $([\frac{1}{2}]^N) \otimes ([\frac{1}{2}]^N) \otimes \cdots \otimes ([\frac{1}{2}]^N)$, with p factors, is adopted. Then each irreducible representation with $\frac{1}{2}p \geq L_1 \geq L_2 \geq \ldots \geq L_N \geq 0$ (and $\frac{1}{2}p - L_r$ integral) occurs with a definite multiplicity in the corresponding representation space, which we shall denote by H_p. In particular, the ordinary parastatistics representation $([\frac{1}{2}p]^N)$ occurs once.[15] If $w = \frac{1}{2}p$ when p is even, but $w = \frac{1}{2}(p-1)$ when p is odd, and M_j is the number of L_r not less than j $(j = 1, 2, \ldots, w)$, the representations $(L_1, L_2, \ldots, L_N)$ may also be labeled $[M_1, M_2, \ldots, M_w]$, where $N \geq M_1 \geq M_2 \geq \cdots \geq M_w \geq 0$ (and M_j is integral).

The L_r and M_j can be regarded as operators in H_p with eigenvalues constant within any irreducible representation of $SO(2N+1)$. In this sense they are connected with the Casimir invariants of $SO(2N+1)$, constructed from the generators a_ρ and $\alpha_{\rho\sigma}$. For example, the quadratic invariant of $SO(2N+1)$ is[16]

$$\begin{aligned} \sigma_2(2N+1) &= \alpha^\rho{}_\sigma\alpha^\sigma{}_\rho + a^\rho a_\rho \\ &= 2\sum_r L_r(L_r + 2N + 1 - 2r) \\ &= 2\Big(N(\tfrac{1}{2}p - w)(\tfrac{1}{2}p - w + N) \\ &\quad - \sum_i M_i(M_i - p + 2w + 1 - 2i - 2N)\Big). \end{aligned} \tag{3}$$

The L_r (or M_i), of course, cannot distinguish between isomorphic representations of $SO(2N+1)$ contained in H_p.

Within each representation of $SO(2N+1)$, there are generally several irreducible representations of $SO(2N)$, with generators $\alpha_{\rho\sigma}$. Any one of these is labeled by its highest weight $(\lambda_1, \lambda_2, \ldots, \lambda_N)$, which occurs once and only once within the representation $(L_1, L_2, \ldots, L_N)$ of $SO(2N+1)$, provided $L_r - \lambda_r$, $\lambda_r - L_{r+1}$ and $L_{r+1} - \lambda_{r+1}$ are nonnegative integers for all $r < N$ and $|\lambda_N| \leqslant L_N$. It follows that if the representation $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ occurs, so does the representation $(\lambda_1, \lambda_2, \ldots, -\lambda_N)$. This pair of irreducible representations of $SO(2N)$ (or single irreducible representation in the case $\lambda_N = 0$) forms an irreducible representation of $O(2N)$, which can also be labeled by $[\mu_1, \mu_2, \ldots, \mu_w]$, where μ_j is the number of $|\lambda_r|$ not less than j $(j = 1, 2, \ldots w)$. Then $N \geqslant \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_w \geqslant 0$. The λ_r and μ_j are related to the Casimir invariants of $SO(2N)$ and $O(2N)$, constructed from the $\alpha_{\rho\sigma}$, with the quadratic invariant being given by[16]

$$\begin{aligned}\sigma_2(2N) &= \alpha^{\rho}{}_{\sigma}\alpha^{\sigma}{}_{\rho} \\ &= 2\sum_r \lambda_r(\lambda_r + 2N - 2r) \\ &= 2\Big(N(\tfrac{1}{2}p - w)(\tfrac{1}{2}p - w + N - 1) \\ &\quad - \sum_i \mu_i(\mu_i - p + 2w + 2 - 2i - 2N)\Big). \end{aligned} \tag{4}$$

The λ_r (or μ_j) cannot distinguish between isomorphic representations of $SO(2N)$ or $O(2N)$ contained in H_p.

Within each representation of $SO(2N)$, there are generally many irreducible representations of $U(N)$, with generators

$$a^r{}_s = \tfrac{1}{2}([a^r, a_s] + p\delta^r{}_s).$$

Any such representation is labeled by its highest weight $(l_1, l_2, \ldots, l_N)$, where [since the weights of $SO(2N+1)$ and $SO(2N)$ are sets of eigenvalues of $a^1{}_1 - \frac{1}{2}p, a^2{}_2 - \frac{1}{2}p$, etc.] $0 \leqslant l_{r+1} \leqslant l_r \leqslant \lambda_r + \frac{1}{2}p \leqslant L_r + \frac{1}{2}p \leqslant p$. The l_r are integers, which can be regarded as the lengths of the rows in the Young diagram associated in the usual way with the symmetry type of the corresponding tensor representation of $U(N)$. Alternatively, the representation may be labeled $[m_1, m_2, \ldots, m_p]$, where m_i is the number of l_r not less than i $(i = 1, 2, \ldots, p)$, or the length of the ith column in the associated Young diagram. Then $N \geqslant m_1 \geqslant m_2 \geqslant \cdots \geqslant m_p \geqslant 0$. The l_r and m_i are related to the Casimir invariants of $U(N)$, constructed from the $a^r{}_s$, with, for example,[17]

$$\begin{aligned} a^r{}_r &= \sum_r l_r = \sum_i m_i, \\ a^r{}_s a^s{}_r &= \sum_r l_r(l_r + N + 1 - 2r) \\ &= -\sum_i m_i(m_i - N + 1 - 2i). \end{aligned}$$

The l_r (or m_i) cannot distinguish between isomorphic representations of $U(N)$ contained in H_p [nor indeed between isomorphic representations of $U(N)$ contained in an irreducible representation of $SO(2N)$]. We now turn our attention to the problem of describing the multiplicity with which isomorphic representations of $U(N)$ occur in H_p.

One way of defining the generalized parastatistics representation is via the ansatz[8]

$$a_p = \sum_{i=1}^{p} \sigma_p{}^{(i)}, \tag{5}$$

in which the $\sigma_p{}^{(i)}$ are fermion creation and annihilation generators for a fixed value of i, but commute for different values of i. The only way we shall make use of this ansatz, however, is by the properties with which it endows the operators

$$c^{(ij)} = \sigma^{r(i)}\sigma_r{}^{(j)},$$

in particular

$$\begin{aligned} &[c^{(ij)}, a^r{}_s] = 0, \\ &\{c^{(ij)}, c^{(jk)}\} = c^{(ik)} \quad (i \neq j \neq k), \\ &[c^{(ij)}, c^{(ji)}] = c^{(ii)} - c^{(jj)} \quad (i \neq j), \\ &[c^{(ii)}, c^{(jj)}] = 0 \quad (i \neq j), \\ &[c^{(ii)}, c^{(ij)}] = c^{(ij)} \quad (i \neq j), \\ &[c^{(ij)}, c^{(kl)}] = 0 \quad (i \neq j \neq k \neq l) \end{aligned} \tag{6}$$

and

$$\begin{aligned} \sum_{i,j} c^{(ij)} &= \tfrac{1}{2}(a^p a_p - Np), \\ \sum_{i=1}^{p} c^{(ii)} &= a^r{}_r = \sum_r l_r = \sum_i m_i, \\ \sum_{i,j} c^{(ij)}c^{(ji)} &= -a^r{}_s a^s{}_r + (N+p)\, a^r{}_r \\ &= -\sum_r l_r(l_r - p - 2r + 1) \\ &= \sum_i m_i(m_i + p - 2i + 1). \end{aligned}$$

