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Zitterbewegung and the internal geometry of the electron

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Schrödinger's work on the *Zitterbewegung* of the free electron is reexamined. His proposed "microscopic momentum" vector for the *Zitterbewegung* is rejected in favor of a "relative momentum" vector, with the value $\vec{P} = mc\vec{\alpha}$ in the rest frame of the center of mass. His oscillatory "microscopic coordinate" vector is retained. In the rest frame, it takes the form $\vec{Q} = -i(\hbar/2mc)\beta\vec{\alpha}$, and the *Zitterbewegung* is described in this frame in terms of $\vec{P}$, $\vec{Q}$, and the Hamiltonian $mc^2\beta$, as a finite three-dimensional harmonic oscillator with a compact phase space. The Lie algebra generated by $\vec{Q}$ and $\vec{P}$ is that of SO(5), and in particular $[Q_i, P_j] = -i\hbar\delta_{ij}\beta$. It is argued that the simplest possible finite, three-dimensional, isotropic, quantum-mechanical system requires such an SO(5) structure, incorporates a fundamental length, and has harmonic-oscillator dynamics. Dirac's equation is derived as the wave equation appropriate to the description of such a finite quantum system in an arbitrary moving frame of reference, using a dynamical group SO(3,2) which can be extended to SO(4,2). Spin appears here as the orbital angular momentum associated with the internal system, and rest-mass energy appears as the internal energy in the rest frame. Possible generalizations of these ideas are indicated, in particular those involving higher-dimensional representations of SO(5).

I. INTRODUCTION

It is widely known that Schrödinger[1] examined the behavior in time of the coordinate operator $\vec{x}$ associated with Dirac's equation,[2] and discovered the highly oscillatory, microscopic motion with velocity c which he called the "*Zitterbewegung*." Although Dirac[3] has argued that such a motion does not contradict relativity or quantum mechanics, very little attention has been given over the years to the physics of the *Zitterbewegung*.[4] Most studies have dealt with the behavior of expectation values of operators in wave packets, or with the question of attributing the *Zitterbewegung* to the mixing of positive- and negative-energy states, without going into the details of the microscopic motion. In contrast, Schrödinger attempted a precise description in terms of "microscopic" dynamical variables (coordinates, momenta, and angular momenta), distinct from "macroscopic" variables associated with the mean motion.

The general view might be that his attempt can now be seen as somewhat irrelevant. For it is often said now that when Dirac's equation is being interpreted at the one-particle level, rather than as a field equation, the only Hermitian operators which can represent observable quantities are those which leave separately invariant the spaces of positive- and negative-energy solutions of the equation.[5] The *Zitterbewegung* is then apparently relegated to the position of an unobservable mathematical curiosity, as it involves the motion in time of an operator $\vec{x}$ which does not satisfy this criterion.

However, the *Zitterbewegung does* have observable effects. These will appear, for example, if one calculates the matrix elements of $(\vec{x}\cdot\vec{x})$ between positive-energy states.[6] Moreover, it is also true that one cannot fully describe the problem of an electron in all possible external weak and electromagnetic fields without employing the whole algebra of operators associated with Dirac's equation. So although one may argue that some of these operators do not represent observables in the presence of a given field, one cannot deny them some physical significance.

The interpretation of $\vec{x}$ as the "position of the charge" of the electron seems to us logically inescapable[6] unless one is to regard the minimal-coupling formula [see Eq. (26) below] as a mere mathematical device, devoid of any direct physical interpretation. Furthermore, it is this coordinate vector $\vec{x}$ which forms a four-vector with the time t, a fact to which great significance must surely be attached in a relativistic theory. Should we assume that the very coordinates of the space-time manifold, within which Dirac's equation is formulated, cannot be directly associated with observable quantities; and if so, why should we attach any significance at all to Lorentz transformations among such coordinates?

We are well aware of the popular and contrary point of view—that t has a privileged role even in relativistic quantum mechanics, and so should not be considered conjointly with one-particle coordinate operators; and that the latter should not be expected to transform simply with respect to the homogeneous Lorentz group.[7] This point of view is largely based on the work of Newton and Wigner.[8] In our opinion, the noncausal properties[9] (not often mentioned) of operators such as the "mean-position" operator of Foldy and Wouthuysen[10] make it doubtful that they can have a *primary* significance in a relativistic theory.[6]

For these reasons, we feel that there are still aspects of Dirac's equation not fully understood, and that Schrödinger's analysis is worthy of reappraisal and further development. As Schrödinger did, we work in what follows in the space of arbitrary superpositions of positive- and negative-energy states of the "one-particle system," and we deal with operators whose eigenvectors are such superpositions. We give no further justification for this, except to say that there is a very interesting geometrical and dynamical substructure associated with Dirac's equation, which cannot be revealed unless these superpositions are considered.

It will become clear that the substructure of the electron as revealed here admits natural generalizations. Furthermore, the analysis of this substructure may throw new light on the theory of the electron itself. For if the electron is not a point particle, but a massless charge performing a complicated motion around a center of mass, such a picture cannot possibly be without implications for the self-energy and renormalization problems of the electron.

II. SCHRÖDINGER'S *ZITTERBEWEGUNG* PAPER

The Hamiltonian for the free electron-positron system according to Dirac is

$$H = c\vec{\alpha}\cdot\vec{p} + mc^2\beta \ , \tag{1}$$

where $\vec{\alpha}$ and β satisfy

$$\begin{aligned}&\{\alpha_i , \alpha_j\} = 2\delta_{ij} I \quad (i,j = 1, 2, 3),\\ &\{\alpha_i , \beta\} = 0, \quad \beta^2 = I \ .\end{aligned} \tag{2}$$

The momentum vector $\vec{p}$ and a conjugate coordinate vector $\vec{x}$ are taken to satisfy

$$\begin{aligned}&[x_i , x_j] = 0 = [p_i , p_j],\\ &[x_i , p_j] = i\hbar\delta_{ij} I \ ,\end{aligned} \tag{3}$$

and to commute with $\vec{\alpha}$ and β.

