The point form of quantum dynamics and a 4-vector coordinate operator for a spinless particle

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We construct the analog in the quantum mechanics of a free spinless particle, of Dirac’s formula for the generators of space–time translations in his point form of classical dynamics, where one takes as fundamental variables the generators of homogeneous Lorentz transformations and the coordinate 4-vector of the point where the world line of the particle meets one sheet of a two-sheeted hyperboloid in space–time. A 4-vector coordinate operator is determined for such a particle, with commuting Hermitian components. The corresponding observable is the analog of the coordinate 4-vector of the point on the hyperboloid. This operator bears the same relation to such a surface as the Newton–Wigner operator does to an instant.

1. INTRODUCTION

Dirac\(^1\) has shown that in classical mechanics there are many ways to set up a dynamical description of a free point particle with nonzero rest mass, consistent with the requirements of the special theory of relativity. An observer in any inertial frame of reference need make use only of variables specifying the condition of the particle at the point in space–time where its world line crosses an arbitrarily chosen spacelike surface. (Even a surface on which every two points are separated by either a spacelike or a null interval may be chosen.) These dynamical variables will include the energy–momentum 4-vector for the particle, \(P\); the relativistic angular–momentum tensor \(J_{\mu
u} (= -J_{\nu\mu})\); and three coordinates specifying the location of the particle on the surface.

A Poisson bracket \([A,B]\) must be introduced for every pair of dynamical variables \(A,B\) in such a way that

\[
\{P_\mu, J_{\mu
u}\} = \varepsilon_{\delta\nu} P_\delta - \varepsilon_{\delta\mu} P_\nu,
\]

\[
\{J_{\mu\nu}, J_{\rho\sigma}\} = \varepsilon_{\mu\rho} J_{\nu\sigma} + \varepsilon_{\mu\sigma} J_{\nu\rho} - \varepsilon_{\nu\rho} J_{\mu\sigma} - \varepsilon_{\nu\sigma} J_{\mu\rho},
\]

in order that the ten variables \(P\) and \(J\) will generate a group of transformations isomorphic to \(\hat{\rho}\), the inhomogeneous Lorentz group.

If the observer adopts a system of space–time coordinates \(x_\alpha\), an obvious choice for the spacelike surface is an instant, say \(x_0 = 0\). The three coordinates \(q_\alpha\) for the particle may then be taken to form a 3-vector. Following Dirac, one may choose to regard this as the 3-vector part of a 4-vector \(q_\alpha\) associated with the particle, and subject to the constraint \(q_0 = 0\); and one may further suppose the existence of a 4-vector \(p_\alpha\) conjugate to \(q_\alpha\), so that one has the Poisson bracket relations

\[
\{q_\alpha, p_\beta\} = 0 = \{p_\alpha, p_\beta\}, \quad \{q_\alpha, q_\beta\} = \varepsilon_{\alpha\beta}.
\]

However, the constraint \(q_0 = 0\) is required to be invariant under canonical transformations generated by all dynamical variables of physical importance—called physical variables by Dirac—and it follows that these can be functions only of the \(q_\alpha\) and \(p_\alpha\). In particular, Dirac argued that \(p_\alpha\) and \(J_{\mu\nu}\) are given by

\[
P_\alpha = p_\alpha, \quad J_{\mu\nu} = P_\mu q_\nu - q_\mu p_\nu,
\]

\[
P_\alpha = (p_\alpha p_\alpha + m^2 c^4)^{1/2},
\]

where \(m\) is the rest mass of the particle, and \(c\) the speed of light. It is well known that these definitions lead, in consequence of (2), to the required relations (1), as well as to the relations

\[
P_\alpha p_\alpha = m^2 c^2,
\]

\[
P_\alpha \geq mc,
\]

\[
\varepsilon_{\alpha\beta\mu\nu} J_{\mu\nu} p_\alpha = 0,
\]

which characterize a system with rest mass \(m\), positive energy, and no internal angular momentum. As Dirac pointed out, the choice of the surface \(x_0 = 0\) singles out the Euclidean subgroup of \(\hat{\rho}\), because coordinate transformations in that subgroup leave this surface invariant. Some consequences of this are the relatively simple expressions for the associated generators \(P_\mu\) and \(J_{\mu\nu}\) in (3) as compared with those for \(P_\alpha\) and \(J_{\mu\nu}\), and a complicated transformation law for \(q_\alpha\) with respect to Lorentz boosts.

Dirac called this the “instant” form of dynamics, and presented corresponding results for two other forms of classical dynamics:

(I) The “point” form, in which the selected surface is taken to be one sheet of a two-sheeted hyperboloid or cone, such as \(x_0x^4 - k^2, \quad x_0 \geq k \geq 0\).

(II) The “front” form, in which the selected surface is a plane light wave front, such as \(x_0 = x_3\).

We are concerned with the point form of relativistic quantum dynamics (in the case \(k > 0\)) for a free, spinless particle with nonzero rest mass and positive energy, in particular as it bears upon the much-discussed question of the definition of position operators for such a particle. We do not consider the special limiting case \(k = 0\), which has been discussed from a slightly different viewpoint by Peres,\(^2\) and when we speak of the point form below, we shall generally be referring to the case \(k > 0\).