Now the operator $c^{(ii)}$ has integral eigenvalues, and it is evident from (6) that

$$\theta_i = \cos(\pi c^{(ii)}),$$

which has unit square, anticommutes with $c^{(ij)}$ and $c^{(ji)}$ provided $j \neq i$, and commutes with all other $c^{(kl)}$. Let us define

$$\begin{aligned} b_{jk} &= (i)^{j+k}\theta_a\theta_{a+1}\cdots\theta_b c^{(jk)}, \quad j < k, \\ b_{kj} &= -(i)^{j+k}\theta_a\theta_{a+1}\cdots\theta_b c^{(kj)}, \quad j < k, \\ b_{jj} &= c^{(jj)}, \end{aligned} \tag{7}$$

where the subscripts $a, a+1, \ldots, b$ of the θ's include all values between the odd integer a, equal to j or $j+1$, and the even integer b, equal to k or $k-1$. Thus $b_{12} = -i\theta_1\theta_2 c^{(12)}$, $b_{23} = ic^{(23)}$, and $b_{51} = \theta_1\theta_2\theta_3\theta_4 c^{(51)}$. Then it is easy to verify that

$$[b_{ij}, b_{kl}] = \delta_{jk} b_{il} - \delta_{il} b_{kj},$$

the commutation relations characteristic of $U(p)$. It follows that an irreducible representation of the b_{ij}, and of the $c^{(ij)}$ also, defines an irreducible representation of $U(p)$. Moreover, since the b_{ij} are $U(N)$ invariants, H_p carries a representation of $U(p) \otimes U(N)$, with generators $b_{ij}, a^r{}_s$. Now

$$\begin{aligned} &\sum_i b_{ii} = \sum_i c^{(ii)} = \sum_i m_i, \\ &\sum_{i,j} b_{ij}b_{ji} = \sum_{i,j} c^{(ij)}c^{(ji)} = \sum_i m_i(m_i + p - 2i + 1), \cdots \end{aligned}$$

and the expressions on the right sides of these equations are precisely those adopted by the Casimir invariants of $U(p)$, when the irreducible representations are labeled $(m_1, m_2, \ldots, m_p)$. In other words, each irreducible representation of $U(p) \otimes U(N)$ contained in H_p is labeled by the $m_1, m_2, \ldots, m_p$, which are the lengths of the rows in the Young diagram corresponding to the $U(p)$ representation $(m_1, m_2, \ldots, m_p)$, and the lengths of the columns in the Young diagram corresponding to the associated $U(N)$ representation $[m_1, m_2, \ldots, m_p]$. Since there are no invariants of $U(N)$ which cannot be expressed in terms of the b_{ij}, no irreducible representation of $U(p) \otimes U(N)$ can occur more than once in H_p. Then it is not difficult to see that every such representation with $N \geqslant m_1 \geqslant m_2 \geqslant \cdots \geqslant m_p \geqslant 0$ must occur just once. We may say that the representation $[m_1, m_2, \ldots, m_p]$ of $U(N)$ occurs in H_p with a multiplicity equal to the dimension of the representation $(m_1, m_2, \ldots, m_p)$ of $U(p)$. This is one way of describing the structure of H_p.

Turning now to the question of the multiplicity with which isomorphic representations of $SO(2N)$ occur in H_p, we note that

$$\gamma^{(ij)} = c^{(ij)} + c^{(ji)} \quad (i \neq j)$$

is an $SO(2N)$ invariant and that, with one exception, all invariants of $SO(2N)$ can be constructed from the $\gamma^{(ij)}$. [The exception is the pseudoscalar $SO(2N)$ invariant associated with the sign of λ_N. More precisely, the $\gamma^{(ij)}$ are $O(2N)$ invariants, from which all invariants of $O(2N)$ can be constructed.] It follows from (6) that

$$\{\gamma^{(ij)}, \gamma^{(jk)}\} = \gamma^{(ik)} \quad (i \neq j \neq k),$$
$$[\gamma^{(ij)}, \gamma^{(kl)}] = 0 \quad (i \neq j \neq k \neq l) \tag{8}$$

and, consequently,

$$(\gamma^{(ik)} \pm \gamma^{(jk)})\gamma^{(ij)} = -(\gamma^{(ij)} \pm 1)(\gamma^{(ik)} \pm \gamma^{(jk)}). \tag{9}$$

We define

$$\gamma = \sum_{j>i} \gamma^{(ij)} = \tfrac{1}{2}(a^\rho a_\rho - Np). \tag{10}$$

By subtraction of (4) from (3) we find

$$a^\rho a_\rho = 2\Big(N(\tfrac{1}{2}p - w) + \sum_i \mu_i(\mu_i - p + 2w + 2 - 2i - 2N) - \sum_i M_i(M_i - p + 2w + 1 - 2i - 2N)\Big),$$

and it follows that the eigenvalues of γ are integral. From consideration of the parastatistics algebra of order 2 contained within the entire algebra, for which γ reduces to $\gamma^{(12)}$, we infer that $\gamma^{(12)}$ has integral eigenvalues, and this conclusion is, of course, independent of the superscripts of $\gamma^{(ij)}$. Hence the commuting invariants $\gamma^{(12)}, \gamma^{(34)}, \ldots, \gamma^{(2w-1,2w)}$ of $O(2N)$ all have integral eigenvalues. If we introduce

$$\theta_{ij} = \cos(\pi\gamma^{(ij)}), \tag{11}$$

it follows further from (9) that θ_{ij} anticommutes with $\gamma^{(ik)}$ and $\gamma^{(jk)}$ when $i \neq j \neq k$. Let us define

$$\beta_{jk} = -(i)^{j+k}\theta_{a,a+1} \cdots \theta_{b-1,b}\gamma^{(jk)}, \quad j < k,$$
$$\beta_{kj} = -\beta_{jk},$$

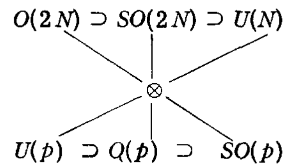
where the subscripts $a, a+1, \ldots, b-1, b$ constitute the same sequence of integers as in (7). Then it is readily verified that

$$[\beta_{ij}, \beta_{kl}] = \delta_{ik}\beta_{jl} + \delta_{jl}\beta_{ik} - \delta_{jk}\beta_{il} - \delta_{il}\beta_{jk},$$

the commutation relations characteristic of $SO(p)$. It follows that an irreducible representation of the β_{ij}, and of the $\gamma^{(ij)}$ also, defines an irreducible representation of $SO(p)$. Since the eigenvalues of $i\beta_{jk}$ are integral, tensor but not spinor representations of $SO(p)$ occur. Indeed, a close examination reveals that $\theta_{ij} = \theta_i\theta_j$, so that

$$\beta_{ij} = b_{ji} - b_{ij},$$

and the $SO(p)$ algebra is a subalgebra of the $U(p)$ algebra already discussed. Now $U(p)$ contains $O(p)$ as well as $SO(p)$. In the present situation, $O(p)$ may be regarded as obtained from $SO(p)$ by the addition of the "reflections" θ_i. Since the β_{ij} are $O(2N)$ invariants, and the θ_i are $SO(2N)$ invariants, it follows that H_p carries representations of $SO(p) \otimes O(2N)$ and $O(p) \otimes SO(2N)$. Recalling that $U(N) \subset SO(2N) \subset O(2N)$, we may characterize the structure of the generalized parastatistics algebra by the diagram



If an irreducible representation of $SO(p)$ is labeled by its highest weight $(\kappa_1, \kappa_2, \ldots, \kappa_w)$, the quadratic invariant of $SO(p)$ is

$$\sigma_2(p) = \sum_{i,j} \beta_{ij}\beta_{ji}$$
$$= 2\sum_{i>j} \gamma^{(ij)2}$$
$$= 2\sum_{ij} \kappa_j(\kappa_j + p - 2j). \tag{12}$$

But

$$\sigma_2(2N) = -2\sum_{j>i} \gamma^{(ij)2} + N(N-1)p + \tfrac{1}{2}Np^2$$
$$= 2N(\tfrac{1}{2}p - w)(\tfrac{1}{2}p - w + N - 1)$$
$$+ 2\sum_j (N - \kappa_j)(\kappa_j + N + p - 2j), \tag{13}$$

and by comparison with (4) above, we see that for p even

$$\kappa_j = N - \mu_{w+1-j} \quad (j = 1, 2, \ldots, w-1),$$
$$|\kappa_w| = N - \mu_1 \tag{14}$$

and for p odd

$$\kappa_j = N - \mu_{w+1-j} \quad (j = 1, 2, \ldots, w). \tag{14'}$$

[Note that if p is even and $\kappa_w \neq 0$, then (14) implies $\mu_1 < N$, which in turn means $\lambda_N = 0$. Thus $\kappa_w\lambda_N = 0$ for p even. Moreover, (14) implies also that λ_N and κ_w cannot both be zero in that case.]

