Positivity condition of Polyhedral realizations of crystal bases

Yuki Kanakubo, Sophia University
(joint work with T. Nakashima)

March 26, 2019 at Osaka City University

Notation

- $\mathfrak{g}$: fin. dim. simple Lie algebra with Cartan matrix $(a_{i,j})$
- $I := \{1, 2, \cdots, r\}$: index set
- $\mathfrak{h} :=< h_j >\subset \mathfrak{g}$
- $\{\alpha_i | i \in I\}$: simple roots
- Linear map $\Lambda_i : \mathfrak{h} \to \mathbb{C}$, $\Lambda_i(h_j) = \delta_{ij}$ is called fundamental weight.
- $P := \bigoplus_{i=1}^{r} \mathbb{Z}\Lambda_i$: weight lattice,
- $P^+ := \bigoplus_{i=1}^{r} \mathbb{Z}_{\geq 0}\Lambda_i$: positive weight lattice.
- W : Weyl group

Outline

$\iota := (\cdots, i_3, i_2, i_1)$: infinite sequence of indices in I
s.t. $i_k \neq i_{k+1}$ $(k \in \mathbb{Z}_{>0})$ and $\sharp\{k \in \mathbb{Z}_{>0} | i_k = j\} = \infty$ (for any $j \in I$).

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Fact(Kashiwara, Nakashima, Zelevinsky)

There exists an embedding of crystal

$$\Psi_\iota : B(\infty) \hookrightarrow \mathbb{Z}^\infty = \{(\cdots, a_3, a_2, a_1) | a_l \in \mathbb{Z},\ a_k = 0\ (k \gg 0)\}.$$

$\mathrm{Im}(\Psi_\iota)$ $(\cong B(\infty))$: **polyhedral realization** of $B(\infty)$.

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Problem1

Find an explicit form of $\mathrm{Im}(\Psi_\iota)$.

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- In the case $\mathfrak{g} = \mathfrak{sl}_{r+1}$, $\iota = (\cdots, r, \cdots, 2, 1, r, \cdots, 2, 1)$, ι satisfies the positivity condition and explicit form of $\mathrm{Im}(\Psi_\iota)$ are given (N,Z).
- For any ι and $\mathfrak{g}$: rank 2, ι satisfies the positivity condition and explicit form of $\mathrm{Im}(\Psi_\iota)$ are given (N,Z).

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- For fin. dim. simple Lie alg $\mathfrak{g}$, $\iota = (\cdots, r, \cdots, 2, 1, r, \cdots, 2, 1)$ satisfies the positivity condition and explicit forms of $\mathrm{Im}(\Psi_\iota)$ are given by Hoshino (2006), J.Kim, D.Shin (2008).

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Problem2

Find a sufficient condition of positivity condition.

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Goal

$\mathfrak{g}$: type A, B, C or D.

- Give a sufficient condition of the positivity condition.
- For ι satisfying the sufficient condition, give an explicit form of $\mathrm{Im}(\Psi_\iota)$.

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Plan

1. Quick review of Crystals
2. Polyhedral realizations
3. Main results
4. Proof

1. Crystals

Crystal

Definition

A **crystal** is a set $\mathcal{B}$ together with the maps $\mathrm{wt} : \mathcal{B} \to P$, $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$ $(i \in I)$ satisfying the followings: For $b, b' \in \mathcal{B}$, $i, j \in I$,

(1) $\varphi_i(b) = \varepsilon_i(b) + \langle \mathrm{wt}(b), h_i \rangle$,

(2) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\tilde{e}_i(b) \in \mathcal{B}$, $\quad \mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}_j(b) - \alpha_i$ if $\tilde{f}_i(b) \in \mathcal{B}$,

(3) $\varepsilon_i(\tilde{e}_i(b)) = \varepsilon_i(b) - 1, \ \ \varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1$ if $\tilde{e}_i(b) \in \mathcal{B}$,

(4) $\varepsilon_i(\tilde{f}_i(b)) = \varepsilon_i(b) + 1, \ \ \varphi_i(\tilde{f}_i(b)) = \varphi_i(b) - 1$ if $\tilde{f}_i(b) \in \mathcal{B}$,

(5) $\tilde{f}_i(b) = b'$ if and only if $b = \tilde{e}_i(b')$,

(6) if $\varphi_i(b) = -\infty$ then $\tilde{e}_i(b) = \tilde{f}_i(b) = 0$.

We call $\tilde{e}_i, \tilde{f}_i$ *Kashiwara operators*.

Tensor product of crystals

Definition

The **tensor product** $\mathcal{B}_1 \otimes \mathcal{B}_2$ of crystals $\mathcal{B}_1$, $\mathcal{B}_2$ is defined to be the set $\mathcal{B}_1 \times \mathcal{B}_2$ whose crystal structure is defined as follows:

(1) $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,

(2) $\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle)$,

(3) $\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \mathrm{wt}(b_2)\rangle)$,

(4) $\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$

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for $i \in I$.

A morphism of crystals

Definition

A **morphism** $\psi : \mathcal{B}_1 \to \mathcal{B}_2$ of crystals $\mathcal{B}_1$, $\mathcal{B}_2$ is a map $\mathcal{B}_1 \bigsqcup \{0\} \to \mathcal{B}_2 \bigsqcup \{0\}$ s.t. $\psi(0) = 0$ and

(1) $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$,

(2) $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$,

(3) $\varphi_i(\psi(b)) = \varphi_i(b)$,

(4) $\psi(\tilde{e}_i(b)) = \tilde{e}_i\psi(b)$ if $\psi(b) \neq 0$ and $\psi(\tilde{e}_i(b)) \neq 0$,

(5) $\psi(\tilde{f}_i(b)) = \tilde{f}_i\psi(b)$ if $\psi(b) \neq 0$ and $\psi(\tilde{f}_i(b)) \neq 0$,

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An injective strict morphism is said to be **strict embedding**.