In order to establish the line of our argument, we describe briefly in Sec. 2 the familiar instant form of quantum dynamics for such a particle. The operator analog of the classical coordinate 3-vector \(q_\alpha\) is easily seen to be the Newton–Wigner operator\(^4\), which therefore corresponds to the measurement of the position of
the particle on an instant. The complicated transformation properties of this operator in respect of Lorentz boosts are from this point of view not to be regarded as a defect when this operator is used to define the concept of localization of the particle (on an instant). On the contrary, they are a necessary consequence of the fact that an instant is not a Lorentz-invariant surface, and they are quite analogous to the transformation properties of the classical 3-vector \( q_r \). (See however the relevant discussion in Ref. 2.) When the papers of Dirac\(^1\) and Newton and Wigner\(^2\) are studied side by side, one's initial reaction may be that the question of localization of an elementary particle (or system) on an instant has been resolved, at least in the spinless case, in a perfectly satisfactory way with due regard for the correspondence principle. One may feel less sure when one remembers that "manifestly covariant" descriptions of particles apparently need to be used if a (field) theory of local interactions is to be developed. Associated with such descriptions one has conserved current densities, which seem to point the way to other concepts of localization. (See for example Barut and Malin\(^3\).)

In Sec. 3 we review Dirac's formula for the generators of space-time translation in the point form of classical mechanics and formulate the problem of finding the analog of this formula in quantum mechanics. We find that a mathematically equivalent problem can also be formulated—that of finding in terms of the group generators an expression for a 4-vector operator which is the analog in quantum mechanics of the coordinate 4-vector of the point where the world line of the classical particle meets the hyperboloid sheet described in (1) above.

Some of the properties of this 4-vector coordinate operator have been summarized by us elsewhere\(^4\) without proof. It has commuting, Hermitian components, and bears the same relation to the surface described in (1) as does the Newton-Wigner operator to the instant; and just as the latter transforms simply under the Euclidean group, but not under Lorentz boosts, so the former transforms simply under the homogeneous Lorentz group (as a 4-vector) but not simply under translations in space or time. No doubt this explains why it does not seem to have been mentioned in the extensive literature on the localization of elementary particles. (See for example Refs. 4, 5, 7-11 and references therein.)

In Sec. 4, we solve the problem formulated in Sec. 3, relegating some proofs to two Appendices. Our conclusions are summarized in Sec. 5.

**Notation:** In what follows, we use the same symbol to denote a classical variable and its operator counterpart in quantum mechanics, relying on context to distinguish the two. Greek indices run over 0, 1, 2, 3 and Latin over 1, 2, 3. The metric tensor is diagonal with \( g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 \), and the alternating tensor is defined with \( \varepsilon_{0123} = 1 \).

### 2. The Instant Form of Relativistic Quantum Dynamics

We proceed by analogy with the development of the classical case in Eqs. (5)–(4) now taking \( p_r \) and \( q_r \) to be operators in a suitable vector space, and replacing the Poisson bracket relations (2) by commutation relations

\[
[q_\lambda, q_\nu] = 0 = [p_\lambda, p_\nu] \quad [p_\lambda, q_\nu] = i\hbar g_{\lambda\nu}
\]

(5)

The constraints \( q_0 = 0 \) must now be interpreted as an operator equation valid in the physically relevant portion of the vector space, and it again restricts the physical variables to be functions only of the \( p_r \) and \( q_r \). In quantum mechanics one also requires that physical variables be Hermitian operators in Hilbert space. Thus \( q_r \) should be taken to be Hermitian, and the assumption that \( p_r \) also is Hermitian guarantees the hermiticity of \( P_r \) and \( J_{r\alpha} \) if we take over the classical formula (3a) to define these variables. Moreover, \( P_r \) as defined in (3b) will then also be Hermitian provided the operator square root is suitably interpreted. However, the formula (3c) for \( J_{r\alpha} \) is then not consistent with the hermiticity requirement, and in order to obtain Hermitian \( J_{r\alpha} \) without violating the correspondence principle, we are naturally led to adopt instead the symmetrized expression

\[
J_{r0} = \frac{1}{2} q_r (p_x^2 + m_0^2 c^2)^{1/2} + \frac{1}{2} p_x q_r (p_x^2 + m_0^2 c^2)^{1/2} q_r \quad \text{for } \alpha = 0
\]

(6)

It is then readily checked that with these definitions, \( P_r \) and \( J_{r\alpha} \) satisfy the required relations

\[
[P_{r\alpha}, P_{\lambda\nu}] = 0 \quad \{P_{r\alpha}, J_{\lambda\nu}\} = i\hbar\{\delta_{\alpha\lambda} P_{\nu} - \delta_{\lambda\nu} P_{\alpha} \}
\]

(7)

as well as the "representation relations" (4), now interpreted as operator equations.

As \( q_r \) and \( p_r \) are by assumption Hermitian operators satisfying the canonical commutation relations, one may take the Hilbert space to be that of square-integrable functions \( \phi(p_1, p_2, p_3) \) with scalar product

\[
(\phi, \psi) = \int \int \phi(x_1, x_2, x_3) \psi(x_1, x_2, x_3) x_3^2 \, dx_1 \, dx_2 \, dx_3
\]

(8)

and take \( q_\alpha = i\hbar \partial / \partial p_\alpha \). It is well-known (see, for example, Folland\(^5\)) that in this space the operators \( P_r \) and \( J_{r\alpha} \) defined above generate the unitary representation \( (m_0 c)^2 \) of \( \mathfrak{p} \), appropriate to the description of the Heisenberg picture of a positive-energy spinless particle with rest mass \( m_0 \). The operators \( q_r \) are seen to form the Newton-Wigner 3-vector position operator (at \( x_0 = 0 \)), which is thus revealed as the analogue in quantum mechanics of the position vector of the classical particle where its world line meets an instant.