In the Heisenberg picture, all these relations hold at any one time, and the time derivative of any one of these operators which do not depend explicitly on time, say A, is given by

$$\frac{dA}{dt} = i[H, A]/\hbar \ . \tag{4}$$

Thus

$$\frac{d\vec{p}}{dt} = \vec{0}, \quad \frac{dH}{dt} = 0 \ , \tag{5}$$

while

$$\frac{d\vec{x}}{dt} = c\vec{\alpha} \tag{6}$$

and

$$-i\hbar\frac{d\vec{\alpha}}{dt} = [H, \vec{\alpha}] = -\{H, \vec{\alpha}\} + 2H\vec{\alpha} = -2c\vec{p} + 2H\vec{\alpha} \ . \tag{7}$$

Writing this last equation as

$$-i\hbar\frac{d\vec{\alpha}}{dt} = 2H\vec{\eta} \ , \tag{8}$$

with

$$\vec{\eta} = \vec{\alpha} - cH^{-1}\vec{p} \ , \tag{9}$$

Schrödinger noted that

$$-i\hbar\frac{d\vec{\eta}}{dt} = -i\hbar\frac{d\vec{\alpha}}{dt} = 2H\vec{\eta} \tag{10}$$

so that

$$\vec{\eta}(t) = e^{2iHt/\hbar}\vec{\eta}_0 \ . \tag{11}$$

Here $\vec{\eta}_0$ is a constant operator:

$$\vec{\eta}_0 = \vec{\eta}(0) = \vec{\alpha}(0) - cH^{-1}\vec{p} \ . \tag{12}$$

It is easily checked that

$$\{H, \vec{\eta}\} = 0 = \{H, \vec{\eta}_0\} \tag{13}$$

so that one can also write, from Eq. (11),

$$\vec{\eta}(t) = \vec{\eta}_0\, e^{-2iHt/\hbar} \ . \tag{11'}$$

Combining Eqs. (7), (9), and (11′), Schrödinger obtained

$$\frac{d\vec{x}}{dt} = c\vec{\alpha} = c^2H^{-1}\vec{p} + c\vec{\eta}_0\, e^{-2iHt/\hbar} \tag{14}$$

which he integrated again to get

$$\vec{x}(t) = \vec{a} + c^2H^{-1}\vec{p}t + \tfrac{1}{2}i\hbar c\vec{\eta}_0 H^{-1}e^{-2iHt/\hbar}, \tag{15}$$

with $\vec{a}$ a constant (operator) of integration

$$\vec{a} = \vec{x}(0) - \tfrac{1}{2}i\hbar c\vec{\alpha}(0)H^{-1} + \tfrac{1}{2}i\hbar c^2H^{-2}\vec{p} \ . \tag{16}$$

Now Eq. (15) can be written as

$$\vec{x}(t) = \vec{x}_A(t) + \vec{\xi}(t) \tag{17}$$

with

$$\vec{x}_A(t) = \vec{a} + c^2H^{-1}\vec{p}t \ , \tag{18}$$

the form one might expect for the "position" operator of a relativistic point mass, by analogy with the classical result. The remaining contribution to $\vec{x}$ is

$$\vec{\xi}(t)=\tfrac{1}{2}i\hbar c\vec{\eta}_0 H^{-1}e^{-2iHt/\hbar}=\tfrac{1}{2}i\hbar c\vec{\eta}H^{-1}\ , \tag{19}$$

which describes a microscopic, high-frequency *Zitterbewegung* superimposed on the macroscopic motion associated with $\vec{x}_A$. The *Zitterbewegung* has a characteristic amplitude $\hbar/2mc$, half the Compton wavelength of the electron, and a characteristic angular frequency $2mc^2/\hbar$ (see below).

Schrödinger went on to note, as Dirac had done,[11] that if the spin vector $\vec{S}$ and the orbital angular momentum vector $\vec{L}$ are introduced, as

$$\vec{S}=-\tfrac{1}{4}i\hbar\vec{\alpha}\times\vec{\alpha},\quad \vec{L}=\vec{x}\times\vec{p}, \tag{20}$$

then $(\vec{L}+\vec{S})$ is a constant of the motion, while neither $\vec{L}$ nor $\vec{S}$ alone is constant. Rather, Schrödinger found that

$$\begin{aligned}\vec{S}(t)&=\vec{S}_A-\vec{\xi}(t)\times\vec{p}\ ,\\ \vec{L}(t)&=\vec{L}_A+\vec{\xi}(t)\times\vec{p}\ ,\end{aligned} \tag{21}$$

where $\vec{S}_A$ and $\vec{L}_A$ are constants, and $\vec{\xi}\times\vec{p}$ is oscillatory. This is easily seen from Eqs. (17) and (18):

$$\begin{aligned}\vec{L}&=\vec{x}\times\vec{p}\\ &=\vec{x}_A\times\vec{p}+\vec{\xi}\times\vec{p}\\ &=\vec{L}_A+\vec{\xi}\times\vec{p}\ ,\end{aligned}$$

where

$$\begin{aligned}\vec{L}_A&=\vec{x}_A\times\vec{p}\\ &=\vec{a}\times\vec{p}\end{aligned} \tag{22}$$

is constant. Since $(\vec{L}+\vec{S})$ is constant, it follows that $\vec{S}$ has the form given in the first equation of Eqs. (21), with $\vec{S}_A$ constant.

Schrödinger next introduced a "microscopic momentum" vector to be associated with the coordinate vector $\vec{\xi}$. From Eq. (9) he found

$$\vec{\eta}\,\frac{H}{c}+\vec{p}=\vec{\alpha}\,\frac{H}{c}=\frac{d\vec{x}}{dt}\,\frac{H}{c^2}\ , \tag{23}$$

which he interpreted as a statement that the sum of the oscillating microscopic momentum $(=\vec{\eta}H/c)$ and the constant macroscopic momentum $(=\vec{p})$ should equal the "total" momentum, assumed to be given by $(d\vec{x}/dt)H/c^2$ (by analogy with the classical value for a relativistic point mass). He then showed that

$$\vec{S}_A=\tfrac{1}{2}\vec{\xi}\times\vec{\eta}\,\frac{H}{c}\quad(=-\tfrac{1}{4}i\hbar\vec{\eta}\times\vec{\eta})\,. \tag{24}$$

The proof is straightforward. From Eq. (9) one has

$$\begin{aligned}\vec{S}&=-\tfrac{1}{4}i\hbar\vec{\alpha}\times\vec{\alpha}\\ &=-\tfrac{1}{4}i\hbar\,(\vec{\eta}+cH^{-1}\vec{p})\times(\vec{\eta}+cH^{-1}\vec{p})\\ &=-\tfrac{1}{4}i\hbar(\vec{\eta}\times\vec{\eta}+2c\,\vec{\eta}H^{-1}\times\vec{p})\ ,\end{aligned}$$

using Eq. (13). Now using Eq. (19) one has

$$\vec{S}=\tfrac{1}{2}\vec{\xi}\times\vec{\eta}\,\frac{H}{c}-\vec{\xi}\times\vec{p}$$

and a comparison with Eq. (21) yields Eq. (24).