Examples of Crystal

$U_q(\mathfrak{g}) = U_q(\mathfrak{g})^- U_q(\mathfrak{g})_0 U_q(\mathfrak{g})^+$: triangular decomp. of the quantum group

- Crystal base $B(\lambda)$ of the fin. dim. irr. repr $V(\lambda)$ of $U_q(\mathfrak{g})$ is a crystal ($\lambda \in P^+$).
- Crystal base $B(\infty)$ of $U_q(\mathfrak{g})^-$ is a crystal.

$\exists\ u_\lambda \in B(\lambda)$, $u_\infty \in B(\infty)$,

$$B(\lambda) = \{\tilde{f}_{j_1} \cdots \tilde{f}_{j_l} u_\lambda | l \in \mathbb{Z}_{\geq 0},\ j_1, \cdots j_l \in I\},$$

$$B(\infty) = \{\tilde{f}_{j_1} \cdots \tilde{f}_{j_l} u_\infty | l \in \mathbb{Z}_{\geq 0},\ j_1, \cdots j_l \in I\}.$$

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- $\mathrm{wt}((n)_i) = n\alpha_i$, $\varepsilon_i((n)_i) = -n$, $\varphi_i((n)_i) = n$,
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$$\cdots (-2)_i \xrightarrow{\tilde{e}_i} (-1)_i \xrightarrow{\tilde{e}_i} (0)_i \xrightarrow{\tilde{e}_i} (1)_i \xrightarrow{\tilde{e}_i} (2)_i \xrightarrow{\tilde{e}_i} \cdots$$

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$\Rightarrow B_i$ is a crystal.

A crystal structure of $\mathbb{Z}_\iota^\infty$

$$\mathbb{Z}^\infty := \{\mathbf{x} = (\cdots, x_4, x_3, x_2, x_1) | x_k \in \mathbb{Z},\ x_l = 0 (l \gg 0)\}.$$

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s.t. $i_k \neq i_{k+1}$ $(k \in \mathbb{Z}_{>0})$ and $\sharp\{k \in \mathbb{Z}_{>0} | i_k = j\} = \infty$ (for any $j \in I$).

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$$\sigma_k(\mathbf{x}) := x_k + \sum_{j>k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j \qquad (k \in \mathbb{Z}_{\geq 1},\ \mathbf{x} \in \mathbb{Z}^\infty).$$

By $x_j = 0$ $(j \gg 0)$, $\sigma_k(\mathbf{x})$ is well-defined for $\mathbf{x} \in \mathbb{Z}^\infty$, and we get

$$\sigma_k(\mathbf{x}) = 0 \quad (k \gg 0). \tag{1}$$

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Next, $\sigma^{(i)}(\mathbf{x}) := \max_{k \in \mathbb{Z}_{\geq 1};\ i_k = i} \sigma_k(\mathbf{x})$ $(i \in I)$. By (1), $\sigma^{(i)}(\mathbf{x}) \geq 0$.

$$\sigma_k(\mathbf{x}) = 0 \quad (k \gg 0). \tag{1}$$

We also set

$$M^{(i)} = M^{(i)}(\mathbf{x}) := \{k \in \mathbb{Z}_{\geq 1} | i_k = i,\ \sigma_k(\mathbf{x}) = \sigma^{(i)}(\mathbf{x})\}.$$

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Note that by (1),

$$\max M^{(i)} < \infty \Leftrightarrow \sigma^{(i)}(\mathbf{x}) > 0.$$

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Now we define $\tilde{f}_i : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ and $\tilde{e}_i : \mathbb{Z}^\infty \to \mathbb{Z}^\infty \cup \{0\}$ as

$$(\tilde{f}_i(\mathbf{x}))_k := x_k + \delta_{k, \min M^{(i)}},\ \ (\tilde{e}_i(\mathbf{x}))_k := \begin{cases} x_k - \delta_{k, \max M^{(i)}} & \text{if } \sigma^{(i)}(\mathbf{x}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

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We also define

$$\mathrm{wt}(\mathbf{x}) = -\sum_{j\in\mathbb{Z}_{\geq 1}} x_j \alpha_{i_j},$$

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Theorem (Nakashima, Zelevinsky)

$(\mathbb{Z}^\infty, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \mathrm{wt})$ is a crystal. We denote it by $\mathbb{Z}^\infty_\iota$.

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Next, we will see $B(\infty) \hookrightarrow \mathbb{Z}_\iota^\infty$.
The image of this embedding is called a **polyhedral realization**.

2. Polyhedral realizations

Kashiwara embedding

Theorem (Kashiwara)

For $\forall i \in I$, there uniquely exists a strict embedding

$$\Psi_i : B(\infty) \hookrightarrow B(\infty) \otimes B_i, \quad u_\infty \mapsto u_\infty \otimes (0)_i.$$

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Using this theorem repeatedly, for a sequence $i_l, \cdots, i_1 \in I$, we obtain

$$\Psi_{i_l,\cdots,i_1} = \Psi_{i_l} \circ \Psi_{i_{l-1}} \circ \cdots \circ \Psi_{i_1} : B(\infty) \hookrightarrow B(\infty) \otimes B_{i_l} \otimes B_{i_{l-1}} \otimes \cdots \otimes B_{i_1}.$$

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Fact

We suppose that $\sharp\{1 \le k \le l | i_k = j\} \gg 0$ ($\forall j \in I$). For $b \in B(\infty)$,

$$\Psi_{i_l,\cdots,i_1}(b) \in u_\infty \otimes B_{i_l} \otimes B_{i_{l-1}} \otimes \cdots \otimes B_{i_1}.$$

Polyhedral realization

$\iota := (\cdots, i_3, i_2, i_1)$: infinite sequence of I
s.t. $i_k \neq i_{k+1}$ $(k \in \mathbb{Z}_{>0})$ and $\sharp\{k \in \mathbb{Z}_{>0} | i_k = j\} = \infty$ $(\forall j \in I)$.