Since the operators \( P_r \) and \( J_{r\alpha} \) do generate an irreducible representation of \( \mathfrak{p} \) one may hope that any given operator on the Hilbert space can be expressed in terms of those generators. This is true for the operator \( q_\lambda \); it can be seen from (3a), (3b), and (6) that

\[
q_\lambda = J_{r\lambda}(P_\lambda)^{1/2} + \frac{i\hbar}{2} P_\lambda (P_\lambda)^{1/2}.
\]

(9)

Moreover it is not hard to see that if one takes (9) as a definition of \( q_\lambda \), assuming the commutation relations (7) and representation relations (4), then one can deduce the relations

\[
[q_\lambda, q_\nu] = 0 \quad [q_\lambda, P_\nu] = i\hbar J_{r\nu} \quad J_{r\nu} = q_\lambda P_\nu - q_\nu P_\lambda
\]

(10)

and

\[
J_{r\alpha} = \frac{1}{2} q_r (p_x^2 + m_0^2 c^2)^{1/2} + \frac{1}{2} p_x q_r (p_x^2 + m_0^2 c^2)^{1/2} q_r \quad \text{for } \alpha \neq 0.
\]

From this point of view, it is the group generators which play the more fundamental role, the variables \( q_r \),
being derived quantities. In such a formulation one does not need to introduce the unphysical variables $p_0$ and $q_0$.

3. THE POINT FORM OF RELATIVISTIC DYNAMICS

The preceding discussion should have its counterpart for each of the other forms of classical dynamics described by Dirac— and in principle, for the form corresponding to an arbitrary choice of spacelike surface. Fleming has found the counterpart for the case of a general spacelike hyperplane, in which case the constraint $q_0 = 0$ is replaced by

$$q_0^n = \tau,$$

where $\tau$ and $\eta^a$ are real constants, with $\eta_a \eta^a = 1$. In the case of a nonplanar surface, such as that defining the point form, the counterpart is more difficult to discover.

In the point form, the three coordinates specifying the point at which the world line of the classical particle meets the surface $x^a \eta^a = k^2$, $x_0 \geq k > 0$, may be written as a 4-vector $q_a$ subject to the constraints

$$q_0^2 = k^2, \quad q_0 \geq k.$$  

(11)

Again introducing variables $p_0$ conjugate to $q_a$ as in (2), and noting these constraints, we expect that in this case physical variables can be functions only of $q_0$ and $J_{ab}$, where

$$J_{ab} = \{q_a, p_b\}.$$  

(12)

The Poisson bracket $\{q_a, p_b\}$ [and that the Poisson bracket $\{q_a, p_b\}$] has the form

$$\{q_a, p_b\} = -\delta_{ab} + q_a P_b (q_0 P_0)^{-1},$$

(13)

so that

$$\{q_a, p_b\} = -3.$$  

(14)

There are a remarkable symmetry between the roles of the timelike 4-vectors $P_a$ and $q_a$ in this form of dynamics. Apart from Eqs. (13–15), one sees that as a result of Eqs. (4a), (9), (12), and (13)

$$k^2 P_a = -\frac{1}{2} q_a q_0 - (m^2 c^2 k^2 - \frac{1}{2} \delta_{ab} p_b p_a)^{1/2},$$

(16)

$$m^2 c^2 q_a = k - (m^2 c^2 k^2 - \frac{1}{2} \delta_{ab} p_b p_a)^{1/2}.$$  

(17)

All equations in the point form of dynamics, such as (13)–(17) above, can be expressed in a manifestly covariant way using four-dimensional tensor notation. As Dirac stressed, this reflects the special role chosen for the homogeneous Lorentz subgroup of $\beta$ by the observer's choice of a special surface which is invariant under Lorentz transformations of his coordinate system. It is this feature of the point form, and the associated fact that this subgroup is simple in the mathematical sense, which makes it attractive and will be responsible for any advantages it may have over the other forms.

Turning now to the point form of quantum dynamics for a spinless particle, we proceed initially (as in the case of the instant form (Sec. 2), introduce a set of operators $q_{\alpha}, p_\alpha$ satisfying the commutation relations (5), and impose the operator constraints

$$q_0 q_0^2 = k^2, \quad q_0 \geq k > 0,$$

(18)

now to be satisfied by Hermitean operators $q_0, p_0$ on a Hilbert space $H$.

We also take $J_{ab} = \{q_a, p_b\}$ to be Hermitean operators in $H$, and identify $J_{ab} = J_{ab}$. Then we wish to find the analog in the quantum theory of the formula (16), defining Hermitian $P_{\alpha}$ in terms of $q_0$ and $J_{ab}$ in such a way that the $P_{\alpha}$ and $J_{ab}$ satisfy the relations (4) and (7).

A suitable realization of the space $H$ is that of functions $f(q_0, q_0, q_0, q_0)$ defined on the sheet $q_0 \geq k > 0$ of the hyperboloid $q_0 q_0^2 = k^2$, with scalar product

$$\langle \phi, \psi \rangle = \int d^2 \nu \langle q_0, q_0, q_0, q_0 \rangle \bar{\psi} (q_0, q_0, q_0, q_0)\, dq_0 dq_0 dq_0 dq_0,$$

(19)

where the integral is over the whole sheet, and $d^2 \nu = dq_0 dq_0 dq_0 dq_0$ is the Lorentz-invariant volume element on the sheet. Then $q_0$ and also $J_{ab}$, which is now given by

$$J_{ab} = \delta_{ab} - q_0 \frac{\partial q_0}{\partial q_0} + q_0 \frac{\partial q_0}{\partial q_0},$$

(20)

and $P_{\alpha}$, defined in such a way that the classical formula (16), and consequently (16), can be recovered in the classical limit.