From these relationships between the invariants of $SO(p)$ and $SO(2N)$, we can deduce the $SO(p) \otimes SO(2N)$ structure of H_p. Each irreducible representation of $SO(p) \otimes SO(2N)$ in H_p is labeled $\{(\kappa_1, \kappa_2, \ldots, \kappa_w), (\lambda_1, \lambda_2, \ldots, \lambda_N)\}$, where the κ_j are integers related to the μ_j as in (14) or (14′), the μ_j being defined in terms of the $|\lambda_r|$ as before. Every such representation with

$$N \geqslant \kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant |\kappa_w| \geqslant 0 \tag{15}$$

for p even, but

$$N \geqslant \kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_w \geqslant 0 \tag{15'}$$

for p odd, occurs just once in H_p. This is another way of describing the structure of that space.

A description of the $SO(p) \otimes O(2N)$ or $O(p) \otimes SO(2N)$ structure of the space is rather involved, especially for

p even. For p odd, it follows from what we have said that if the representation $\{(\kappa_1, \kappa_2, \ldots, \kappa_w), (\lambda_1, \lambda_2, \ldots, \lambda_N)\}$ occurs, so does the representation $\{(\kappa_1, \kappa_2, \ldots, \kappa_w), (\lambda_1, \lambda_2, \ldots, -\lambda_N)\}$. Each such pair forms an irreducible representation of $SO(p) \otimes O(2N)$, which we may label $\{(\kappa_1, \kappa_2, \ldots, \kappa_w), [\mu_1, \mu_2, \ldots, \mu_w]\}$. Then every such representation of $SO(p) \otimes O(2N)$ occurs just once in H_p, provided the conditions (14′) and (15′) are satisfied. On the other hand, each member of this pair forms an irreducible representation of $O(p) \otimes SO(2N)$. On the first member, the inversion operator $\zeta = \theta_1\theta_2 \cdots \theta_p$, which extends $SO(p)$ to $O(p)$, has the value

$$\zeta = \exp[i\pi(\lambda_1 + \lambda_2 + \cdots + \lambda_N + \tfrac{1}{2}Np)], \tag{16}$$

as may be seen by evaluating it on the state of highest weight with respect to $SO(2N)$. On the second member of the pair, the value of ζ is

$$\exp[i\pi(\lambda_1 + \lambda_2 + \cdots + \lambda_{N-1} - \lambda_N + \tfrac{1}{2}Np)],$$

which is opposite in sign to (16), since $2\lambda_N$ is an odd integer when p is odd. Irreducible representations of $O(p) \otimes SO(2N)$ may therefore be labeled $\{(\kappa_1, \kappa_2, \ldots, \kappa_w; \zeta), (\lambda_1, \lambda_2, \ldots, \lambda_N)\}$, and every such representation occurs just once in H_p, provided the conditions (14′), (15′), and (16) are satisfied.

Now we come to the question of the multiplicity with which isomorphic representations of $SO(2N+1)$ occur in H_p. Let us define

$$C^{(ij)} = (\tfrac{1}{2} - \gamma^{(ij)})\theta_{ij}. \tag{17}$$

From (14) and (10) we see that

$$\alpha_2(2N+1) = -2\sum_{j>i} (C^{(ij)2} - \tfrac{1}{4}) + Np(N + \tfrac{1}{2}p). \tag{18}$$

By considering the application of this result to parastatistics of order 2, it is evident that $C^{(12)2}$ is an $SO(2N+1)$ invariant, and the same must be true of $C^{(ij)2}$. We can therefore resolve a_ρ into two parts:

$$a_\rho = (2\gamma^{(ij)} - 1)([\gamma^{(ij)}, a_\rho] + \{\gamma^{(ij)} - \tfrac{1}{2}, a_\rho\})$$

the first of which anticommutes, and the second of which commutes, with both $\gamma^{(ij)} - \frac{1}{2}$ and θ_{ij}. Hence $C^{(ij)}$ commutes with a_ρ, and is itself an invariant of $SO(2N+1)$. From (8) it follows that

$$\{C^{(ij)}, C^{(jk)}\} = C^{(ik)} \quad (i \neq j \neq k),$$
$$[C^{(ij)}, C^{(kl)}] = 0 \quad (i \neq j \neq k \neq l).$$

Although these relations are the same as those satisfied by the $\gamma^{(ij)}$, the fact that the eigenvalues of the $C^{(ij)}$ are half-integral and not integral ensures very different properties. If we denote by $C(p)$ the algebra of the $C^{(ij)}$, then in view of (11) and (17), $C(p)$ is a subalgebra of the enveloping algebra of $SO(p)$, which relationship we denote by $SO(p) \to C(p)$. In general an irreducible representation of $SO(p)$ defines a reducible representation of $C(p)$. We shall show that irreducible representations of $C(p)$ in H_p may be labeled $(K_1, K_2, \ldots, K_w)$, where the K_j are related to the M_j already defined. Moreover, each irreducible representation of $C(p)$ provides a (usually reducible) representation of the symmetric group $S(p)$, i.e., $C(p) \to S(p)$. However, we have not been able, for general values of p, to set up a one-to-one correspondence between irreducible representations of $C(p)$ and irreducible representations of some Lie algebra, in the way that irreducible representations of the $c^{(ij)}$ can be associated with those of $U(p)$, and irreducible representations of the $\gamma^{(ij)}$ with those of $SO(p)$. [For $p = 3$, we shall see in Sec. 4 that it is possible to set up such a correspondence between $C(3)$ and $SU(2)$.]

Since the eigenvalues of $\gamma^{(ij)}$ are $0, \pm 1, \pm 2, \ldots$, the eigenvalues of $C^{(ij)}$ form a series $\frac{1}{2}, -\frac{3}{2}, \frac{5}{2}, -\frac{7}{2}, \ldots$, with a maximum value $(-1)^k(k + \frac{1}{2})$ in a particular irreducible representation. We shall suppose that, in an irreducible representation, the maximum eigenvalue of $|C^{(12)}|$ is $K_1 + \frac{1}{2}$ and that the maximum eigenvalue of $|C^{(2j-1)2j}|$ is $K_j + \frac{1}{2}$, when the $|C^{(2i-1\,2i)}|$ $(i < j)$ already have their maxima. Then the K_j $(j = 1, \ldots, w)$ may be used to label the representation. Let

$$P_{ij} = \cos[\tfrac{1}{2}\pi(C^{(ij)} - \tfrac{1}{2})].$$

Then it follows from the identities analogous to (9) satisfied by the $C^{(ij)}$ that P_{ij} commutes with $C^{(ik)} + C^{(jk)}$ but anticommutes with $C^{(ik)} - C^{(jk)}$, and hence

$$P_{ij}C^{(ik)} = C^{(jk)}P_{ij}, \quad i \neq j \neq k.$$

Since also $P_{ij}^2 = 1$, the P_{ij} provide a representation of the symmetric group $S(p)$.