Let us construct a map
$\Psi_\iota : B(\infty) \hookrightarrow \mathbb{Z}_\iota^\infty := \{(\cdots, a_l, \cdots, a_2, a_1) | a_l \in \mathbb{Z}, a_k = 0 (k \gg 0)\}$
as follows:

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Then we set $\Psi_\iota(b) := (\cdots, 0, 0, a_m, \cdots, a_1)$.

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Theorem (Nakashima, Zelevinsky)

The above map $\Psi_\iota : B(\infty) \hookrightarrow \mathbb{Z}_\iota^\infty$ is a unique strict embedding of crystals s.t. $\Psi_\iota(u_\infty) = (\cdots, 0, 0, 0)$.

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Definition

$\mathrm{Im}\Psi_\iota (\cong B(\infty))$ is called a **polyhedral realization** of $B(\infty)$.

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$\mathrm{Im}\Psi_\iota (\cong B(\infty))$ is called a **polyhedral realization** of $B(\infty)$.

Problem

Find an explicit form of $\mathrm{Im}(\Psi_\iota)$.

Polyhedral realization

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For $\iota = (\cdots, i_3, i_2, i_1)$ and $k \in \mathbb{Z}_{\geq 1}$,

$$k^- := \max(\{l \in \mathbb{Z}_{\geq 1} | k > l, \ i_k = i_l\} \cup \{0\}),$$
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ex) $\iota = (\cdots, 2, 1, 2, 1, 2, 1) \Rightarrow 1^- = 0$, $2^- = 0$, $3^- = 1$, $4^- = 2$, $5^- = 3$, $6^- = 4$, $\quad 1^+ = 3$, $2^+ = 4$, $3^+ = 5$, $4^+ = 6$.

Calculations of Polyhedral realization

- For $k \in \mathbb{Z}_{\geq 1}$, we define $x_k \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\infty}, \mathbb{Z})$ as $x_k(\cdots, a_3, a_2, a_1) := a_k$.

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$$\beta_k = x_k + \sum_{k<j<k^+} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_{k^+}.$$

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- For $\varphi = \sum c_k x_k \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^\infty, \mathbb{Z})$ and $k \in \mathbb{Z}_{\geq 1}$, we define $S_k(\varphi) \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^\infty, \mathbb{Z})$ as

$$S_k(\varphi) := \begin{cases} \varphi - c_k \beta_k & \text{if } c_k \geq 0, \\ \varphi - c_k \beta_{k^-} & \text{if } c_k < 0. \end{cases}$$

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where we set $\beta_0 := 0$. We also define

$$\Xi_\iota := \{S_{j_l} \cdots S_{j_1} x_{j_0} | l \geq 0, j_0, j_1, \cdots, j_l \geq 1\}.$$

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If $k^- = 0$ ($k \in \mathbb{Z}_{\geq 1}$) then $c_k \geq 0$ for any $\varphi = \sum c_k x_k \in \Xi_\iota$.

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Theorem (Nakashima, Zelevinsky)

If ι satisfies the positivity condition then

$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0, \quad \forall \varphi \in \Xi_\iota\}.$$

An example of the polyhedral realization

Example) $\mathfrak{g}$: type A_2, $\iota = (\cdots, 2, 1, 2, 1, 2, 1)$.

$$1^- = 2^- = 0, \;\; k^- > 0 \; (k > 2).$$

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We write a vector $(\cdots, x_6, x_5, x_4, x_3, x_2, x_1)$ as

$$(\cdots, x_{3,2}, x_{3,1}, x_{2,2}, x_{2,1}, x_{1,2}, x_{1,1}). \qquad (x_{l,1} = x_{2l-1},\ x_{l,2} = x_{2l})$$

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Recall) positivity condition $\Leftrightarrow$ the coefficients of $x_1 = x_{1,1}$ and $x_2 = x_{1,2}$ in each $\varphi \in \Xi_\iota$ are non-negative.

Similarly, we write $S_{l,1} = S_{2l-1}, \ S_{l,2} = S_{2l}$.

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For $k \in \mathbb{Z}_{\geq 1}$, we obtain

$$x_{k,1} \underset{S_{k+1,1}}{\overset{S_{k,1}}{\rightleftarrows}} x_{k,2} - x_{k+1,1} \underset{S_{k+1,2}}{\overset{S_{k,2}}{\rightleftarrows}} -x_{k+1,2},$$

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and other actions are trivial. Thus

$$\Xi_\iota = \{x_{k,1},\ x_{k,2} - x_{k+1,1},\ -x_{k+1,2},\ x_{k,2},\ x_{k+1,1} - x_{k+1,2}, -x_{k+2,1} | k \geq 1\}.$$

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The coefficients of $x_{1,1}$ and $x_{1,2}$ in each $\varphi \in \Xi_\iota$ are non-negative.
$\therefore$ ι **satisfies the positivity condition** and