Combining Eqs. (21), (22), and (24) one has

$$\vec{L}+\vec{S}=\vec{L}_A+\vec{S}_A=\vec{x}_A\times\vec{p}+\tfrac{1}{2}\vec{\xi}\times\vec{\eta}\,\frac{H}{c}\ . \tag{25}$$

Apart from the factor $\frac{1}{2}$, by which Schrödinger admitted to being puzzled, this formula has an attractive unifying appearance, as a sum of two "moments of momenta," one macroscopic and one microscopic. However, Schrödinger's choice of $\vec{\eta}H/c$ as the microscopic momentum can be criticized on several grounds.

In the first place, one could just as easily have multiplied Eq. (9) by H/c on the left to obtain

$$\frac{H}{c}\vec{\eta}+\vec{p}=\frac{H}{c^2}\,\frac{d\vec{x}}{dt}\ , \tag{23$'$}$$

which might suggest the adoption of $(H/c)\vec{\eta}$ as the microscopic momentum. One can then also write

$$\vec{L}_A+\vec{S}_A=-\vec{p}\times\vec{x}_A-\frac{1}{2}\frac{H}{c}\vec{\eta}\times\vec{\xi}\ , \tag{25$'$}$$

a formula just as attractive as (25). But these two choices for the microscopic momentum are incompatible, one being the negative of the other by virtue of Eq. (13). In the second place, the operator $\vec{\eta}H/c$, unlike $\vec{x}_A$, $\vec{p}$, and $\vec{\xi}$, is not Hermitian, and the attempt to make it so by symmetrizing fails, again because of Eq. (13). In the third place, one can argue that $\vec{p}$ itself should represent the total momentum of the system—it is a constant, unlike $(d\vec{x}/dt)H/c^2$—and one should not look for a formula in which $\vec{p}$ appears as a component of a greater total momentum.

It is sometimes said that one should distinguish two "centers" for the electron-positron system: one a center of charge, the other of mass. The coordinate $\vec{x}$ should be the center of charge, since the four-vector potential is evaluated at this $\vec{x}$ when it appears in the minimal electromagnetic coupling prescription

$$\begin{aligned}\vec{p}&\to\vec{p}-e\vec{A}(\vec{x},t)\ ,\\ H&\to H-e\varphi(\vec{x},t)\ .\end{aligned} \tag{26}$$

In the present context the logical candidate for the center of mass is clearly $\vec{x}_A$. However, there are certain ambiguities in the definition of center of mass in relativistic mechanics (quantum or classical),[12] and it is important and interesting to note that $\vec{x}_A$ is *not* the mean-position operator $\vec{X}$ of Foldy and Wouthuysen.[10] That operator $\vec{X}$, like $\vec{x}_A$, satisfies

$$\frac{d\vec{X}}{dt} = c^2 H^{-1}\vec{p} , \tag{27}$$

but $\vec{X}$ differs from $\vec{x}_A$ by a constant of the motion

$$\vec{x}_A - \vec{X} = [m(E+mc^2)]^{-1}(\vec{p}\times\vec{S}_A) , \tag{28}$$

where

$$E = c(\vec{p}^2 + m^2c^2)^{1/2} . \tag{29}$$

The operators $\vec{x}_A$ and $\vec{X}$ are two of several "mass centers" described by Pryce[12] for Dirac's equation (he denoted $\vec{x}_A$ by $\vec{q}$ and $\vec{X}$ by $\tilde{\vec{q}}$). Among these operators, $\vec{X}$ is distinguished by the fact that its components commute, whereas those of $\vec{x}_A$, in particular, do not. Instead, we have

$$[x_{Ai}, x_{Aj}] = -i\hbar c^2 E^{-2}\epsilon_{ijk} S_{Ak} . \tag{30}$$

We have seen little discussion of the occurrence of $\vec{x}_A$ rather than $\vec{X}$ as the coordinate vector relative to which the *Zitterbewegung* takes place. Foldy and Wouthuysen incorrectly claimed that $(\vec{x}-\vec{X})$ is "oscillating rapidly about zero," whereas Schrödinger's analysis shows clearly that it is $\vec{\xi}$ $(=\vec{x}-\vec{x}_A)$ which is the oscillating coordinate. It then follows from Eq. (28) that in general the center of the Zitterbewegung should be thought of as displaced by a constant amount from what Foldy and Wouthuysen called the "mean" position. The magnitude of this displacement, i.e., $|\vec{x}_A - \vec{X}|$, can be seen from Eq. (28) to be of the order $\frac{1}{2}\lambda(c|\vec{p}|E^{-1})$. Tani[13] made some observations on this matter, and Davydov[14] correctly distinguishes $\vec{x}_A$ as the "average-position" operator for the electron. Following Schrödinger[1] and Pauli,[2] Pryce noted that

$$\vec{x}_A = \tfrac{1}{2}(\vec{x} + HE^{-1}\vec{x}HE^{-1}) , \tag{31}$$

so that $\vec{x}_A$ is that part of $\vec{x}$ which commutes with the "sign of the energy" operator HE^{-1}. Then $\vec{\xi}$ is the remaining part of $\vec{x}$, and anticommutes with HE^{-1}. For this reason Pryce called $\vec{x}_A$ the "observable part" of $\vec{x}$. Similarly, $\vec{L}_A$ and $\vec{S}_A$ are in this sense the observable parts of $\vec{L}$ and $\vec{S}$. We emphasize that $\vec{L}_A$ is *not* equal to $\vec{X}\times\vec{p}$, called by Foldy and Wouthuysen the mean orbital angular momentum operator. Likewise, $\vec{S}_A$ is not their mean spin operator. The commutation relations satisfied by $\vec{L}_A$ and $\vec{S}_A$ are not the usual SO(3) relations. For example, one can deduce, using Eqs. (22) and (30), that

$$[L_{Ai}, L_{Aj}] = i\hbar\epsilon_{ijk}[L_{Ak} - c^2(\vec{S}_A\cdot\vec{p})p_k E^{-2}] . \tag{32}$$

If we think of $\vec{x}_A$ as a center-of-mass coordinate, and $\vec{\xi}$ as the coordinate of a motion relative to that of the mass center, then we are led to think of $\vec{p}$ as the total or center-of-mass momentum of the system, and to look for an independent "internal" or "relative" momentum vector $\vec{P}_{\rm rel}$. This is to be contrasted with the view adopted by Schrödinger, as described above. In the rest frame of the center of mass, such a relative momentum vector is not difficult to identify.