Peres has solved the corresponding problem in the limiting case $k = 0$ (for particles with spin 0 or 1/2). However, his solution is not expressed in a manifestly covariant way, so that the peculiar advantage of the point form is to some extent lost in his treatment. While Kubo, Kuroda, and Jackiw, 13 Sommerfield, 14 and Gromes, Rothe, and Stech 15 have considered the initial-value problem for quantum or classical fields with the surface $x^a \eta^a = k^2, x_0 \geq k > 0$ as Cauchy surface, no one to our knowledge has tackled the specific problem posed above, although Kubo et al. make passing reference to its difficulty.

Supposing that a solution $P_{\alpha}$ exists, it is clear that $q_0 P_{\alpha} - q_0 P_{\alpha} = P_\alpha q_0 - P_\alpha q_0$ can differ from $J_{ab}$ only by terms which in some sense vanish in the classical limit. We shall see that it is possible to find a solution $P_{\alpha}$ with

$$J_{ab} = \frac{1}{2} (q_0 P_{\alpha} - q_0 P_{\alpha}) + \frac{1}{2} (P_\alpha q_0 - P_\alpha q_0),$$

(21)

but not with $J_{ab}$ equal to either of the asymmetrical forms. The solution is uniquely determined if we require in addition the analogue of (15), viz.

$$[q_0, P_\alpha] = -3 \delta_{ab},$$

(22)

In the representation space $H$ the $P_{\alpha}$ and $J_{ab}$ must satisfy the representation relations (4) as well as the commutation relations (7). We note that in $H$ we shall have the relations (18), and also
\[ \epsilon_{\lambda \sigma} J^{\mu \nu} \sigma^\rho = 0 \]  

(21)

by virtue of (2). Furthermore, it is clear that in \( \mathcal{H} \)

\[ [q_{\lambda}, q_{\sigma}] = 0 \]

\[ [q_{\lambda}, J_{\mu \nu}] = i \hbar (q_{\lambda \mu} q_{\nu} - q_{\lambda \nu} q_{\mu}) \quad (22) \]

Comparing (18) and (21) with (4a)–(4c), and (22) with (7), we see that the Hermitean operators \( q_{\lambda} \) and \( J_{\mu \nu} \) may be regarded as generators in \( \mathcal{H} \) of a unitary irreducible representation \( (\hbar^2, 0, \pm) \) of a group isomorphic to \( \beta \).

Just as in the case of the instant form of dynamics, one can look at the problem posed above for the point form from a different point of view, supposing the generators \( P_{\lambda}, J_{\mu \nu} \) of the representation \( (m^2 c^2, 0, \pm) \) to be given, and arguing that other operators in the Hilbert space, such as \( q_{\lambda} \), might be expressible in terms of those generators. From this point of view, one has Hermitean operators \( P_{\lambda}, J_{\mu \nu} \) satisfying (4, 7), and wishes to express in terms of them, Hermitean operators \( q_{\lambda} \) via a formula reducing to (17) in the classical limit.

One sees that mathematically, this problem is essentially the same as the former one. In its most abstract form, each problem can be formulated as follows:

One has a set of Hermitean operators \( a_{\lambda}, J_{\mu \nu} \) which generate in a Hilbert space \( \mathcal{H} \) a unitary irreducible representation \( (1, 0, \pm) \) of a group isomorphic to \( \beta \), and hence satisfy

\[ \begin{align*}
[\alpha_{\lambda}, q_{\sigma}] &= 0, \\
[\alpha_{\lambda}, J_{\mu \nu}] &= i \hbar (q_{\lambda \mu} q_{\nu} - q_{\lambda \nu} q_{\mu}), \quad (23a, b) \\
[J_{\mu \nu}, J_{\rho \sigma}] &= i \hbar (q_{\mu \rho} J_{\nu \sigma} + q_{\mu \sigma} J_{\nu \rho} - q_{\mu \nu} J_{\sigma \rho} - q_{\mu \rho} J_{\nu \sigma}), \quad (23c) 
\end{align*} \]

as well as

\[ a_{\lambda} a_{\sigma} = 1, \quad a_{\lambda} a_{\sigma} a_{\lambda} a_{\sigma} = 0; \quad (24a, b, c) \]

and one wishes to find, in terms of \( a_{\lambda} \) and \( J_{\mu \nu} \), Hermitean operators \( b_{\lambda} \) satisfying the same relations as \( a_{\lambda} \) in (23) and (24). In addition, one wants to obtain in the classical limit

\[ a_{\lambda} b_{\nu} - a_{\nu} b_{\lambda} = \alpha J_{\lambda \nu} \quad (25) \]

and (consequently)

\[ b_{\lambda} = c a_{\lambda} a_{\nu} a_{\nu} a_{\lambda} b_{\lambda}, \quad (\beta + 1) a_{\lambda} a_{\nu} a_{\nu} a_{\lambda} b_{\lambda} = \frac{1}{2} (1 - \alpha / J_{\lambda \nu})^{1/2}, \quad (26) \]

where \( \alpha \) is a nonzero constant with the dimensions of \( m^{-2} \). In the first problem posed, \( \alpha = 1/m c \), \( a_{\lambda} = q_{\lambda} / k \), \( b_{\lambda} = P_{\lambda} / m c \); in the second \( \alpha = -1/m c \), \( a_{\lambda} = P_{\lambda} / m c \), \( b_{\lambda} = q_{\lambda} / k \).