$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0, \quad \forall \varphi \in \Xi_\iota\}.$$

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- $x_{1,2}-x_{2,1}\geq 0\Rightarrow x_{1,2}\geq x_{2,1}$.
- $x_{2,1}-x_{2,2}\geq 0\Rightarrow x_{2,1}\geq 0$.

$$\begin{aligned}\therefore\qquad \mathrm{Im}(\Psi_\iota)&=\{\mathbf{x}\in\mathbb{Z}^\infty|\varphi(\mathbf{x})\geq 0,\quad \forall\varphi\in\Xi_\iota\}\\&=\{\mathbf{x}\in\mathbb{Z}^\infty|x_{k+1,2}=x_{k+2,1}=0\ (k\geq 1),\ x_{1,2}\geq x_{2,1}\geq 0,\ x_{1,1}\geq 0\}.\end{aligned}$$

An example which does not satisfy the positivity condition

Example) $\mathfrak{g}$: type A_3, $\iota = (\cdots, 2, 1, 2, 3, 2, 1)$.

$$x_1 \overset{S_1}{\to} -x_5 + x_4 + x_2 \overset{S_2}{\to} -x_5 + x_3 \overset{S_5}{\to} -x_4 + x_3 - x_2 + x_1.$$

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Problem

Find a sufficient condition of the positivity condition.

3. A sufficient condition of the positivity condition (Main results)

Infinite sequences adapted to A

$A = (a_{i,j})_{i,j\in I}$: The Cartan matrix of $\mathfrak{g}$

Definition

If ι satisfies the following condition, we say ι is **adapted to** A :
For $i, j \in I$ with $i \neq j$ and $a_{i,j} \neq 0$, the subsequence of ι consisting of i, j is

$$(\cdots, i, j, i, j, i, j, i, j) \quad \text{or} \quad (\cdots, j, i, j, i, j, i, j, i).$$

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Example) $\mathfrak{g}$: type A_3, $\iota = (\cdots, 2, 1, 3, 2, 1, 3, 2, 1, 3)$

- subsequence consisting of 1, 2 : $(\cdots, 2, 1, 2, 1, 2, 1)$
- subsequence consisting of 2, 3 : $(\cdots, 2, 3, 2, 3, 2, 3)$
- Since $a_{1,3} = 0$ we do not need consider the pair 1, 3.

Thus, ι is adapted to A.

Infinite sequences adapted to A

Example) $\mathfrak{g}$: type A_3, $\iota = (\cdots, 3, 2, 1, 3, 2, 3, 1, 2, 3, 1, 2, 1)$

- subsequence consisting of 1, 2 : $(\cdots, 2, 1, 2, 1, 2, 1)$
- subsequence consisting of 2, 3 : $(\cdots, 3, 2, 3, 2, 3, 2)$

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Infinite sequences adapted to A

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- subsequence consisting of 1, 2 : $(\cdots, 2, 1, 2, 2, 1)$

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The following theorem is an answer of the above Problem:

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Theorem (K, Nakashima)

Let $\mathfrak{g}$ be of type A,B,C or D. If ι is adapted to the Cartan matrix of $\mathfrak{g}$ then ι satisfies the positivity condition.

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Theorem (K, Nakashima)

Let $\mathfrak{g}$ be of type A,B,C or D. If ι is adapted to the Cartan matrix of $\mathfrak{g}$ then ι satisfies the positivity condition.

Example) The following ι satisfy the positivity condition ($\mathfrak{g}$:rank 4):

- $\iota = (\cdots, 1, 2, 3, 4, 1, 2, 3, 4)$,
- $\iota = (\cdots, 3, 1, 4, 2, 3, 1, 4, 2, 3, 1, 4, 2)$,
- $\iota = (\cdots, 4, 3, 1, 2, 4, 1, 3, 2, 1, 4, 3, 2)$.

Tableaux description of Ξ_ι

Next, let us consider the following problem:

Problem

Find an explicit form of $\mathrm{Im}(\Psi_\iota)$.

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In what follows, we suppose that ι is adapted to the Cartan matrix of $\mathfrak{g}$ and let us construct **an explicit form** of Ξ_ι by using Young tableaux.

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Recall

$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0, \ \ \forall \varphi \in \Xi_\iota\} \ (\cong B(\infty))$$

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Recall

ι is **adapted to** $A = (a_{ij}) \overset{def}{\Leftrightarrow}$
For $i, j \in I$ $(i \neq j,\ a_{i,j} \neq 0)$, the subsequence of ι consisting of i, j is

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Let $(p_{i,j})_{i \neq j,\ a_{i,j} \neq 0}$ be the set of integers s.t.

$$p_{i,j} = \begin{cases} 1 & \text{if } (\cdots, j, i, j, i, j, i), \\ 0 & \text{if } (\cdots, i, j, i, j, i, j). \end{cases}$$

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For k $(2 \leq k \leq r)$, we set

$$P(k) := \begin{cases} p_{2,1} + p_{3,2} + \cdots + p_{r-2,r-3} + p_{r,r-2} & \text{if } k = r,\ \mathfrak{g} : \text{type } \mathrm{D}_r, \\ p_{2,1} + p_{3,2} + p_{4,3} + \cdots + p_{k,k-1} & \text{if o.w.} \end{cases}$$

Tableaux descriptions

$$\iota = (\cdots, i_k, \cdots, i_3, i_2, i_1).$$

For $k \in \mathbb{Z}_{\geq 1}$, we write

$$x_k = x_{s,j}, \quad S_k = S_{s,j},$$

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if $i_k = j$ and j is appearing s times in $i_k, i_{k-1}, \cdots, i_1$.