III. RELATIVE MOTION IN THE CENTER-OF-MASS REST FRAME

Here $\vec{p}=\vec{0}$, and the Hamiltonian and its inverse become

$$H_r = mc^2\beta, \quad H_r^{-1} = \beta/mc^2 , \tag{33}$$

where $\beta=\beta(0)$ is now a constant of the motion. The center-of-mass coordinate $\vec{x}_A$ is not well defined in this frame, in the sense that it does not commute with $\vec{p}$. [The same is true of $\vec{X}$, but in passing we note from Eq. (28) that the difference between $\vec{x}_A$ and $\vec{X}$ is well defined and equal to $\vec{0}$.] The relative coordinate $\vec{\xi}$ is well defined, and takes the form

$$\begin{aligned}\vec{\xi}(t)|_{\vec{p}=\vec{0}} &= \tfrac{1}{2}i\hbar c\vec{\eta}(t)|_{\vec{p}=\vec{0}}\, H_r^{-1}\\ &= \tfrac{1}{2}i\frac{\hbar}{mc}\vec{\alpha}(t)\beta\\ &= \tfrac{1}{2}i\lambda\vec{\alpha}(0)\beta e^{-2i\beta ct/\lambda}\\ &= \vec{Q}(t), \quad \text{say.}\end{aligned} \tag{34}$$

Here λ is the Compton wavelength.

The velocity $d\vec{x}/dt$ of the center of charge is also well defined and is different from zero even though $\vec{p}=\vec{0}$. We have

$$\frac{d\vec{x}_A}{dt} = \vec{0}, \text{ but } \frac{d\vec{x}}{dt} = \frac{d\vec{\xi}}{dt} = c\vec{\alpha} , \tag{35}$$

and we are led to suggest the identification of the relative momentum in this frame as

$$\begin{aligned}\vec{P}_{\rm rel}|_{\vec{p}=\vec{0}} &= mc\vec{\alpha}(t)\\ &= \vec{P}(t), \quad \text{say.}\end{aligned} \tag{36}$$

In other words, we suppose that

$$\vec{p}=\vec{0} \Rightarrow \vec{P}_{\rm rel} = \vec{P}_{\rm charge} . \tag{37}$$

The relative motion or *Zitterbewegung* is then described by the variables $\vec{Q}(t)$, $\vec{P}(t)$, with Hamiltonian H_r. We have the commutation relations

$$[Q_i, H_r] = \tfrac{1}{2} i\lambda mc^2[\alpha_i\beta, \beta] = i\frac{\hbar}{m}P_i\ ,$$
$$[P_i, H_r] = m^2c^3[\alpha_i, \beta] = -4\frac{ic\hbar^2}{\lambda^3}Q_i\ , \tag{38}$$

so that

$$\frac{d\vec{Q}}{dt} = \frac{1}{m}\vec{P},\quad \frac{d\vec{P}}{dt} = -\frac{4c\hbar}{\lambda^3}\vec{Q}\ , \tag{39}$$

and we have a harmonic oscillator:

$$\left.\begin{aligned} &\frac{d^2\vec{Q}}{dt^2} + \omega^2\vec{Q} = \vec{0}\ ,\\ &\frac{d^2\vec{P}}{dt^2} + \omega^2\vec{P} = \vec{0}\ , \end{aligned}\right\}\ \omega = \frac{2c}{\lambda} = \frac{2mc^2}{\hbar}\ . \tag{40}$$

The general solution of these equations is

$$\vec{Q}(t) = \vec{Q}(0)\cos\omega t + \frac{\lambda^2}{2\hbar}\vec{P}(0)\sin\omega t\ ,$$
$$\vec{P}(t) = \vec{P}(0)\cos\omega t - \frac{2\hbar}{\lambda^2}\vec{Q}(0)\sin\omega t\ . \tag{41}$$

The operators $\vec{Q}$ and $\vec{P}$ do not satisfy the canonical commutation relations. Instead we have (at any time)

$$[Q_i, P_j] = -i\hbar\delta_{ij}\beta\ ,$$
$$[Q_i, Q_j] = \frac{i\lambda^2}{\hbar}\epsilon_{ijk}S_k\ ,\quad [P_i, P_j] = \frac{4i\hbar}{\lambda^2}\epsilon_{ijk}S_k\ , \tag{42}$$

where

$$\vec{S} = -\tfrac{1}{4}i\hbar\,\vec{\alpha}\times\vec{\alpha}\ . \tag{43}$$

It can be seen that the Lie algebra generated by the Q_i and P_i closes on the ten-dimensional algebra of SO(5), with $(1/\lambda)\vec{Q}$, $(\lambda/\hbar)\vec{P}$, β, and $(1/\hbar)\vec{S}$ forming a basis. The remaining commutation relations are

$$[Q_i, \beta] = i\frac{\lambda^2}{\hbar}P_i\ ,\quad [P_i, \beta] = -\frac{4i\hbar}{\lambda^2}Q_i\ ,$$
$$[Q_i, S_j] = i\hbar\epsilon_{ijk}Q_k\ ,\quad [P_i, S_j] = i\hbar\epsilon_{ijk}P_k\ , \tag{44}$$
$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k\ ,\quad [\beta, S_i] = 0\ .$$