To establish the relation between the K_j and the M_j, we note that the vector $|n\rangle$ corresponding to the maximum eigenvalue $(-1)^n(n + \frac{1}{2})$ of $C^{(ij)}$ satisfies

$$(C^{(ik)} - (-1)^n C^{(jk)})\,|n\rangle = 0 \quad (i \neq j \neq k),$$
$$(C^{(ik)2} + C^{(jk)2})\,|n\rangle = (n + \tfrac{1}{2})\,|n\rangle,$$

and hence compute the unique eigenvalue of the invariant

$$\sum_{j>i} C^{(ij)2} = \sum_{j=1}^{w} (K_j + \tfrac{1}{2})(K_j + p - 2j + \tfrac{1}{2})$$

within the irreducible representation considered. From this result and (18) we obtain the value

$$\alpha_2(2N+1) = Np(N + \tfrac{1}{2}p) - 2\sum_j K_j(K_j + p - 2j + 1) \tag{19}$$

for the quadratic invariant of $SO(2N+1)$. By comparison with (3) we see that

$$K_j = N - M_{w+1-j} \quad (j = 1, 2, \ldots, w). \tag{20}$$

Since the $C^{(ij)}$ are $SO(2N+1)$ invariants, H_p carries a representation of $C(p) \otimes SO(2N+1)$. The result (20) shows that *each* irreducible representation of $C(p) \otimes SO(2N+1)$ in H_p is completely characterized by the K_j. It is easily seen that H_p contains, just once, each such representation of $C(p) \otimes SO(2N+1)$ with

$$N \geqslant K_1 \geqslant K_2 \geqslant \cdots \geqslant K_w \geqslant 0.$$

We may say that the representation $[M_1, M_2, \ldots, M_w]$ of $SO(2N+1)$ occurs in H_p with a multiplicity equal to the dimension of the representation $(K_1, K_2, \ldots, K_w)$ of $C(p)$, where $K_j = N - M_{w+1-j}$. This is a third way of describing the structure of H_p.

Since $SO(p) \to C(p) \to S(p)$ and $SO(2N+1) \supset O(2N)$, we may enlarge the characteristic diagram to

$$\begin{array}{c} SO(2N+1) \supset O(2N) \supset SO(2N) \supset U(N) \\ \otimes \\ U(p) \supset O(p) \supset SO(p) \to C(p) \to S(p) \end{array}$$

Because no representation of $U(p) \otimes U(N)$, $O(p) \otimes$

$SO(2N)$, $SO(p)\otimes O(2N)$, or $C(p)\otimes SO(2N+1)$ is contained more than once in H_p, it is possible to make certain deductions from the results described above. In particular, the representation $[m_1, m_2, \ldots, m_p]$ of $U(N)$ is contained in the representation $[M_1, M_2, \ldots, M_w]$ of $SO(2N+1)$ the same number of times as the representation $(N-M_w, N-M_{w-1}, \ldots, N-M_1)$ of $C(p)$ is contained in the representation $(m_1, m_2, \ldots, m_p)$ of $U(p)$. Similarly, if p is odd, the representation $[m_1, m_2, \ldots, m_p]$ of $U(N)$ is contained in the representation $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ of $SO(2N)$, the same number of times as the representation $(\kappa_1, \kappa_2, \ldots, \kappa_w; \zeta)$ of $O(p)$ is contained in the representation $(m_1, m_2, \ldots, m_p)$ of $U(p)$, where the κ_j and ζ are related to the λ_r by (14′) and (16). But it is easily seen that $\zeta = \exp[i\pi(m_1 + m_2 + \cdots + m_p)]$ throughout the representation $(m_1, m_2, \ldots, m_p)$ of $U(p)$. Hence the representation $[m_1, m_2, \ldots, m_p]$ of $U(N)$ cannot occur in the representation $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ of $SO(2N)$ unless

$$\exp[i\pi(m_1 + m_2 + \cdots + m_p)] = \exp[i\pi(\lambda_1 + \lambda_2 + \cdots + \lambda_N + \tfrac{1}{2}Np)].$$

We turn next to the problem of finding a suitable complete set of labeling operators in the space H_p. Such a set should contain the M_j (or K_j), since these characterize irreducible representations in H_p of the ordinary parafermion $[SO(2N+1)]$ algebra. It should also contain the m_i, since these label irreducible representations of $U(N)$, each of which corresponds to a collection of state vectors with a fixed member of particles present and with a definite symmetry type.[13] So it is appropriate to try and set up a $SO(2N+1) \supset O(2N) \supset SO(2N) \supset U(N)$ $[\supset U(N-1) \supset \cdots \supset U(1)]$ basis. The set of Casimir invariants of $SO(2N+1) \supset O(2N) \supset SO(2N) \supset U(N)$ is not in general a complete set of commuting $U(N)$ invariants in H_p. We have seen that these invariants are directly related to those of $U(p) \supset O(p) \supset SO(p) \to C(p)$, in associated representations. Then the problem of completing the former set by the addition of further suitable $U(N)$ invariants is precisely that of completing, in the associated representations of $U(p)$, the set of labeling operators provided by the Casimir invariants of $U(p) \supset O(p) \supset SO(p) \to C(p)$. It is well known[18] that it is extremely difficult to find an operator suitably to complete, in general representations of $U(3)$, the set of labeling operators provided by the invariants of $U(3) \supset SO(3)$, so the problems facing us here for $p \geq 3$ are formidable indeed. One complete set of commuting operators in a representation of $U(p)$ is provided by the invariants of the chain $U(p) \supset U(p-1) \supset \cdots \supset U(1)$. In the present situation, the corresponding subalgebras of the b_{ij} algebra are obtained by restricting the ranges of the subscripts i and j to the values 1 to $p-1$, then 1 to $p-2$, and so on. The corresponding orthonormal basis is quite unsuitable for our purposes, as it is a basis in which the K_j and κ_j are not diagonal. However, there must exist at least one orthonormal basis in which they *are* diagonal, and this basis must be related to the former one by a unitary transformation T. If we define $e_{ij} = T b_{ij} T^*$, then the e_{ij} will generate an equivalent set of $U(p)$ representations, and the invariants of the chain $U(p) \supset U(p-1) \supset \cdots \supset U(1)$ defined now in terms of the e_{ij} rather than the b_{ij}, will again be a complete set of commuting operators. In the corresponding orthonormal basis, the K_j and κ_j will be diagonal. If we can find the operator T, or at least the e_{ij}, and can identify the functional dependence of the K_j (in particular) on the invariants of $U(p) \supset U(p-1) \supset \cdots \supset U(1)$ (defined in terms of the e_{ij}), then we shall have a satisfactory solution to the labeling problem.

For $p = 2$, the problem may be solved more directly. We have in that case

$$b_{11} = c^{(11)}, \quad b_{12} = -i\theta_1\theta_2 c^{(12)}, \qquad b_{21} = i\theta_1\theta_2 c^{(21)}, \quad b_{22} = c^{(22)}, \tag{21}$$

and while the operators $\sum_i b_{ii}$, $\sum_{i,j} b_{ij}b_{ji}$, and b_{11} do not comprise a suitable complete commuting set, the operators $\sum_i b_{ii}$, $\sum_{i,j} b_{ij}b_{ji}$, and $\rho = i(b_{12} - b_{21})$ do. For $\rho = \theta_{12}\gamma^{(12)} = \gamma\cos\pi\gamma$, whence $\gamma = \rho\cos\pi\rho$, and the $SO(2)$ and $C(2)$ labels κ_1 and K_1 are related to γ by (12), (17), (18), and (19), which yield

$$(\kappa_1)^2 = \gamma^2, \quad K_1(K_1 + 1) = \gamma(\gamma - 1).$$

However, one can also give explicit expressions for the e_{ij} in this case, following Govorkov.[9] He found

$$\begin{aligned} e_{11} &= \tfrac{1}{2}(c^{(11)} + c^{(12)} + c^{(21)} + c^{(22)}), \\ e_{12} &= \tfrac{1}{2}(-c^{(11)} + c^{(12)} - c^{(21)} + c^{(22)}), \\ e_{21} &= \tfrac{1}{2}(-c^{(11)} - c^{(12)} + c^{(21)} + c^{(22)}), \\ e_{22} &= \tfrac{1}{2}(c^{(11)} - c^{(12)} - c^{(21)} + c^{(22)}), \end{aligned} \tag{22}$$

and the operators $\sum_i e_{ii}$, $\sum_{i,j} e_{ij}e_{ji}$, and e_{11} are a suitable complete commuting set, since $e_{11} - e_{22} = \gamma$.