As indicated above, in order to specify a solution uniquely, we shall find it necessary to require

\[ [\alpha_{\lambda}, b_{\mu}] = -3 i \hbar \alpha, \quad (27) \]

the analog of the classical relation \( (\alpha_{\lambda}, b_{\mu}) = -3 \alpha \).

4. SOLUTION OF THE PROBLEM

It is known \(^{16} \) that the operator \( J_{\lambda \nu} a_{\lambda} \) has in \( \mathcal{H} \) a continuous spectrum of points \( (1 + \beta^2) \), \( 0 \leq \beta < \infty \); we introduce an Hermitean Lorentz–scalar operator \( B \) satisfying

\[ B J_{\mu \nu} = 0, \quad J_{\lambda \nu} B_{\mu} = - (1 + \beta^2) \mathbf{B} \quad (28a, b) \]

(Here and below, a numerical multiple of the identity operator on \( \mathcal{H} \) is denoted by the corresponding complex number.) The specification of \( B \) is completed by the requirement that its spectrum be of points \( \beta, \quad 0 \leq \beta < \infty \), corresponding to that of \( J_{\lambda \nu} J_{\mu \nu} \) in the obvious way.

Then \( \mathbf{B} \) is an analog in quantum theory of the positive square root of the positive classical quantity \( (1 + \beta^2) \).

The identity

\[ J_{\lambda \nu} J_{\mu \nu} a_{\lambda} - 2 i \hbar J_{\mu \nu} a_{\nu} + (1 / 2) J_{\lambda \nu} J_{\mu \nu} a_{\lambda} = 0 \quad (29) \]

holds as a consequence of (23c) and (24c), as can be checked by substitution of index values. This is a special case of a more general type of identity, discussed in detail elsewhere. \(^{11} \)

Combining (28b) and (29) we have

\[ J_{\lambda \nu} - (i + B) \mathbf{B} a_{\nu} = 0, \quad (30) \]

so that

\[ a_{\lambda} = v_{\lambda}^{(\lambda)} \quad (31) \]

Then it follows from (30) that

\[ J_{\lambda \nu} v_{\lambda}^{(\mu)} = i (i + B) \mathbf{B} v_{\lambda}^{(\lambda)} \quad (32) \]

Now according to (31), \( v_{\lambda}^{(\lambda)} \) is a 4-vector operator, so that

\[ [v_{\lambda}^{(\lambda)}, J_{\mu \nu}] = i \hbar (q_{\lambda \mu} v_{\lambda}^{(\mu)} - q_{\lambda \nu} v_{\lambda}^{(\nu)}), \quad (33) \]

and hence

\[ [v_{\lambda}^{(\lambda)}, J_{\mu \nu}] = 2 i \hbar J_{\lambda \nu} v_{\lambda}^{(\mu)} + 3 i \hbar^2 v_{\lambda}^{(\mu)} \quad (34) \]

Combining (28b), (33), and (35) we have

\[ [v_{\lambda}^{(\lambda)}, (1 + B^2)^2] = (-i + 2 i B) v_{\lambda}^{(\lambda)} \quad (35) \]

that is

\[ (B^2 - 1)^2 v_{\lambda}^{(\lambda)} = v_{\lambda}^{(\lambda)} B^2 \quad (36) \]

In Appendix A we show the validity of the (apparently) stronger result

\[ B_{\lambda} v_{\lambda}^{(\lambda)} = v_{\lambda}^{(\lambda)} (B + \mathbf{1}), \quad (37) \]

and in Appendix B that (37) implies with (32), the hermiticity of \( v_{\lambda}^{(\lambda)} \). It must be emphasized that (37) is consistent with the hermiticity of \( v_{\lambda}^{(\lambda)} \) and \( B \), although formally it seems to imply that \( v_{\lambda}^{(\lambda)} \) shifts an eigenvector of \( B \) corresponding to a real eigenvalue \( \beta \), to one corresponding to the complex eigenvalue \( (\beta + i) \). The point is that \( B \) has no eigenvectors in \( \mathcal{H} \), and \( \mathbf{B} \) none in the domain of \( v_{\lambda}^{(\lambda)} \). [A similar situation occurs in those unitary representations of the conformal group corresponding to massless particles, where the Hermitean dilatation generator \( D \) and translation generators \( P_{\lambda} \) satisfy \( D P_{\lambda} = P_{\lambda} (D - \mathbf{1}) \).]

The operators \( v_{\lambda}^{(\lambda)} \) have several remarkable properties which we now list, putting derivations in Appendix B, and they play a central role in what follows. One has