If we write

$$\theta_{ij} = \frac{1}{\hbar}\epsilon_{ijk}S_k\ ,\quad \theta_{i4} = -\theta_{4i} = \frac{1}{\lambda}Q_i\ ,$$
$$\theta_{i5} = -\theta_{5i} = \frac{\lambda}{2\hbar}P_i\ ,\quad \theta_{45} = -\theta_{54} = -\tfrac{1}{2}\beta\ , \tag{45}$$

then the relations (42) and (44) can be written in a standard form for SO(5):

$$[\theta_{ab}, \theta_{cd}] = i(\delta_{ac}\theta_{bd} + \delta_{bd}\theta_{ac} - \delta_{bc}\theta_{ad} - \delta_{ad}\theta_{bc})\ , \tag{46}$$

where a, b, c, and d run over 1, 2, 3, 4, 5. The operators θ_{ab} are all Hermitian. We can say that *the phase space of the Zitterbewegung is curved and compact*. The SO(5) algebra can be compared with the seven-dimensional nilpotent Heisenberg algebra generated by $\vec{Q}$ and $\vec{P}$ in the case of canonical commutation relations, or better, with the ten-dimensional algebra spanned by I, $\vec{Q}$, $\vec{P}$, and $\vec{L} = \vec{Q}\times\vec{P}$, in that case. For the vector $\vec{S}$ can suitably be called the angular momentum vector for the *Zitterbewegung* in this frame, where it is a constant of the motion in view of the last equation of Eqs. (44). We cannot write $\vec{S}$ as the moment of the momentum in the usual sense, i.e.,

$$\vec{S} \neq \vec{Q}\times\vec{P}\ ,$$

but this is not surprising in a curved geometry. We can, however, write

$$\vec{S} = \vec{Q}\times\zeta\vec{P} = -\vec{P}\times\zeta\vec{Q}\ , \tag{47}$$

where ζ is a kind of metric operator for this space:

$$\zeta = -\tfrac{1}{2}\beta = \theta_{45}\ . \tag{48}$$

Again, the Hamiltonian H_r cannot be written in the form $(a_1/m)\vec{P}^2 + a_2 m\omega^2\vec{Q}^2$, as both $\vec{P}^2$ and $\vec{Q}^2$ are c numbers, but we can write

$$H_r = \frac{a_1}{m}\vec{P}\cdot\zeta\vec{P} + a_2 m\omega^2\vec{Q}\cdot\zeta\vec{Q} \tag{49}$$

provided

$$3(a_1 + a_2) = 2\ . \tag{50}$$

We note further that

$$\{Q_i, Q_j\} = \tfrac{1}{2}\lambda^2\delta_{ij}I\ ,\quad \{P_i, P_j\} = 2\left(\frac{\hbar}{\lambda}\right)^2\delta_{ij}I\ ,$$
$$\beta Q_i = -Q_i\beta = -\frac{i\lambda^2}{2\hbar}P_i\ ,\quad \beta P_i = -P_i\beta = 2i\frac{\hbar}{\lambda^2}Q_i\ , \tag{51}$$

and we see that each component of $\vec{Q}$ has just two eigenvalues $\pm\frac{1}{2}\lambda$, and each component of $\vec{P}$ has just two eigenvalues $\pm\hbar/\lambda$. We cannot simultaneously diagonalize $\vec{Q}$ and $\vec{P}$, nor even two different components of $\vec{Q}$ or of $\vec{P}$, but we can at one instant diagonalize Q_1 and P_2, for example. Thus at $t = 0$ we can introduce the four basis states ψ_{++}, ψ_{+-}, ψ_{-+}, ψ_{--}, with

$$Q_1(0)\psi_{+-} = +\tfrac{1}{2}\lambda\,\psi_{+-}\ ,\quad P_2(0)\psi_{+-} = -\frac{\hbar}{\lambda}\psi_{+-}\ , \tag{52}$$

etc. These are not "stationary states," and from Eqs. (41) we can see that at time t we have

$$\left(Q_1(t)\cos\omega t-\frac{\lambda^2}{2\hbar}P_1(t)\sin\omega t\right)\psi_{+-}=+\tfrac{1}{2}\lambda\psi_{+-},$$
$$\left(P_2(t)\cos\omega t+\frac{2\hbar}{\lambda^2}Q_2(t)\sin\omega t\right)\psi_{+-}=-\frac{\hbar}{\lambda}\psi_{+-}. \quad (53)$$

Although each component of $d\vec{Q}/dt\,[=(1/m)\vec{P}]$ has only two eigenvalues $\pm c$, we see from these equations that it is not until $t=\pi/\omega$ that

$$Q_1(t)\psi_{+-}=-\tfrac{1}{2}\lambda\psi_{+-}.$$

Thus the mean speed in the Q_1 direction, when the system is in the state ψ_{+-}, is only $\lambda/(\pi/\omega)$, or c/π.

Stationary states can be defined by diagonalizing H_r and one component of $\vec{S}$. It is clear from the fact that $\vec{Q}$ and $\vec{P}$ anticommute with H_r [see the last of Eqs. (51)] that the expectation values $\langle\vec{Q}\rangle$, $\langle\vec{P}\rangle$ of $\vec{Q}$ and $\vec{P}$ vanish at all times in such states. If we define as usual the uncertainties ΔQ_i, ΔP_i by, for example,

$$(\Delta Q_i)^2=\langle Q_i^2\rangle-\langle Q_i\rangle^2, \quad (54)$$

then it follows from Eqs. (51) that in any stationary state

$$\Delta Q_i=\tfrac{1}{2}\lambda,\ \Delta P_i=\frac{\hbar}{\lambda}. \quad (55)$$

In general, we also have the uncertainty relations

$$\Delta Q_i\Delta Q_j\geq\frac{1}{2}\frac{\lambda^2}{\hbar}|\epsilon_{ijk}\langle S_k\rangle|,$$
$$\Delta P_i\,\Delta P_j\geq\frac{2\hbar}{\lambda^2}|\epsilon_{ijk}\langle S_k\rangle|, \quad (56)$$
$$\Delta Q_i\,\Delta P_j\geq\tfrac{1}{2}\hbar|\langle\beta\rangle|\delta_{ij}.$$

We could now recover the full set of states and algebra of operators associated with Dirac's equation by suitably "boosting" the states and operators associated with the *Zitterbewegung* in the rest frame. However, we prefer to proceed in a slightly different way, beginning with a discussion from "first principles" of a finite, charged, quantum system, and then arriving at Dirac's equation by a quasideductive argument.

IV. A DERIVATION OF DIRAC'S EQUATION

We imagine a system having a compact phase space, with three degrees of freedom for a point carrying a charge e. In quantum mechanics, the associated Hilbert space can be taken to be finite dimensional, and there will act in it a set of coordinate operators Q_i and a set of momentum operators P_i $(i=1,\ 2,\ 3)$ for the charged point. We take these operators to be Hermitian.