In this paper, we are concerned mainly with the case $p = 3$. Rather than attempt to write down simple closed expressions for *all* the operators e_{ij} in that case, we shall define some of them explicitly, and the rest in a rather implicit, but nevertheless complete way. Essentially, our method involves the identification of all the states in a suitable $U(3) \supset U(2) \supset U(1)$ basis, which then defines a suitable set of $U(3)$ operators e_{ij}. We shall show how all states in H_p can be built up by applying creation operators to certain "vacuumlike" states, and subsequently how each state so constructed can be allotted $U(3) \supset U(2) \supset U(1)$ quantum numbers, depending on its mode of construction. So we have an "operational" definition of the required complete set of labeling operators. In the next section, we discuss the structure and multiplicity of "vacuumlike" states in H_p and the way in which other states in H_p can be constructed from them. These observations form the basis of our treatment of the case $p = 3$, given in the following section, and should be useful if a complete solution of the labeling problem for larger values of p ever becomes desirable.

3. PARTICLE AND ANTIPARTICLE STATES

We shall call an eigenvector in H_p of each of the $U(N)$ generators $a^r{}_r$ a state vector, or state, provided it belongs to an irreducible representation of $U(N)$. It will be called a basic state vector, or basic state, provided it also belongs to an irreducible representation of $SO(2N+1)$.

In a theory in which both particles and antiparticles are present, we assume that $N = 2W$ is even, and that, for $r \leq W$, a^r creates a particle, but for $r > W$, a_r creates an antiparticle. The vacuum state vector then belongs to the representation of $U(N)$ labeled (p^W, O^W) and is defined by

$$\begin{aligned} a_r|\rangle &= 0, \quad r \leq W, \\ a^r|\rangle &= 0, \quad r > W, \end{aligned} \tag{23}$$

together with the conditions

$$(c^{(ij)} - W\delta_{ij})\,|\rangle = 0.$$

This last condition ensures that the vacuum state is also a basic state, belonging to the representation of $SO(2N+1)$ labeled $([\frac{1}{2}p]^N)$. Other vectors are formed by applying a sequence of creation operators and invariants of $U(N)$ to the vacuum state vector; the resulting tensors, e.g., $a^q a^r c^{(12)} a_s c^{(23)} a^u |\ \rangle$ (where $s > W$), may be resolved into components belonging to irreducible representations of $U(N)$ which are, by definition, state vectors. We wish to describe in detail how this decomposition is to be effected.

The conditions (23) alone are insufficient to define the vacuum (unless $p = 1$), and there may be many basic states, which we call reservoir states, corresponding to vectors $|K'\rangle$ satisfying

$$a_r |K'\rangle = 0, \quad r \leqslant W,$$
$$a^r |K'\rangle = 0, \quad r > W. \tag{24}$$

Within a given representation of $SO(2N+1)$, such states correspond to weights which are in the same equivalence class as the highest weight, i.e., their weights are obtained from the highest weight by certain permutations and changes of sign of its elements. Since all such weights are simple,[21] it follows that any representation of $U(N)$ containing a reservoir state can occur at most once within a given representation of $SO(2N+1)$. Moreover, it can be seen, again from the weights, that all such representations of $U(N)$ are contained in the same representation of $SO(2N)$, labeled $(L_1, L_2, \ldots, L_{N-1}, L_N)$ or $(L_1, L_2, \ldots, L_{N-1}, -L_N)$, according as W is even or odd. Supposing that $|K'\rangle$ belongs to an irreducible representation of $U(N)$ labeled $(l_1', l_2', \ldots, l_N')$, or $[k_1', k_2', \ldots, k_p']$, within the representation $(L_1, L_2, \ldots, L_N)$ of $SO(2N+1)$, we shall next determine the limitations on the values of the l_r' and k_i'.

By applying a suitable product of particle creation operators a^r to the reservoir state $|K'\rangle$, we can attain a vector of highest weight, in the same representation of $SO(2N+1)$, belonging to the representation of $U(N)$ labeled $(L_1 + \frac{1}{2}p, L_2 + \frac{1}{2}p, \ldots, L_N + \frac{1}{2}p)$. Since only W of the l_r' are changed in this process, at least W of the l_r' must have values not less than $\frac{1}{2}p$. Again, by applying a suitable product of antiparticle creation operators a_r to such a vector $|K'\rangle$ we can attain a vector of lowest weight in the same representation of $SO(2N+1)$, belonging to the representation of $U(N)$ labeled $(\frac{1}{2}p - L_N, \frac{1}{2}p - L_{N-1}, \ldots, \frac{1}{2}p - L_1)$. Since only W of the l_r' are changed in this process, at least W of the l_r' must have values not greater than $\frac{1}{2}p$. Hence, $p \geqslant l_1' \geqslant \cdots \geqslant l_W' \geqslant \frac{1}{2}p \geqslant l_{W+1}' \geqslant \ldots \geqslant l_N' \geqslant 0$, and $N \geqslant k_1' \geqslant \ldots \geqslant k_w' \geqslant W \geqslant k_{p-w+1}' \geqslant \ldots \geqslant k_p' \geqslant 0$, where, as before, $w = \frac{1}{2}p$ if p is even, but $w = \frac{1}{2}(p-1)$ if p is odd and in that event $k_{w+1}' = W$.

In general, the number of antiparticles in a reservoir state is different from zero. However, corresponding to any reservoir state $|K'\rangle$ there exists another reservoir state $|K\rangle$, within the same representation of $SO(2N)$, in which the antiparticles have been replaced by particles. Explicitly,

$$|K\rangle = \prod_{r'} (\alpha^{r\,r-W})\, |K'\rangle$$

where the product $\Pi_{r'}$ is over all values of r (greater than W) associated with antiparticles in $|K'\rangle$. Suppose that $|K\rangle$ belongs to the representation of $U(N)$ labeled $[W + k_1, W + k_2, \ldots, W + k_p]$, where $k_i \leqslant 0$ when $i > w$. If

$$S = \prod_{r=W+1}^{N} (a_r)^p$$

is the operator which creates antiparticles to saturate every level, then $S|K\rangle$ is a basic state, in the same representation of $SO(2N+1)$ as the reservoir states $|K\rangle$ and $|K'\rangle$, and is labeled $[k_1, k_2, \ldots, k_p]$. Since k_i must here be nonnegative,

$$k_i = 0, \quad i = w+1, \ldots, p.$$

By multiplying $S|K\rangle$ with a contravariant tensor operator formed entirely from the a^r (i.e., without factors a_r or $c^{(ij)}$) with a symmetry corresponding to the irreducible representation $[j_1, j_2, \ldots, j_p]$ of $U(N)$, we obtain a vector $|J, K\rangle$ of the direct product of the irreducible representations $[j_1, j_2, \ldots, j_p]$ and $[k_1, k_2, \ldots, k_p]$. This vector $|J, K\rangle$ can be resolved into basic vectors $|M(J,K)\rangle$ belonging to irreducible representations $[m_1, m_2, \ldots, m_p]$ of $U(N)$, where[17]

$$\max(j_1 + k_p, k_1 + j_p) \leqslant m_1 \leqslant j_1 + k_1,$$
$$\max(j_1 + j_2 + k_p + k_{p-1}, k_1 + k_2 + j_p + j_{p-1}) \leqslant m_1 + m_2 \leqslant j_1 + j_2 + k_1 + k_2,$$
$$\cdots$$
$$m_1 + m_2 + \cdots + m_p = j_1 + k_1 + j_2 + k_2 + \cdots + j_p + k_p. \tag{25}$$

Each of the vectors $|M(J,K)\rangle$ belongs to the same irreducible representation of $SO(2N+1)$ as $|K\rangle$.