\[ \begin{align*}
\langle v_{\lambda}^{(\lambda)}, v_{\lambda}^{(\lambda)} \rangle &= \hbar \mathbf{B} \quad (38a) \\
v_{\lambda}^{(\lambda)}, v_{\lambda}^{(\lambda)} &= 0 \quad (38b) \\
v_{\lambda}^{(\lambda)}, v_{\lambda}^{(\lambda)} &= - (B + \mathbf{1}) / 2 B \quad (38c) \\
[v_{\lambda}^{(\lambda)}, v_{\lambda}^{(\lambda)}] &= 0 \quad (38d)
\end{align*} \]
We now tackle the problem posed in Sec. 3, seeking an expression for the operator \( b_i \) in the form
\[
b_i = F^{(i)}(B) n^{(i)} + F^{(-i)}(B) n^{-i},
\]
where \( F^{(i)}(B) \) is a Lorentz-scaler operator-valued function of \( B \). We shall think of \( F^{(i)} \) as an "analytic" function with, for example on suitable vectors in \( H \),
\[
F^{(+i)}(B) = \sum_{\sigma=0}^{\infty} \sigma a_\sigma (B - b)^{\sigma},
\]
which we write as \( F^{(i)*}(B) \). Thus
\[
F^{(i)}(B)^* = F^{(i)*}(B),
\]
In what follows, we repeatedly make use of relations like
\[
F^{(+i)}(B)n^{(+i)} = n^{(+i)}F^{(i)}(B + i),
\]
which are taken to follow from (37) and (40). Taking the Hermitian conjugate of (39), we have
\[
b_i^* = n^{(+i)} F^{(+i)}(B) + n^{-i} F^{(-i)}(B)
\]
as so that \( b_i \) is Hermitian if and only if
\[
F^{(i)}(B) = F^{(i)*}(B + i).
\]
Next we note from (39)
\[
b_i b_i = F^{(+i)}(B) F^{(+i)}(B - i) n^{(+i)} n^{(i)} + F^{(+i)}(B) F^{(-i)}(B - i) n^{(-i)} n^{(i)} + F^{(-i)}(B) F^{(+i)}(B + i) n^{(-i)} n^{(i)} + F^{(-i)}(B) F^{(-i)}(B + i) n^{(-i)} n^{(i)} u.
\]
Then as a consequence of (38b, c) we have
\[
b_i b_i = - (B - i) F^{(+i)}(B) F^{(-i)}(B - i) + (B + i) F^{(+i)}(B + i) F^{(-i)}(B) - 2B
\]
so that \( b_i b_i = 1 \) if and only if
\[
F^{(+i)}(B) F^{(-i)}(B - i) + (B + i) F^{(+i)}(B + i) F^{(-i)}(B) = 2B.
\]
Furthermore, (43) also implies, with (38d, e), that
\[
[b_i, b_i] = - (F^{(+i)}(B) F^{(-i)}(B - i) + F^{(+i)}(B + i) F^{(-i)}(B)) J_{\mu u}/2\hbar B,
\]
so that \( [b_i, b_i] = 0 \) if and only if
\[
F^{(+i)}(B) F^{(-i)}(B - i) = F^{(+i)}(B + i) F^{(-i)}(B).\]
Combining (44) and (45) we have at once
\[
F^{(+i)}(B) F^{(-i)}(B - i) = -1.
\]
Thus a four-vector operator of the form (39) is Hermitian, with commuting components and unit length, if and only if (42) and (46) hold. [In particular this is so for \( a_i \) itself, for which \( F^{(+i)}(B) = - F^{(-i)}(B) = 1 \).]
so that
\[ F^{(4i)}(B) = G(B) + [1 + G^2(B)]^{1/2}, \]
where \( [1 + G^2(B)]^{1/2} \) denotes some Lorentz-scalar square root of \( [1 + G^2(B)] \).

Then (46) implies that
\[ F^{(4i)}(B - i) = G(B) - [1 + G^2(B)]^{1/2}, \]
that is
\[ F^{(4i)}(B) = G(B + i) - [1 + G^2(B + i)]^{1/2}. \]

Conversely, if \( G(B) \) satisfies (56) and (58), then \( F^{(4i)}(B) \) and \( F^{(4i)}(B) \), defined as in (59), (60), can be seen to satisfy (42), (46), and (55), and so to define via (39) a 4-vector operator \( b_\xi \) with the required properties of hermiticity, unit length, commuting components, and satisfaction of (47c). Since (56) and (58) evidently do not uniquely determine \( G(B) \), the solution to the problem at hand is not yet specified completely.

In order to remove this ambiguity, we now impose (27). From (32) and (39), we get with the help of (38b, c),
\[ a_i b_i = \frac{[(B + i)F^{(4i)}(B + i) - (B - i)F^{(4i)}(B - i)]}{2B}, \]
and
\[ b_\xi a_\xi = \frac{[(B + i)F^{(4i)}(B) - (B + i)F^{(4i)}(B)]}{2B}, \]
so that (27) implies
\[ G(B + i) - G(B - i) = 3i\alpha h B. \]

Then (56) and (61) imply
\[ G(B) = -\alpha h (B - \frac{1}{2}i), \]
which is seen to be consistent with (58).

Substituting in (59), (60), we arrive at the expressions
\[ F^{(4i)}(B) = -\alpha h (B + \frac{1}{2}i) \pm H^{(4)}(B), \]
where
\[ H^{(4)}(B) = [1 + \alpha^2 h^2 (B + \frac{1}{2}i)]^{1/2}. \]

Substituting from (62) in (39), and recalling the definition (31) of \( n^{(4)} \), we finally have
\[ b_\xi = \frac{1}{2} - \alpha (H^{(4)}(B) - H^{(-4)}(B))/2hB^2 \tau_{\alpha\nu} \sigma^\nu \]
\[ + \frac{(3\alpha h B + (B - i)H^{(4)}(B) + (B + i)H^{(-4)}(B))}{2B} \tau_{\alpha\nu} \sigma^\nu, \]
and (26) is recovered from (64), provided the point \( H^{(4)}(B) \) in the spectrum of \( H^{(4)}(B) \), corresponding to the point \( \beta \) in the spectrum of \( B \), has positive real part \((-1/2\alpha h^2 \beta^2)\) when \( \beta > 0 \). This is achieved by taking \( H^{(4)}(B) \) to be the principal square root of \( 1 + \alpha^2 h^2 (B + \frac{1}{2}i)^2 \).