Example) $\iota = (\cdots, 2, 1, 3, 2, 1, 3, 2, 1, 3)$
$(\cdots, x_7, x_6, x_5, x_4, x_3, x_2, x_1) = (\cdots, x_{3,3}, x_{2,2}, x_{2,1}, x_{2,3}, x_{1,2}, x_{1,1}, x_{1,3})$

Example) $\iota = (\cdots, 1, 2, 3, 1, 2, 1)$
$(\cdots, x_6, x_5, x_4, x_3, x_2, x_1) = (\cdots, x_{3,1}, x_{2,2}, x_{1,3}, x_{2,1}, x_{1,2}, x_{1,1})$

Tableaux descriptions

$\mathfrak{g} = \mathrm{A}_r$ case

For $1 \leq j \leq r+1$ and $s \in \mathbb{Z}$, we set

$$\boxed{j}^{\mathrm{A}}_{s} := x_{s+P(j),j} - x_{s+P(j-1)+1,j-1} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\infty}, \mathbb{Z}).$$

$(x_{m,0} = x_{m,r+1} = 0$ for $m \in \mathbb{Z}$, and $x_{m,i} = 0$ $(m \leq 0, i \in I))$.

Tableaux descriptions

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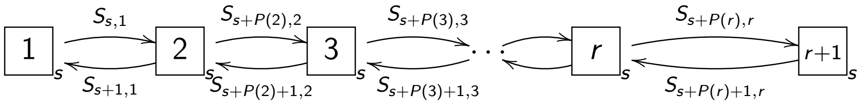
$(x_{m,0} = x_{m,r+1} = 0$ for $m \in \mathbb{Z}$, and $x_{m,i} = 0$ $(m \le 0, i \in I))$.

$\boxed{j}_s = \boxed{j}^{\mathrm{A}}_s$ are obtained from $\boxed{1}_s = x_{s,1}$ by operators $S_{m,j}$ $(1 \le s)$:



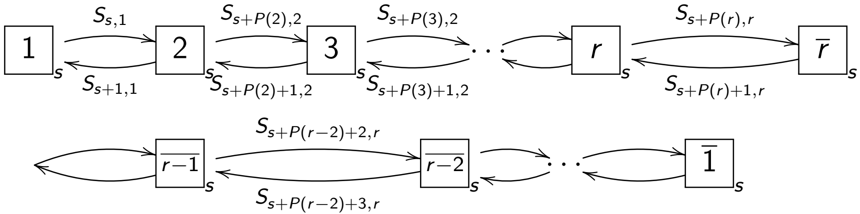
Tableaux descriptions

$\mathfrak{g} = \mathrm{B}_r$ case

For $1 \le j \le r$ and $s \in \mathbb{Z}$, we set

$$\boxed{j}^{\mathrm{B}}_{s} := x_{s+P(j),j} - x_{s+P(j-1)+1,j-1},$$

$$\boxed{\bar{j}}^{\mathrm{B}}_{s} := x_{s+P(j-1)+r-j+1,j-1} - x_{s+P(j)+r-j+1,j}.$$



Tableaux descriptions

$\mathfrak{g} = \mathrm{C}_r$ case

For $1 \le j \le r-1$ and $s \in \mathbb{Z}$, we set

$$\boxed{j}^{\mathrm{C}}_s := x_{s+P(j),j} - x_{s+P(j-1)+1,j-1},$$

$$\boxed{r}^{\mathrm{C}}_s := 2x_{s+P(r),r} - x_{s+P(r-1)+1,r-1},$$

$$\boxed{\bar{r}}^{\mathrm{C}}_s := x_{s+P(r-1)+1,r-1} - 2x_{s+P(r)+1,r},$$

$$\boxed{\bar{j}}^{\mathrm{C}}_s := x_{s+P(j-1)+r-j+1,j-1} - x_{s+P(j)+r-j+1,j}.$$

Tableaux descriptions

$\mathfrak{g} = \mathrm{D}_r$ case

For $s \in \mathbb{Z}$, we set

$$\boxed{j}^{\mathrm{D}}_s := x_{s+P(j),j} - x_{s+P(j-1)+1,j-1}, \ (1 \le j \le r-2, \ j = r),$$

$$\boxed{r-1}^{\mathrm{D}}_s := x_{s+P(r-1),r-1} + x_{s+P(r),r} - x_{s+P(r-2)+1,r-2},$$

$$\boxed{\overline{r}}^{\mathrm{D}}_s := x_{s+P(r-1),r-1} - x_{s+P(r)+1,r},$$

$$\boxed{\overline{r-1}}^{\mathrm{D}}_s := x_{s+P(r-2)+1,r-2} - x_{s+P(r-1)+1,r-1} - x_{s+P(r)+1,r},$$

$$\boxed{\overline{j}}^{\mathrm{D}}_s := x_{s+P(j-1)+r-j,j-1} - x_{s+P(j)+r-j,j}, \ (1 \le j \le r-2).$$

Tableaux descriptions

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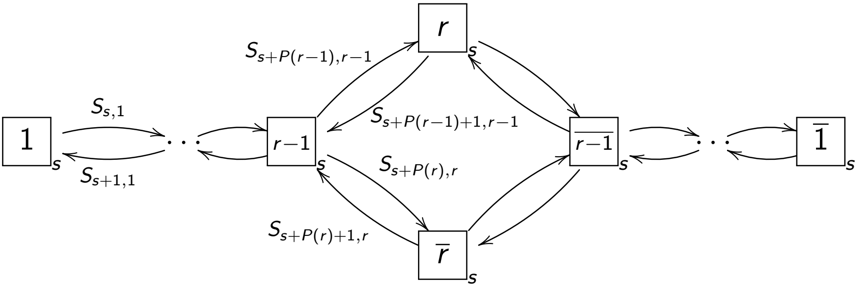