The above discussion suggests that isomorphic representations of $U(N)$ in H_p may be distinguished by the associated sets of values of the j_i and the k_i, and the eigenvalues of the commuting invariants of $SO(2N+1)$ (constructed from the $C^{(ij)}$) which resolve the multiplicity of representations with the same k_i, among the reservoir states. In the next section, we shall confirm that this is so for $p = 3$. However, it should be pointed out that we have no guarantee that every set of values of the j_i and k_i consistent with a given set of values of the m_i will correspond to a different isomorphic representation of $U(N)$ in general. In fact, it is easy to confirm that the only completely symmetric tensor involving p operators $a^r, a^s, \ldots, a^x$ is $\{a^r, \{a^s, \{\ldots, a^x\}\ldots\}\}$, and even in expressions where other creation and annihilation operators are present, such a symmetrized product can be separated in this form. *Hence it may be assumed that* $j_p = m_p$.

4. PARASTATISTICS OF ORDER 3

According to what has been said in Sec. 2, the representation $[m_1, m_2, m_3]$ of $U(N)$ occurs in H_3 a number of times equal to the dimension of the representation (m_1, m_2, m_3) of $U(3)$, and there act in H_3 a corresponding set of $U(3)$ generators e_{ij} $(i, j = 1, 2, 3)$ such that the Casimir invariants of the chain $U(3) \supset U(2) \supset U(1)$ form a complete set of commuting invariants of $U(N)$. Moreover, the elements of this set may be assumed to commute also with the operators K_1 (= K below) and κ_1 (= κ below), which label representations of $C(3) \otimes SO(2N+1)$ and $SO(3) \otimes O(2N)$ in H_3, so that K and κ are functions of the elements of that set.

Rather than the notation e_{ij} for all the $U(3)$ generators, we shall use the more familiar $I_3 = \frac{1}{2}(e_{11} - e_{22})$, $I_+ = I_1 + iI_2 = e_{12}$, $I_- = I_1 - iI_2 = e_{21}$, and $Y = (e_{11} + e_{22} - 2e_{33})/3$, together with e_{13}, e_{31}, e_{23}, and e_{32}. We seek to identify or characterize these operators, and to express K (in particular) as a function of the Casimir invariants of the chain $U(3) \supset U(2) \supset U(1)$, i.e., of m_1, m_2, m_3, Y, I, and I_3 (where $I^2 = I(I+1)$, $I \geqslant 0$).

By considering the reduction of the representation $([\frac{1}{2}]^N) \otimes ([\frac{1}{2}]^N) \otimes ([\frac{1}{2}]^N)$, we find that the irreducible representations of $SO(2N+1)$ contained in H_3 are labeled $([\frac{3}{2}]^{N-K}, [\frac{1}{2}]^K)$, or alternatively $[N-K]$, where $K = 0, 1, \ldots, N$, and there are $(K+1)$ isomorphs corresponding to a particular value of K. These may be distinguished by the eigenvalues $\frac{1}{2}, -\frac{3}{2}, \ldots (-1)^K(K+\frac{1}{2})$ of $C^{(12)}$. The corresponding representation of $C(3)$ is thus $(K+1)$-dimensional, and is labeled (K). Each such irreducible representation of $SO(2N+1)$ decomposes with respect to $SO(2N)$ as $([\frac{3}{2}]^{N-K-1}, [\frac{1}{2}]^K, \pm\frac{1}{2}) \oplus ([\frac{3}{2}]^{N-K}, [\frac{1}{2}]^{K-1}, \pm\frac{1}{2})$, with the exception of $([\frac{3}{2}]^N)$, which gives $([\frac{3}{2}]^{N-1}, \pm\frac{1}{2}) \oplus ([\frac{3}{2}]^{N-1}, \pm\frac{3}{2})$, and $([\frac{1}{2}]^N)$, which gives $([\frac{1}{2}]^{N-1}, \pm\frac{1}{2})$ only. Thus in H_3 there are $(2\kappa+1)$ irreducible representations of $SO(2N)$ labeled $([\frac{3}{2}]^{N-\kappa}, [\frac{1}{2}]^\kappa)$, where $\kappa = 0, 1, \ldots, N$ and a similar number with the sign of the last weight reversed; alternatively there are $(2\kappa+1)$ irreducible representations of $O(2N)$ labeled $[N-\kappa]$. The isomorphs may be distinguished by the eigenvalues $0, \pm 1, \ldots, \pm\kappa$ of $\gamma^{(12)}$, and the corresponding representation of $SO(3)$ is labeled (κ), and is $(2\kappa+1)$-dimensional. Clearly, when κ has a value κ_0, then $K = \kappa_0$ or $\kappa_0 - 1$, except that $K=0$ if $\kappa = 0$. In other words, the $(2\kappa_0+1)$-dimensional representation $(\kappa = K_0)$ of $SO(3)$ reduces into irreducible representations of $C(3)$ of dimension κ_0, labeled $(K = \kappa_0 - 1)$, and κ_0+1, labeled $(K = \kappa_0)$; except that the representation (0) of $SO(3)$ yields only the representation (0) of $C(3)$. Now it is easily seen that if a representation of $U(3)$ contains representations of $SO(3)$ labeled $(\kappa_1), (\kappa_2), \ldots$, of dimension $(2\kappa_1+1), (2\kappa_2+1), \ldots$, then it contains representations of $SU(2)$ corresponding to $I = \frac{1}{2}\kappa_1, \frac{1}{2}(\kappa_1 - 1); \frac{1}{2}\kappa_2, \frac{1}{2}(\kappa_2-1); \cdots$, of dimension $\kappa_1+1, \kappa_1; \kappa_2+1, \kappa_2; \cdots$. It follows that the irreducible representation (K) of $C(3)$ appears in any given representation of $U(3)$ the same number of times as the irreducible representation of $SU(2)$ with $I = \frac{1}{2}K$, which has the same dimension $(K+1)$. We may therefore choose the e_{ij} in such a way that I is *equal* to $\frac{1}{2}K$, and the enveloping algebras of $C(3)$ and $SU(2)$ (with generators I_1, I_2, and I_3) are the same. Then we see from (18) and (19) that

$$4\mathbf{I}^2 = K(K+2) = C^{(12)2} + C^{(23)2} + C^{(31)2} - \tfrac{3}{4}. \qquad (26)$$

It also follows from this identification of $C(3)$ and $SU(2)$ representations that the isospin generators I_1, I_2, and I_3 are $SO(2N+1)$ invariants, commuting with all a^r and a_r. Thus these parafermion operators are associated with isoscalar particles. Note that the functional dependence of K on m_1, m_2, m_3, Y, I, and I_3 has now been fixed as simply $K = 2I$. The most natural characterization of the isospin in this formalism is by the $C(3)$ operators $C_1 = C^{(23)}$, $C_2 = C^{(31)}$, and $C_3 = C^{(12)}$, which satisfy

$$\{C_1, C_2\} = C_3 \qquad (27)$$

etc., rather than by the components of $\mathbf{I}$, which are generators of $SU(2)$. However, we shall show later in this section how the two sets of operators are related.