In other words, in the representation described in Appendix A, where \( B \) is realized as multiplication by the nonnegative quantity \( \beta \), \( H^{(4)}(B) \) will be realized as multiplication by the principal square root \( [1 + \alpha^2 h^2 (\beta + \frac{1}{2}i)]^{1/2} \). In this connection, it is worth mentioning that for \( \alpha^2 h^2 \equiv 4 \), the two curves \( z^{(4)}(\beta) = [1 + \alpha^2 h^2 (\beta + \frac{1}{2}i)^2], \beta \equiv 0 \), in the complex \( z \) plane meet the branch cut of the principal square root \( z^{1/2} \) at \( \beta = 0 \). It may be that this means that \( b_\xi \), as defined in (64) is not self-adjoint in such cases, but with our algebraic approach we cannot determine this. Note that \( \alpha^2 h^2 < 4 \) can be written in the very suggestive form
\[ (mc)k > \frac{1}{2}h. \]

5. Conclusion

We have derived for a spinless particle with nonzero rest mass \( m \) and positive energy, the analog in quantum mechanics of Dirac's formula (10) defining the point form of classical dynamics. Substituting \( \alpha = 1/mck \), \( a_i = p_i/k \), \( b_\xi = P_\xi/mck \) in (64) we have
\[ \hbar^2 P_\xi = \frac{1}{2}(1 + mckH^{(4)}(B) - H^{(-4)}(B))/2\hbar B \tau_{\alpha\nu} \sigma^\nu \]
\[ + \frac{(3\alpha h B + mckB - iH^{(4)}(B))}{2B} \tau_{\alpha\nu} \sigma^\nu, \]
where
\[ H^{(4)}(B) = [1 + \alpha^2 h^2 (B + \frac{1}{2}i)]^{1/2}. \]

By the same means, we have derived a formula for the 4-vector coordinate operator \( q_\xi \) in terms of the generators of the inhomogeneous Lorentz group \( \mathcal{P} \).

Substituting \( \alpha = -1/mck \), \( a_i = P_i/mc \), \( b_\xi = q_\xi/k \) in (64) we have
\[ m^2 c^2 q_\xi = \frac{1}{2}(1 + mckH^{(4)}(B) - H^{(-4)}(B))/2\hbar B \tau_{\alpha\nu} \sigma^\nu \]
\[ + \frac{(3\alpha h B + mckB - iH^{(4)}(B))}{2B} \tau_{\alpha\nu} \sigma^\nu, \]
with \( H^{(4)}(B) \) as in (66).

This 4-vector operator satisfies \( q_\xi q_\xi = h^2, q_\xi \equiv k > 0 \), and has commuting components which are Hermitian. We have made implicit in the text certain reasonable but unproven assumptions about domains of definition of operators, and we certainly cannot claim to have proved the stronger condition of self-adjointness of \( q_\xi \). In particular, we have suggested that there is some question as to the self-adjointness of \( q_\xi \) when one does not have
\[ (mc)k > \frac{1}{2}h. \]

The operator \( q_\xi \) is the analogue in quantum mechanics of the coordinate 4-vector of the point where the world line of the classical particle meets the positive sheet of a two-sheeted hyperboloid, and as such certainly qualifies to be called a coordinate operator. It bears the same relation to this surface as the Newton-Wigner 3-vector operator does to an instant. Only transformations in the Euclidean subgroup of \( \mathcal{P} \) leave an instant invar-
ant, and accordingly the Newton–Wigner operator transforms simply under this subgroup but not simply under Lorentz boosts. Similarly, the sheet of the hyperboloid is Lorentz invariant, and \( q_s \) transforms simply under the homogeneous Lorentz group (as a 4-vector) but not simply under translations in space and time. In particular, the canonical relations

\[
[q_a, p_s] = i\hbar \delta_a^s,
\]

do not hold. We have so far been unable to evaluate the commutator \([q_a, p_s]\) in terms of simpler expressions, in order to find the analog of the classical equation (14).

It could reasonably be argued that the formula (67) is so complicated that one cannot hope to manipulate readily or usefully with the operator \( q_s \). Our main purpose has been to indicate the existence of this 4-vector operator, which we have seen is defined by the conditions that it has Hermitian, commuting components, and satisfies

\[
q_s q_s^\dagger = k^2, \quad q_s \geq k > 0,
\]

\[
(q_s p_s - q_s p_s) + (p_s q_s - p_s q_s) = 2q_j a^j,
\]

\[
[q_s, p_s^\dagger] = -3\hbar. \tag{68}
\]

Given that, we hope it will prove possible to investigate its properties further (for example, to find the common generalized eigenvectors of its components in, say, the momentum representation) by other means.