Tableaux descriptions

For $\mathrm{X} = \mathrm{A}$, B, C or D,

$$\begin{array}{|c|}\hline j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_{k-1} \\ \hline j_k \\ \hline \end{array}^{\mathrm{X}}_{s} := \boxed{j_k}^{\mathrm{X}}_{s} + \boxed{j_{k-1}}^{\mathrm{X}}_{s+1} + \cdots + \boxed{j_2}^{\mathrm{X}}_{s+k-2} + \boxed{j_1}^{\mathrm{X}}_{s+k-1}$$

Partial order set

Let us define the following posets:

- $J_A := \{1, 2, \cdots, r, r+1\}, \quad 1 < 2 < \cdots < r < r+1.$
- $J_B = J_C := \{1, 2, \cdots, r, \overline{r}, \cdots, \overline{2}, \overline{1}\},$

$$1 < 2 < \cdots < r < \overline{r} < \cdots < \overline{2} < \overline{1}.$$

- $J_D := \{1, 2, \cdots, r, \overline{r}, \cdots, \overline{2}, \overline{1}\},$

$$1 < 2 < \cdots < r-1 < \begin{matrix} r \\ \overline{r} \end{matrix} < \overline{r-1} < \cdots < \overline{2} < \overline{1}.$$

For $j \in \{1, 2, \cdots, r\}$, we set $|j| = |\overline{j}| = j$.

Tableaux descriptions

For $\mathrm{X} = \mathrm{A}, \mathrm{B}$,

$$\mathrm{Tab}_{\mathrm{X}} := \{ \boxed{\begin{array}{c} j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \end{array}}^{\mathrm{X}}_{s} \mid k \in I,\ j_i \in J_X,\ 1 - P(k) \leq s,\ (*)^{\mathrm{X}}_k \}$$

$(*)^{\mathrm{A}}_k : 1 \leq j_1 < j_2 < \cdots < j_k \leq r+1,$

$$(*)^{\mathrm{B}}_k : \begin{cases} 1 \leq j_1 < j_2 < \cdots < j_k \leq \overline{1} & \text{for } k < r, \\ 1 \leq j_1 < j_2 < \cdots < j_r \leq \overline{1}, \quad |j_l| \neq |j_m|\ (l \neq m) & \text{for } k = r. \end{cases}$$

Tableaux descriptions

$$\mathrm{Tab_C} := \left\{ \boxed{\begin{array}{c} j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \end{array}}^{\mathrm{C}}_{s} \middle| k \in \{1, \cdots, r-1\},\ j_i \in J_C,\ 1 - P(k) \le s,\ (*)^{\mathrm{C}}_k \right\}$$

$$\cup \left\{ \boxed{\begin{array}{c} \overline{r+1} \\ \hline j_1 \\ \hline \vdots \\ \hline j_t \end{array}}^{\mathrm{C}}_{s} \middle| 0 \le t \le r,\ j_i \in J_C,\ 1 - P(r) \le s,\ (*)^{\mathrm{C}}_r \right\},$$

where $\boxed{\overline{r+1}}^{\mathrm{C}}_{s} := x_{s+P(r),r}$, $(*)^{\mathrm{C}}_k$: $\begin{cases} 1 \le j_1 < \cdots < j_k \le \overline{1} & \text{for } k < r, \\ \overline{r} \le j_1 < \cdots < j_t \le \overline{1} & \text{for } k = r. \end{cases}$

Tableaux descriptions

$$\mathrm{Tab_D} := \{\boxed{\begin{array}{c} j_1 \\ \vdots \\ j_k \end{array}}^{\mathrm{D}}_{s} \mid k \in \{1, \cdots, r-2\},\ j_i \in J_D,\ 1 - P(k) \leq s,\ (*)^{\mathrm{D}}_{k}\}$$

$$\cup \{\boxed{\begin{array}{c} \overline{r+1} \\ j_1 \\ \vdots \\ j_t \end{array}}^{\mathrm{D}}_{s} \mid 0 \leq t \leq r,\ j_i \in J_D,\ 1 - P(r-1) \leq s,\ (*)^{\mathrm{D}}_{r-1}\}$$

$$\cup \{\boxed{\begin{array}{c} \overline{r+1} \\ j_1 \\ \vdots \\ j_t \end{array}}^{\mathrm{D}}_{s} \mid 0 \leq t \leq r,\ j_i \in J_D,\ 1 - P(r) \leq s,\ (*)^{\mathrm{D}}_{r}\}.$$

Tableaux descriptions

$$\boxed{\overline{r+1}}^{\mathrm{D}}_{s} := x_{s+P(r),r},$$

$$(*)^{\mathrm{D}}_{k} : \begin{cases} 1 \leq j_1 < j_2 < \cdots < j_k \leq \overline{1} & \text{for } k < r-1, \\ \overline{r} \leq j_1 < \cdots < j_t \leq \overline{1},\ t \text{ is odd} & \text{for } k = r-1, \\ \overline{r} \leq j_1 < \cdots < j_t \leq \overline{1},\ t \text{ is even} & \text{for } k = r. \end{cases}$$

Explicit forms of Ξ_ι via tableaux descriptions

Theorem (K, Nakashima)

$\mathfrak{g}$: type X (=A, B, C, or D), ι : adapted to the Cartan matrix of $\mathfrak{g}$.

$$\Xi_\iota = \mathrm{Tab}_{\mathrm{X}}.$$

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$\Rightarrow$An explicit form of Ξ_ι via Young tableaux.