Because of the finite dimensionality, it cannot be true that

$$[Q_i,P_j]=i\hbar\delta_{ij}I,$$

as one can see by taking the trace of each side of this equation. However, we *can* suppose that

$$[Q_i,P_j]=-i\hbar\delta_{ij}\beta, \quad (57)$$

where β is traceless and Hermitian, and

$$\beta^2=I. \quad (58)$$

If we suppose further that the Q_i and the P_i form three-vectors, then we require in addition that there act in the Hilbert space three Hermitian operators S_i generating a representation of SO(3) and satisfying

$$[S_i,S_j]=i\hbar\epsilon_{ijk}S_k,\quad [S_i,\beta]=0,$$
$$[S_i,Q_j]=i\hbar\epsilon_{ijk}Q_k,\quad [S_i,P_j]=i\hbar\epsilon_{ijk}P_k. \quad (59)$$

Now what can we take for the commutation relations between the components of $\vec{Q}$? They cannot be taken to commute, because $(1/\lambda)\vec{Q}$ and $(1/\hbar)\vec{S}$ would then satisfy the defining relations of the Lie algebra of the Euclidean group E(3), which has no (faithful) finite-dimensional unitary representations. The next simplest possibility is to have

$$[Q_i,Q_j]=+i\frac{\lambda^2}{\hbar}\epsilon_{ijk}S_k, \quad (60)$$

where λ is a positive constant with dimensions of length. The appearance of such a constant is inevitable in a finite-dimensional Hilbert space. In particular, each of the Q_i will satisfy a polynomial (Cayley-Hamiltonian) identity, with constant, dimensional coefficients. Note that the plus sign could not be replaced by a minus sign on the right-hand side of Eq. (60), since $(1/\lambda)\vec{Q}$ and $(1/\hbar)\vec{S}$ would then satisfy the defining relations of the Lie algebra of the Lorentz group SO(3,1), which also has no (nontrivial) finite-dimensional unitary representations.

By a similar argument, we cannot have the components of $\vec{P}$ commuting, and the next simplest possibility is to have

$$[P_i,P_j]=+ib^2\frac{\hbar}{\lambda^2}\epsilon_{ijk}S_k, \quad (61)$$

where b is a positive dimensionless constant.

Consistency of the relations (57), (59), (60), and (61) with the Jacobi identity requires that

$$[\beta,\ Q_i] = -i\frac{\lambda^2}{\hbar}P_i\,,\quad [\beta,\ P_i] = i\,\frac{b^2\hbar}{\lambda^2}\,Q_i\,, \tag{62}$$

and we arrive at the Lie algebra of SO(5), spanned by $(1/\lambda)\vec{Q}$, $(\lambda/b\hbar)\vec{P}$, $(1/b)\beta$, and $(1/\hbar)\vec{S}$. It is easily checked[15] that the only irreducible representation of SO(5) in which Eq. (58) can hold is the four-dimensional representation and then only if $b=2$. Then the anticommutation relations (51) also hold, and we arrive at the *Zitterbewegung* phase-space algebra of Eqs. (42), (44), and (51).

We remark that Santhanam[16] has recently written about the modification of the canonical commutation relations appropriate to finite-dimensional Hilbert spaces. Following Weyl's idea[17] that one should approach this question by consideration of finite-dimensional, unitary, projective representations of finite Abelian groups, he derived possible commutation relations between one Q and one P in a finite-dimensional space. Our results here provide a particular generalization for the case of three Q's and P's. We note in this connection that the operators

$$A_j = ie^{(i\pi/\lambda)Q_j}\,,\quad B_j = ie^{(i\pi\lambda/2\hbar)P_j} \tag{63}$$

generate under multiplication a representation of the appropriate Weyl group in this case. We have

$$\{A_j, A_k\} = \delta_{jk}I = \{B_j, B_k\}\,,\quad \{A_j, B_k\} = 0\,. \tag{64}$$

To return to our line of reasoning, we next ask what are the possible dynamics of this system with four-dimensional Hilbert space. We know from our familiarity with the Dirac matrices that the only three-scalar Hermitian Hamiltonians one can construct from the $\vec{Q}$, $\vec{P}$, β, and $\vec{S}$ have the form

$$H_r = \frac{\hbar c}{\lambda}(u\beta + vI)\,, \tag{65}$$

where u (which can be assumed non-negative without loss of generality) and v are real, dimensionless constants. [Pseudoscalar contributions like $(c/\lambda^2)\vec{Q}\times\vec{Q}\cdot\vec{P}$ and $(c/\hbar)\vec{P}\times\vec{P}\cdot\vec{Q}$ are excluded from consideration.] Since we then have

$$[Q_i, H_r] = iuc\lambda\, P_i\,,\quad [P_i, H_r] = -4iu\,\frac{c\hbar^2}{\lambda^3}\,Q_i\,, \tag{66}$$

the only possible nontrivial dynamics is that of a harmonic oscillator, with angular frequency

$$\omega = u\,\frac{2c}{\lambda}\,. \tag{67}$$

We note that we then have $d\vec{Q}/dt = uc(\lambda/\hbar)\vec{P}$, so that each dQ_i/dt has eigenvalues $\pm uc$. Furthermore, if $v \geq u$, the eigenvalues of H_r are non-negative. However, we find that if the model we have described is to admit to a relativistic interpretation, we must set $v=0$, $u=1$, so that each dQ_i/dt has eigenvalues $\pm c$, and positive and negative energies appear symmetrically.

We now suppose that the dynamical system we have described is that associated with the internal dynamics in the rest frame of a relativistic "particle," with energy-momentum four-vector $(H/c, \vec{p})$. Thus we suppose that $H = H_r$ when $\vec{p} = \vec{0}$.

Then the quantity (rest mass squared) of the particle has the form

$$M^2 \equiv c^{-4}(H^2 - c^2\vec{p}^{\,2})$$
$$= c^{-4}H_r^{\,2} = \frac{\hbar^2}{\lambda^2 c^2}\,[(u^2+v^2)I + 2uv\beta] \tag{68}$$

in the rest frame. But if $v \neq 0$, we would find it impossible to boost the rest-frame states and operators in such a way that the operator $[(u^2+v^2)I + 2uv\beta]$ does represent the form of an invariant in the rest frame, while $(\hbar c/\lambda)\,(u\beta + vI)$ represents the form of the fourth component of a four-vector (energy-momentum) there. We are forced to take $v=0$, and then we have

$$H_r = mc^2\beta\,,\quad M^2 = m^2 I\,,\ \text{where } m = \frac{u\hbar}{\lambda c}\,. \tag{69}$$

Note that we do not assign any mass to the point charge at $\vec{Q}$. Rather we prescribe the constant λ (which may be thought of as determining the "curvature" of the internal phase space), and the rest-mass energy of the system as a whole, i.e., the "particle," appears as simply the energy of the internal motion in the rest frame.