Next we come to the identification of the hypercharge operator Y. It is convenient to introduce the $O(2N)$ invariant γ defined in (10), which reduces here to

$$\begin{aligned}\gamma &= \tfrac{1}{2}a^\rho a_\rho - 3N/2\\ &= \gamma^{(12)} + \gamma^{(13)} + \gamma^{(23)}\end{aligned} \qquad (28)$$

and which may be seen from (13), (18), and (19) to have the eigenvalues $(K+2)$ or $-K$ in the representation $[N-K]$ of $SO(2N+1)$, and $(\kappa+1)$ or $-\kappa$ in the representation $[N-\kappa]$ of $O(2N)$. Hence $K = \kappa - 1$ if the eigenvalue of γ is positive and otherwise $K = \kappa$. Consider now the reservoir states, which, according to the analysis of the preceding section, belong to $U(N)$ representations of the form $[W+k_1, W, W+k_3]$. Here k_1 is the number of particles, and $-k_3$ the number of antiparticles in the reservoir. From (24) and (28) we find that the corresponding eigenvalues of γ are $-(k_1 - k_3)$. Thus $I = \frac{1}{2}(k_1 - k_3)$ on such reservoir states. A general M representation of $U(N)$, labeled $[m_1, m_2, m_3]$, may be regarded as belonging to the decomposition of the direct product of a J representation $[j_1, j_2, j_3]$, associated with an appropriately symmetrized product of operators a^r, and a K representation $[k, 0, 0]$, to a state of which that product is applied. Such basic states in $[k, 0, 0]$ are obtained by adding all possible antiparticles to reservoir states $|K\rangle$, which contain k particles only, and are therefore labeled $[W+k, W, W]$. (The "preceding" vectors of Govorkov[9] belong to representations labeled $[k, 0, 0]$.)

Bearing in mind that I commutes with a_ρ, one sees that if a particular M representation is associated with a certain value of I, then the corresponding K representation has $k = 2I$. Moreover, in view of the inequalities (25), the maximum isospin in a set of isomorphic M representations will be $\frac{1}{2}(m_1 - m_3)$. This can be resolved into two Casimir operators $\frac{1}{2}(m_1 - m_2)$ and $\frac{1}{2}(m_2 - m_3)$, corresponding to the isospins in the submultiplets of greatest and least hypercharge, respectively. The hypercharge Y itself should vary between a minimum value $(m_2 + m_3 - 2m_1)/3$ attained when $k = m_2 - m_3$, $j_2 = j_3$ and $j_1 = m_1$, and a maximum value $(m_1 + m_2 - 2m_3)/3$ attained when $k = m_1 - m_2, j_1 = j_2$, and $j_3 = m_3$. Thus

$$Y = \tfrac{1}{3}(m_1 + m_2 + m_3) + j_2 - j_1 - j_3.$$

As we pointed out at the end of the last section, multiplets corresponding to different values of the j_i are not always independent, and for $p = 3$ the ambiguity is most simply removed by imposing the condition

$$j_3 = m_3 \qquad (29)$$

so that the formula for the hypercharge and isospin may be written

$$\begin{aligned}Y &= \tfrac{1}{3}(m_1 + m_2 - 2m_3) + j_2 - j_1,\\ I &= \tfrac{1}{2}k = \tfrac{1}{2}(m_1 + m_2) - \tfrac{1}{2}(j_1 + j_2).\end{aligned} \qquad (30)$$

Within a given representation $[m_1, m_2, m_3]$ of $U(N)$, it is evident that the values of the j_i are completely determined by Y and I, when the condition (29) is adopted; moreover, the values of Y and I allowed are just those which occur in the $U(3)$ multiplet (m_1, m_2, m_3). Thus, we have verified that the condition (29) does not exclude any states contained within the representations of the generalized parastatistics algebras.

The operators m_1, m_2, m_3, Y, I, and I_3 (which we have not yet defined) form a complete set of commuting $U(N)$ invariants in H_3, and by fixing their eigenvalues on a set of basic states we implicitly define the e_{ij} completely, since all their matrix elements are then determined. Although these $U(3)$ generators do not provide the simplest characterization of the algebraic structure—we have already seen that $C(3)$ arises more naturally than $SU(2)$—we wish to show how they can be constructed if required.

The choice of the isospin $SU(2)$ generators $I_\pm$ and I_3 is not, of course, unique, but can be made so by requiring that

$$UI_+U^* = \omega I_+, \quad UI_-U^* = \omega^2 I_-, \quad UI_3U^* = I_3, \tag{31}$$

where ω is a complex cube root of 1, and U is the operator inducing a cyclic transformation of the ansatz components. Thus

$$Ua'_\rho U^* = \omega a'_\rho, \quad Ua''_\rho U^* = \omega^2 a''_\rho, \quad Ua_\rho U^* = a_\rho, \tag{32}$$

where we have defined

$$a'_\rho = \sigma_\rho^{(1)} + \omega\sigma_\rho^{(2)} + \omega^2\sigma_\rho^{(3)},$$
$$a''_\rho = \sigma_\rho^{(1)} + \omega^2\sigma_\rho^{(2)} + \omega\sigma_\rho^{(3)}. \tag{33}$$

in terms of the psuedofermion operators appearing in the ansatz (5). This U is in the symmetric group $S(3)$ discussed in Sec. 2. Indeed, if

$$U_j = \cos[\tfrac{1}{2}\pi(C_j - \tfrac{1}{2})],$$

it follows from (27) that, e.g., $U_3C_1 = C_2U_3$ and $U_3C_2 = C_1U_3$, and since U must be an $SO(2N+1)$ invariant, we have

$$U = U_1U_2 = U_2U_3 = U_3U_1.$$

We can easily construct one set of $SU(2)$ generators $H_\pm$ and H_3 by writing

$$H_3 = \tfrac{1}{2}[C_3 - \tfrac{1}{2}(-1)^K],$$
$$H_+ = f(C_3)(C_1 + C_2)(C_2 - C_1),$$
$$H_- = (C_2 - C_1)(C_1 + C_2)f(C_3).$$

Since

$$[H_3, H_\pm] = \pm H_\pm,$$

$$H_+H_- = [f(C_3)]^2[(K+1)^2 - (C_3 + \tfrac{1}{2})^2][(K+1)^2 - (C_3 - \tfrac{3}{2})^2],$$

$$H_-H_+ = [f(C_3+1)]^2[(K+1)^2 - (C_3 + \tfrac{3}{2})^2][(K+1)^2 - (C_3 - \tfrac{1}{2})^2],$$

the required commutation relations will be satisfied, provided $f(C_3)$ is defined by

$$4[f(C_3)]^2[(K+1)^2 - (C_3 + \tfrac{1}{2})^2][(K+1)^2 - (C_3 - \tfrac{3}{2})^2] = (K+1)^2 - [C_3 - \tfrac{1}{2}(-1)^K - \tfrac{1}{2}]^2.$$

It is important to note that any $SO(2N)$ vector, i.e., a linear combination of a_ρ, a'_ρ, and a''_ρ with coefficients which may be $SO(2N)$ invariants, can change the eigenvalue of H_3 by at most ± 1; the same will apply to I_3, as defined below.

The H_j and I_j are evidently connected by a unitary transformation, which we next determine. If

$$V_3 = 1 + Uu_3^* + U^*u_3,$$
$$u_3 = \exp(4\pi i H_3/3),$$

it is easy to verify that the relations (31) are satisfied, provided

$$V_3H_j = I_jV_3$$

and hence that

$$I_j = VH_jV^*,$$
$$V = \lim_{\epsilon\to 0}(V_3 + \epsilon)[(V_3^* + \epsilon)(V_3 + \epsilon)]^{-1/2}.$$

The need to take a limit here arises from the fact that V_3 has one accidentally vanishing eigenvalue, in the representation of $C(3)$ corresponding to $K = 2$. In any irreducible representation of $SO(2N+1)$, the matrix elements of U can readily constructed from the normalized eigenvectors of H_1 in a representation in which H_3 is diagonal, and those of V can be derived therefrom.