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$\Rightarrow$An explicit form of Ξ_ι via Young tableaux.

Remark

If X=B, C or D then the tableaux descriptions of the homomorphisms are not unique:

$$\text{Example)} \qquad \boxed{\begin{array}{c} r \\ \hline \overline{r} \end{array}}_{s}^{\mathrm{B}} = \boxed{\begin{array}{c} r-1 \\ \hline \overline{r-1} \end{array}}_{s}^{\mathrm{B}}.$$

Explicit forms of $\mathrm{Im}(\Psi_\iota)$ via tableaux descriptions

Corollay

$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0, \ \forall \varphi \in \mathrm{Tab}_X\}$$

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$$= \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0,\ \forall \varphi \in \mathrm{Tab}^r_X, \quad x_{m,i} = 0\ (\forall i \in I,\ m > r).\}$$

$$\mathrm{Tab}^r_X := \{ \begin{array}{|c|}\hline j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \\ \hline \end{array}{}^{X}_{s} \in \mathrm{Tab}_X | s \leq r \} \ :\ \text{a finite set}$$

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Corollay

$$\begin{aligned} &\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0,\ \forall \varphi \in \mathrm{Tab}_X\} \\ = \ &\{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0,\ \forall \varphi \in \mathrm{Tab}_X^r, \quad x_{m,i} = 0\ (\forall i \in I,\ m > r).\} \end{aligned}$$

$$\mathrm{Tab}_X^r := \left\{ \boxed{\begin{array}{c} j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \end{array}}^{\mathrm{X}}_{s} \in \mathrm{Tab}_X | s \leq r \right\} \ : \ \text{a finite set}$$

Thus, $\mathrm{Im}(\Psi_\iota)$ can be written via finitely many inequalities and finitely many variables $x_{m,i}$.

Explicit forms of Ξ_ι via tableaux descriptions

Example) $\mathfrak{g}$: type A_2, $\iota = (\cdots, 2, 1, 2, 1, 2, 1)$. We have $p_{2,1} = 0$.

$$\begin{aligned}
&\mathrm{Tab}^2_A \\
&= \{\boxed{i}^{\mathrm{A}}_s \,|\, 1 \le i \le 3,\ 1 \le s \le 2\} \cup \{\begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array}^{\mathrm{A}}_s \,|\, 1 \le i < j \le 3,\ 1 \le s \le 2\} \\
&= \{x_{s,i} - x_{s+1,i-1} | 1 \le i \le 3,\ 1 \le s \le 2\} \\
&\quad \cup \{x_{s+1,i} - x_{s+2,i-1} + x_{s,j} - x_{s+1,j-1} | 1 \le i < j \le 3,\ 1 \le s \le 2\}.
\end{aligned}$$

Explicit forms of Ξ_ι via tableaux descriptions

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&\mathrm{Tab}_A^2 \\
&= \{\boxed{i}_s^{\mathrm{A}} \,|\, 1 \le i \le 3,\ 1 \le s \le 2\} \cup \{\begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array}_s^{\mathrm{A}} \,|\, 1 \le i < j \le 3,\ 1 \le s \le 2\} \\
&= \{x_{s,i} - x_{s+1,i-1} | 1 \le i \le 3,\ 1 \le s \le 2\} \\
&\quad \cup \{x_{s+1,i} - x_{s+2,i-1} + x_{s,j} - x_{s+1,j-1} | 1 \le i < j \le 3,\ 1 \le s \le 2\}.
\end{aligned}$$

$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \ge 0,\ \forall \varphi \in \mathrm{Tab}_A^2,\ x_{m,1} = x_{m,2} = 0\ (3 \le m)\}$$

Explicit forms of Ξ_ι via tableaux descriptions

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&= \{x_{s,i} - x_{s+1,i-1} \mid 1 \le i \le 3,\ 1 \le s \le 2\} \\
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\end{aligned}$$

$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty \mid \varphi(\mathbf{x}) \ge 0,\ \forall \varphi \in \mathrm{Tab}^2_A,\ x_{m,1} = x_{m,2} = 0\ (3 \le m)\}$$
$$= \{\mathbf{x} \in \mathbb{Z}^\infty \mid x_{s,i} - x_{s+1,i-1} \ge 0\ (1 \le i \le 3,\ 1 \le s \le 2),\ x_{m,i} = 0\ (3 \le m)\}$$

Rem) $x_{s,i} - x_{s+1,i-1} \ge 0\ (1 \le i \le 3,\ 1 \le s) \Rightarrow$
$x_{s+1,i} - x_{s+2,i-1} + x_{s,j} - x_{s+1,j-1} \ge 0\ (1 \le i < j \le 3,\ 1 \le s)$.

Explicit forms of Ξ_ι via tableaux descriptions

Example) $\mathfrak{g}$: type A_2, $\iota = (\cdots, 2, 1, 2, 1, 2, 1)$. We have $p_{2,1} = 0$.

Furthermore,

$\mathrm{Im}(\Psi_\iota) =$

$\{\mathbf{x} \in \mathbb{Z}^\infty | x_{s,i} - x_{s+1,i-1} \geq 0 \, (1 \leq i \leq 3, \ 1 \leq s \leq 2), \ x_{m,i} = 0 \, (3 \leq m)\}$

$= \{\mathbf{x} \in \mathbb{Z}^\infty | x_{1,2} \geq x_{2,1} \geq 0, \ x_{1,1} \geq 0, \ x_{m+1,2} = x_{m+2,1} = 0, \ \forall m \in \mathbb{Z}_{\geq 1}\}.$

Remark

In this way, when we decide ι (so that $p_{i,j}$), reductions of inequalities are happened. Thus

$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0, \ \forall \varphi \in \mathrm{Tab}^r_X, \ x_{m,i} = 0 \, (\forall i \in I, \ m > r)\}$
includes unnecessary inequalities.