Now let ψ denote an arbitrary normalized four-component state vector of the internal dynamics. Then any state vector of the particle in its rest frame, in a $\vec{p}$ representation, will have the form

$$\chi_r(\vec{p}) = \text{const}\times\psi\,\delta(\vec{p}) \tag{70}$$

for some such ψ. In order to obtain the state vectors for the particle in an arbitrary frame, we want first to identify for the internal dynamics a suitable dynamical algebra, containing the Lie algebra of the homogeneous Lorentz group and a four-vector operator. Then we shall be able to follow essentially the procedure used elsewhere[18] to obtain a wave equation for the hydrogen atom, regarded as a "relativistic particle."

The compact SO(5) Lie algebra described above is unsuitable for this purpose: we need one of its real, noncompact forms. In particular, we seek boost operators M_i acting in the four-dimensional space and satisfying

$$[M_i, M_j] = -i\hbar\epsilon_{ijk}S_k\,,\quad [M_i, S_j] = i\hbar\epsilon_{ijk}M_k\,. \tag{71}$$

Acting with hindsight, we make the choice

$$\vec{M}=(1/uc)H_r\vec{Q}=-\tfrac{1}{2}i\lambda\vec{P}\,. \tag{72}$$

[Other possibilities would be $\vec{M}=+\frac{1}{2}i\lambda\vec{P}$, $\vec{M}=\pm(i\hbar/\lambda)\vec{Q}$. However, these would lead us eventually to wave equations with the property that minimal coupling to an external electromagnetic field could not be made consistent with the charge being "at $\vec{Q}$" in the rest frame.] A suitable dynamical algebra should include the Hamiltonian H_r, the generators $\vec{S}$ of its invariance algebra, and the operators $\vec{M}$. Then the commutator

$$\frac{i}{\hbar}\,[\,\vec{M},\ H_r\,]=2iu\,\frac{c\hbar}{\lambda^2}\,\vec{Q}$$

must be included, and we find that the Lie algebra closes. Defining

$$\begin{aligned}
S_{ij}&=\frac{1}{\hbar}\,\epsilon_{ijk}S_k\,,\\
S_{i5}&=-S_{5i}=-\frac{1}{\hbar}\,M_i=\frac{1}{2}i\,\frac{\lambda}{\hbar}\,P_i\,,\\
S_{i6}&=-S_{6i}=-\frac{i}{\lambda}\,Q_i\,,\\
S_{56}&=-S_{65}=\tfrac{1}{2}\beta=\frac{1}{2mc^2}\,H_r\,,
\end{aligned} \tag{73}$$

we have

$$\begin{aligned}
i[\,S_{AB},S_{CD}]&=g_{AC}S_{BD}+g_{BD}S_{AC}\\
&\quad-g_{BC}S_{AD}-g_{AD}S_{BC}\,,
\end{aligned} \tag{74}$$

where A, B, C, D run over 1, 2, 3, $5\equiv 0$, 6 and g_{AB} is diagonal, with $g_{55}=g_{66}=1$, $g_{11}=g_{22}=g_{33}=-1$. This is therefore the Lie algebra of SO(3, 2). It does contain a four-vector operator as desired:

$$\gamma^\mu\equiv(\gamma^0,\gamma^i)=(2S^{06},\ 2S^{i6})=\left(\beta,\ \frac{2i}{\lambda}\ Q_i\right)\,. \tag{75}$$

The dynamical algebra can be extended to the Lie algebra of SO(4, 2) if we include the pseudoscalars

$$\begin{aligned}
S_{45}&=-S_{54}=\frac{\lambda}{6\hbar^2}\,\vec{P}\times\vec{P}\cdot\vec{Q}\,,\\
S_{46}&=-S_{64}=-\frac{1}{3\lambda\hbar}\,\vec{Q}\times\vec{Q}\cdot\vec{P}\,,
\end{aligned} \tag{76}$$

and the pseudovector with components

$$S_{i4}=-S_{4i}=\frac{1}{2\hbar}\,\epsilon_{ijk}\,Q_jP_k\,. \tag{77}$$

Then the commutation relations are as in Eq. (74), where now A, B, C, D run over 1, 2, 3, 4, 5, 6, and $g_{44}=-1$. The irreducible, nonunitary four-dimensional representation of SO(4, 2) involved here, remains irreducible when restricted to SO(3, 2). [An inequivalent extension from SO(3, 2) to SO(4, 2) would replace S_{45}, S_{46}, and S_{i4} by $-S_{45}$, $-S_{46}$, and $-S_{i4}$ in Eqs. (76) and (77). Alternatively, two inequivalent extensions from SO(3, 2) to SO(3, 3) can be defined by replacing S_{45}, S_{46}, and S_{i4} by $\pm iS_{45}$, $\pm iS_{46}$, and $\pm iS_{i4}$.] In terms of the operators γ^μ and $\gamma_5(=-\gamma^0\gamma^1\gamma^2\gamma^3)$ we can write the operators of Eqs. (73), (76), and (77) as

$$\begin{aligned}
&S_{ij}=\tfrac{1}{4}i[\,\gamma^i,\gamma^j\,]\,,\quad S_{i5}=\tfrac{1}{2}i\gamma^0\gamma^i\,,\quad S_{i6}=-\tfrac{1}{2}\gamma^i\,,\\
&S_{56}=\tfrac{1}{2}\gamma^0,\quad S_{46}=-\tfrac{1}{2}\gamma_5,\quad S_{45}=\tfrac{1}{2}i\gamma_5\gamma^0\,,\\
&S_{i4}=\tfrac{1}{2}i\gamma^i\gamma_5\,.
\end{aligned} \tag{78}$$