The hypercharge changing operators e_{23}, e_{32}, e_{13}, and e_{31} can be split into two parts, one of which increases the isospin I by one-half unit, and the other decreases it by one-half unit; thus

$$e_{ij} = (e_{ij})^+ + (e_{ij})^-,$$
$$(e_{ij})^+ I = (I - \tfrac{1}{2})(e_{ij})^+ \quad (i < j = 3 \text{ or } j < i = 3)$$
$$(e_{ij})^- I = (I + \tfrac{1}{2})(e_{ij})^-.$$

Instead of $(e_{ij})^+$ and $(e_{ij})^-$, we shall first construct operators $(D_{ij})^+$ and $(D_{ij})^-$ which differ from them only in normalization. We shall need to make use of the $U(N)$ invariants m_1, m_2, and m_3, and since our object is to construct all the $SU(3)$ generators at least implicitly from the creation and annihilation operators, we note that the m_j are determined by $m_1 + m_2 + m_3 = a^r{}_r$,

$$2(m_1 + 2m_2 + 3m_3) - (m_1^2 + m_2^2 + m_3^2) = a^r{}_s a^s{}_r - (N-1)a^r{}_r,$$

$$3(m_1 + 4m_2 + 9m_3) - 3(m_1^2 + 2m_2^2 + 3m_3^2) + (m_1^3 + m_2^3 + m_3^3) = a^r{}_s a^s{}_t a^t{}_r - (2N - \tfrac{3}{2})a^r{}_s a^s{}_r + \tfrac{1}{2}a^r{}_r[a^s{}_s + (N-1)(2N-1)]$$

(summation over repeated affixes implied). The operators $a^r{}_{,i} = [m_i, a^r]$ and $a_{r,i} = [a_r, m_i]$ change the eigenvalues of m_i by $+1$ and -1, respectively, leaving the other m_j unchanged. Moreover, they have no effect on I or I_3, since they commute with the $SO(2N+1)$ invariants.

The hypercharge changing operators are $U(N)$ invariants which change the value of both I_3 and I by one-half unit, and must therefore involve a'_ρ and a''_ρ. It follows from (31) and (32) that a'_ρ has components which change I_3 by $+\frac{1}{2}$ and -1, while a''_ρ has components which change I_3 by $-\frac{1}{2}$ and $+1$. To separate the components which change I_3 by $\pm\frac{1}{2}$, we make use of the identity

$$[A^2, [A^2, [A^2, a'_\rho]]] = [2A^4 - A^2, a'_\rho],$$

where we have set $A = \gamma - \frac{1}{2}$. This identity can be verified directly, or deduced from the fact that a'_ρ can have only components which change the eigenvalue A' of A to $\pm A'$, $\pm(A'+1)$ or $\pm(A'-1)$. It is evident from (19) and (28) that the components which change I by $\pm\frac{1}{2}$ are those which change A' to $-A'$ or $A' \pm 1$, and are therefore contained in the vector

$$\alpha'_\rho = \{A, [A^2, a'_\rho]\} - [A, a'_\rho].$$

Of course α''_ρ is defined similarly in terms of a''_ρ. The vectors $\alpha'^r, \alpha''^r (r \leqslant W)$ and $\alpha'_r, \alpha''_r\ (r > W)$ can be used to create particles and antiparticles respectively in the reservoir.

We may now assert that the invariants which increase the hypercharge and change the isospin (I_3, I) by fixed amounts are

$$(D_{13})^+ = [\alpha'^r, a_{r,1}], \quad (D_{23})^+ = [\alpha''^r, a_{r,1}],$$
$$(D_{13})^- = [a^r{}_{,2}, \alpha'_r], \quad (D_{23})^- = [a^r{}_{,2}, \alpha''_r].$$

For these invariants do not alter the values of m_1, m_2, and m_3, and whereas the factors $a_{\rho,i}$ change the values of j_1 and j_2, the factors α'_ρ and α''_ρ cannot. By inspection of the expression (30) for Y, it is clear that the above operators will all increase Y by one unit; their Hermitian conjugates $(D_{31})^-$, $(D_{32})^-$, $(D_{31})^+$, and $(D_{32})^+$ will similarly decrease Y by one unit. Hence we may write

$$(e_{ij})^+ = F_{ij}[(D_{ij})^+(D_{ji})^-]^{-1/2}(D_{ij})^+$$

for $i < j = 3$ and $j < i = 3$, where the F_{ij} are normalization factors known from the work of Baird and Biedenharn[19]:

$$F_{13} = F(Y, -I-1, -I_3),$$
$$F_{23} = F(Y, -I-1, I_3),$$
$$F_{31} = F(Y+1, I-\tfrac{1}{2}, -I_3+\tfrac{1}{2}),$$
$$F_{32} = F(Y+1, I-\tfrac{1}{2}, I_3-\tfrac{1}{2}),$$
$$2(I+1)(2I+1)[F(Y,I,I_3)]^2 = -(I+I_3+1) \times (m_1 - \mu + I - \tfrac{1}{2}Y + 2)(m_2 - \mu + I - \tfrac{1}{2}Y + 1) \times (m_3 - \mu + I - \tfrac{1}{2}Y),$$
$$\mu = \tfrac{1}{3}(m_1 + m_2 + m_3).$$

This completes the determination of the $U(3)$ generators.

Turning now to the physical interpretation implied by the above identifications, we note that there are similarities to Gell-Mann's well-known theory,[4] but also important differences. It is a requirement of Gell-Mann's theory that the fundamental particles are quarks, each of which has a definite isospin and hypercharge and, on account of its fractional charge, cannot be positively identified with any particle so far observed in nature. The generalized parastatistics also requires the hadrons to be composite particles, but the fundamental particles do not carry a definite isospin and hypercharge. The reservoir particles carry the isospin, but are of two kinds, one of which has $I_3 = -1$ or $+\frac{1}{2}$, and the other has $I_3 = -\frac{1}{2}$ or $+1$. The external particles have zero isospin, but, as can be seen from (30), have $Y = -\frac{2}{3}$ or $+\frac{4}{3}$. The indeterminacy of these quantities, and the charge $I_3 + \frac{1}{2}Y$, is resolved by the symmetry type of the state in which the particles appear. There are quarkish states with fractional hypercharge, as in Gell-Mann's theory, which can, however, be excluded by requiring that the particle number should be a multiple of 3. There is also a requirement that strong interactions should involve only the $U(N)$ generators $a^r{}_s$, which conserve isospin and hypercharge and, as Gray[20] has shown, are consistent with the cluster property which is indispensable in a theory of composite particles.

In Table I, we list the well-known hadrons and the corresponding quantum numbers suggested by the present interpretation.

The χ is of course formed by creating an antiparticle and filling the "hole" with a particle. Even excluding the quarkish states, there are obviously some simple assignments of quantum numbers to which no known stable particle can be found to correspond, notably the fermion singlet $m_1 = m_2 = m_3 = W + 1$, which can, however, be identified as a combination of a baryon and meson. If we denote the number of objects of this kind

TABLE I. Table of hadrons and assignments of quantum numbers.

Particle(s)	$m_1 - W$	$m_1 - m_2$	$m_2 - m_3$	$m_1 - j_1$	$m_2 - j_2$	I	Y
χ	0	0	0	0	0	0	0
$(\Lambda, \eta, \bar{\Lambda})$	(2, 1, 0)	1	1	0	0	0	0
$(N, K, \bar{\Xi})$	"	"	"	1	0	$\frac{1}{2}$	1
$(\Xi, \bar{K}, \bar{N})$	"	"	"	0	1	$\frac{1}{2}$	-1
$(\Sigma, \pi, \bar{\Sigma})$	"	"	"	1	1	1	0
Ω	3	3	0	0	0	0	-2
Ξ^*	"	"	"	1	0	$\frac{1}{2}$	-1
Σ^*	"	"	"	2	0	1	0
N^*	"	"	"	3	0	$\frac{3}{2}$	1

by N_1, the numbers of baryon, meson, and antibaryon octet states by N_8, N'_8, and $\bar{N}_8$, we have in general

$$m_1 - W = N_1 - \bar{N}_1 + N'_8 + 2N_8,$$
$$m_2 - W = N_1 - \bar{N}_1 + N_8 - \bar{N}_8,$$
$$m_3 - W = N_1 - \bar{N}_1 - N'_8 - 2\bar{N}_8.$$

All possible values of the m_i for which $m_1 + m_2 + m_3$ is a multiple of 3 can be obtained by suitable substitutions in this formula. It may be noticed that the higher admissible $SU(3)$ multiplets (e.g., the 27-et) can be constructed from octets, and even the decuplet can be constructed in this way.

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