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APPENDIX A

Chakrabarti, Levy–Nahas, and Seneor have described the matrix elements of the operators \( a_s \) in a "Lorentz basis" for the representation \((1, 0, +)\) of \( \rho \). This basis, in which \( B, \frac{1}{2} J_a, J_a^a, a_a, \) and \( a_a \) are diagonalized is not a true basis, as it consists of non-normalizable vectors, but it can be used to define a realization of the underlying abstract structure. Then the Hilbert space is realized as the space of vectors

\[
\phi = (\phi_0(\beta), \phi_1(\beta), \phi_2(\beta), \phi_3(\beta), \phi_4(\beta), \ldots),
\]

\[
0 \leq \beta < \infty,
\]

where

\[
\sum \phi_{lm}(\beta) |^2 < \infty
\]

the sum being over \( m = 1, 2, 3, 4 \). The scalar product of two such vectors is

\[
\langle \phi, \psi \rangle = \sum_{m=1}^{\infty} \phi_{lm}(\beta) \psi_{lm}(\beta)\beta^m.
\]

The results of Chakrabarti et al. [see in particular Eqs. (2, 3), (2, 18), (2, 19), and (2, 29) in Ref. 18] show that in this realization, the action of the operators \( a_s \) and \( B \) is

\[
(a_0 \phi)(1, \beta) = \left[(1 - i \beta)(1 + i \beta + 1)/4 \beta (\beta + 1) \right]^{1/2} \phi_{lm}(\beta + i) + \left[(1 + i \beta)(1 - i \beta + 1)/4 \beta (\beta + 1) \right]^{1/2} \phi_{lm}(\beta - i)
\]

\[
(\beta \phi)(1, \beta) = \phi_{lm}(\beta).
\]

Here the action of \( a_0 \) evidently presupposes certain analyticity properties of the functions \( \phi_{lm}(\beta) \) for vectors \( \phi \) in its domain. Although it is difficult to exhibit explicitly a common invariant dense domain of Hermiticity for \( a_0 \) and \( B \), it is nevertheless clear from (A1) that

\[
a_0 = n_{(+)} - n_{(-)} \tag{A2}
\]

where, on suitable vectors \( \phi \)

\[n_{(+)}(\phi) \backslash \phi \mid (\beta + i) = \left[(1 + i \beta)(1 + i \beta + 1)/4 \beta (\beta + 1) \right]^{1/2} \phi_{lm}(\beta + i) \]

and, consequently

\[B n_{(+)}(\phi) \beta \mid (\beta + i) \tag{A3}
\]

It follows from (A2) and (A3) that

\[n_{(+)} = \frac{1}{2} a_0 + \frac{1}{2} i [a_0, B] \tag{A4}
\]

Defining the 4-vector operator

\[n_{(+)} = \frac{1}{2} a_0 + \frac{1}{2} i [a_0, B] \tag{A5}
\]

so that

\[a_0 = n_{(+)} - n_{(-)} \tag{A6}
\]

we see from (A3) by covariance that

\[B n_{(+)} = n_{(+)} B \tag{A7}
\]

Since it then follows trivially that

\[B (\beta + i) = [n_{(+)} B] (\beta + i) \tag{A8}
\]

we see from (A6), (A8), (32), and (36) that \( n_{(+)} = n_{(+)} \), and (37) is thus justified.

APPENDIX B

Once (37) is established, it follows using (32) that

\[n_{(+)} (\phi) \beta \mid (\beta + i) = \frac{1}{2} a_0 + \frac{1}{2} i [a_0, B] \tag{B1}
\]

Since \( a_0 \) and \( B \) are Hermitian, it follows at once that \( n_{(+)} \) is also Hermitian; that is (38a) holds.

Now note from (B1) and (37) that

\[n_{(+)} (\phi) \beta \mid (\beta + i) = \frac{1}{2} a_0 + \frac{1}{2} i [a_0, B] - \frac{1}{2} (\beta + i) [a_0, B] \tag{B2}
\]

But from (31)

\[n_{(+)} (\phi) \beta \mid (\beta + i) = (2 \hbar B)^{-1} (J_{\alpha} a_{\alpha} a_{\beta} - (\beta + i) a_{\beta} a_{\alpha}) \tag{B3}
\]

since the components of \( a_\alpha \) commute, and \( a_\alpha a_\beta = 1 \). Combining (B2) and (B3) we have

\[n_{(+)} (\phi) \beta \mid (\beta + i) = 0 \tag{B4}
\]

establishing (38b). In a similar way, (38c) is deduced.

Again using (B1) and (37) we have

\[n_{(+)} (\phi) \beta \mid (\beta + i) = \frac{1}{2} (\beta + i) a_0 + \frac{1}{2} i [a_0, B] - \frac{1}{2} (\beta + i) [a_0, B] \tag{B5}
\]

But from (31)

\[n_{(+)} (\phi) \beta \mid (\beta + i) = (2 \hbar B)^{-1} (J_{\alpha} a_{\alpha} a_{\beta} - (\beta + i) a_\beta a_\alpha) \tag{B6}
\]

so that

\[n_{(+)} a_\alpha - n_{(+)} a_\beta = (2 \hbar B)^{-1} (J_{\alpha} a_{\alpha} a_{\beta} - J_{\beta} a_{\beta} a_{\alpha}) \tag{B7}
\]

Now (24c) is equivalent to

\[J_{\alpha} a_\alpha + J_{\beta} a_\beta + J_{\alpha} a_\beta = 0 \tag{B8}
\]

Contracting with \( a_\alpha \) from the right in this equation, and
noting that $a_\nu a^\nu = 1$, we obtain

$$J_\mu a_\nu a^\nu = J_\mu.$$

(37)

Combining (B6) and (B7) we have

$$\gamma^{(3)} \gamma a_\mu \gamma a^\mu = (2\hbar H)^{-1} J_{16},$$

and thus from (B4) that

$$\gamma^{(3)} \gamma \gamma a_\mu \gamma a^\mu \gamma = 0,$$

establishing (38d). In a similar way, (38c) is deduced.

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