Now we use the operators $\vec{M}$ in defining the state vectors of the particle with $\vec{p}=\hbar\vec{k}$. Consider a vector $\chi_{r\pm}$ of the general form in Eq. (70):

$$\chi_{r\pm}=\text{const}\times\psi_\pm\,\delta(\vec{p}) \tag{79}$$

with

$$H_r\chi_{r\pm}=\pm mc^2\chi_{r\pm}\ \text{or}\ \gamma^0\psi_\pm=\pm\psi_\pm\,. \tag{80}$$

An appropriate Lorentz boost operator for the positive- and negative-energy states is

$$\begin{aligned}
B^\pm(\vec{\theta})&=\exp\!\left(\pm\frac{i}{\hbar}\,\vec{\theta}\cdot\vec{M}\right)\\
&=I\cosh(\tfrac{1}{2}\theta)\pm\frac{2i}{\hbar}\ \frac{\vec{\theta}\cdot\vec{M}}{\theta}\ \sinh(\tfrac{1}{2}\theta)\,,
\end{aligned} \tag{81}$$

where

$$\vec{\theta}=\frac{\vec{k}}{|\vec{k}|}\,\text{arctanh}\!\left(\frac{\hbar|\vec{k}|}{(m^2c^2+\hbar^2\vec{k}^2)^{1/2}}\right),\quad \theta=|\vec{\theta}|\,. \tag{82}$$

The $\pm$ sign in $B^\pm(\vec{\theta})$ is necessary because the particle with positive or negative energy and momentum $\hbar\vec{k}$ is "moving with velocity $\pm c\hbar|\vec{k}|(m^2c^2+\hbar^2\vec{k}^2)^{-1/2}$ $(=\pm c\tanh\theta)$ relative to the rest frame." Then the general form of a positive- or negative-energy state with momentum $\hbar\vec{k}$ is

$$\chi_{\vec{k}\pm}(\vec{p})=\text{const}\times B^\pm(\vec{\theta})\psi_\pm\,\delta(\vec{p}-\hbar\vec{k})\,. \tag{83}$$

In view of Eq. (80), we have

$$B^\pm(\vec{\theta})\gamma^0B^\pm(\vec{\theta})^{-1}\chi_{\vec{k}\pm}=\pm\chi_{\vec{k}\pm}\,,$$

and, noting that $B^\pm(\vec{\theta})^{-1}=B^\mp(\vec{\theta})$, we find that this equation reduces to

$$\gamma^0\!\left(\cosh\theta\mp\frac{2i}{\hbar}\ \frac{\vec{\theta}\cdot\vec{M}}{\theta}\ \sinh\theta\right)\!\chi_{\vec{k}\pm}=\pm\chi_{\vec{k}\pm}\,, \tag{84}$$

i.e.,

$$\hbar\,(\gamma^0k_0-\vec{\gamma}\cdot\vec{k})\chi_{\vec{k}\pm}=mc\,\chi_{\vec{k}\pm}, \tag{85}$$

where

$$\hbar k_0 \chi_{\vec{k}\pm} = \pm (m^2c^2 + \hbar^2 \vec{k}^2)^{1/2} \chi_{\vec{k}\pm} . \tag{86}$$

Since the energy of the state $\chi_{\vec{k}\pm}$ is then $c\hbar k_0$, the Hamiltonian of the particle equals

$$(c\hbar \gamma^0 \vec{\gamma} \cdot \vec{k} + mc^2\gamma^0) \tag{87}$$

when $\vec{p} = \hbar \vec{k}$; and in general then

$$H = c\gamma^0\vec{\gamma} \cdot \vec{p} + mc^2\gamma^0 = c\vec{\alpha} \cdot \vec{p} + mc^2\beta , \tag{88}$$

as in Eq. (1).

We can now go to the usual configuration representation via a Fourier transform, and then go to the Schrödinger picture to obtain the familiar Dirac wave functions $\psi(\vec{x}, t)$ satisfying

$$i\hbar \gamma^\mu \partial_\mu \psi = mc\psi, \quad \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) . \tag{89}$$

Although the coordinate vector $\vec{x}$ introduced in this way is *mathematically* conjugate to $\vec{p}$, it need not be—and indeed it is not—the coordinate vector of the center of mass.

If we now couple the particle to an external electromagnetic field via the minimal-coupling prescription (26), we are in effect saying that the *charge* is at $\vec{x}$ at time t. But then in the rest frame, according to Schrödinger's analysis as described in Sec. I, the relative coordinate of the charge has the value

$$\vec{\xi} = \tfrac{1}{2} i \frac{\hbar}{mc} \vec{\alpha}\beta = \tfrac{1}{2} i \frac{\lambda}{u} \vec{\alpha}\beta = \frac{1}{u}\vec{Q} . \tag{90}$$

Since we began by assuming that the relative coordinate of the charge is $\vec{Q}$ in this frame, we must for consistency now set $u = 1$. Then

$$H_r = \frac{\hbar c}{\lambda}\beta = mc^2\beta , \quad m = \frac{\hbar}{\lambda c} , \tag{91}$$

and we conclude that the only dynamics of the finite, charged quantum system which has an interpretation consistent with relativity is the *Zitterbewegung*.

Dirac's equation has been obtained as the relativistically invariant equation appropriate to the description of this system in an arbitrary moving frame of reference.

V. POSSIBLE GENERALIZATIONS

The ideas discussed above suggest new ways of looking for relativistic wave equations for other elementary particles, starting with an analysis of finite quantum systems. One could remove the constraint (58), and consider representations of SO(5) other than the four-dimensional representation: this would evidently lead to the class of wave equations discussed first by Lubański[19] and Bhabha,[20] and later by many others. One could retain Eqs. (57), (58), and (59), but seek to replace Eqs. (60) and (61) by other relations which would make the algebra generated by $\vec{Q}$ and $\vec{P}$ close on a Lie algebra larger than that of SO(5). Or one could consider finite quantum systems with more than three degrees of freedom.

On the other hand, we know that Majorana's equation (and Dirac's[21] and Staunton's[22] positive-energy wave equations, which are closely related to it) can be interpreted as providing the description, in an arbitrary frame, of an internal two-dimensional dynamical system with an infinite-dimensional Hilbert space.[23] In particular, the internal system can be taken to be a two-dimensional harmonic oscillator. This "infinite quantum system" can also be regarded as a generalization of the finite quantum system we have described above. The dynamical group is again SO(3, 2), but infinite-dimensional unitary representations are now involved. Further generalization to a three-dimensional internal motion leads to the infinite-dimensional unitary representations of SO(4, 2).[24]

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