4. Proof

Proof

In the case $\mathfrak{g}$ is of type A_r and ι is adapted to the Cartan matrix of $\mathfrak{g}$, let us prove

- ι satisfies the positivity condition.
- $\Xi_\iota = \mathrm{Tab}_{\mathrm{A}}$.

Proof

Lemma

For $l \in \mathbb{Z}_{\geq 1}$, we have $S_l \mathrm{Tab}_{\mathrm{A}} \subset \mathrm{Tab}_{\mathrm{A}}$.

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[Proof.]
We set $P(m) := p_{2,1} + p_{3,2} + \cdots + p_{m,m-1}$ $(1 \leq m \leq r)$. $(P(1) := 0)$.

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We set $P(m) := p_{2,1} + p_{3,2} + \cdots + p_{m,m-1}$ $(1 \leq m \leq r)$. $(P(1) := 0)$.

Let $T \in \mathrm{Tab}_{\mathrm{A}}$ be the following tableau:

$$T = \boxed{\begin{array}{c} j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \end{array}}_s = \sum_{i=1}^{k} \boxed{j_i}_{s+k-i} = \sum_{i=1}^{k} (x_{s+k-i+P(j_i),j_i} - x_{s+k-i+1+P(j_i-1),j_i-1})$$

with $1 - P(k) \leq s$ and $k \in I$.

Proof

Note that since

$$
\begin{aligned}
& \boxed{j_i}_{s+k-i} + \boxed{j_{i+1}}_{s+k-i-1} \\
= \quad & x_{s+k-i+P(j_i),j_i} - x_{s+k-i+1+P(j_i-1),j_i-1} \\
& + x_{s+k-i-1+P(j_{i+1}),j_{i+1}} - x_{s+k-i+P(j_{i+1}-1),j_{i+1}-1},
\end{aligned}
$$

if $j_{i+1} = j_i + 1$ then we get

$$
\begin{aligned}
& \boxed{j_i}_{s+k-i} + \boxed{j_i+1}_{s+k-i-1} \\
= \quad & x_{s+k-i+P(j_i),j_i} - x_{s+k-i+1+P(j_i-1),j_i-1} \\
& + x_{s+k-i-1+P(j_i+1),j_i+1} - x_{s+k-i+P(j_i),j_i}
\end{aligned}
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if $j_{i+1} = j_i + 1$ then we get

$$\begin{aligned} & \boxed{j_i}_{s+k-i} + \boxed{j_i+1}_{s+k-i-1} \\ = \ & \cancel{x_{s+k-i+P(j_i),j_i}} - x_{s+k-i+1+P(j_i-1),j_i-1} \\ & + x_{s+k-i-1+P(j_i+1),j_i+1} - \cancel{x_{s+k-i+P(j_i),j_i}} \\ = \ & x_{s+k-i-1+P(j_i+1),j_i+1} - x_{s+k-i+1+P(j_i-1),j_i-1}. \end{aligned}$$

Proof

Recall

- $T = \sum_{i=1}^{k} (x_{s+k-i+P(j_i),j_i} - x_{s+k-i+1+P(j_i-1),j_i-1})$
- For $\varphi = \sum c_k x_k \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\infty}, \mathbb{Z})$ and $k \in \mathbb{Z}_{\geq 1}$,

$$S_k(\varphi) := \begin{cases} \varphi - c_k\beta_k & \text{if } c_k \geq 0, \\ \varphi - c_k\beta_{k^-} & \text{if } c_k < 0. \end{cases}$$

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Thus, we have $S_l T = T \in \mathrm{Tab}_{\mathrm{A}}$ except for the following case:

$S_l = S_{m,j}$ and (m, j) belongs to

$$\{(s+k-i+P(j_i), j_i) | i = 1, 2, \cdots, k, \quad j_{i+1} > j_i + 1\}$$
$$\cup \{(s+k-i+1+P(j_i-1), j_i-1) | i = 1, 2, \cdots, k \quad j_i - 1 > j_{i-1}\}.$$

Proof

Recall

$$\boxed{j_i}_{\,s+k-i} \;\underset{S_{s+k-i+P(j_i)+1,j_i}}{\overset{S_{s+k-i+P(j_i),j_i}}{\rightleftarrows}}\; \boxed{j_i+1}_{\,s+k-i}$$

If $S_l = S_{m,j}$ and $(m,j) = (s+k-i+P(j_i), j_i)$ with $j_{i+1} > j_i + 1$ then

$$S_l T = S_l \begin{array}{|c|}\hline j_1 \\ \hline \vdots \\ \hline {\color{red} j_i} \\ \hline j_{i+1} \\ \hline \vdots \\ \hline j_k \\ \hline \end{array}_{\,s} = \begin{array}{|c|}\hline j_1 \\ \hline \vdots \\ \hline {\color{red} j_i+1} \\ \hline j_{i+1} \\ \hline \vdots \\ \hline j_k \\ \hline \end{array}_{\,s} \in \mathrm{Tab}_{\mathrm{A}}.$$

Proof

If $S_l = S_{m,j}$ and $(m, j) = (s + k - i + 1 + P(j_i - 1), j_i - 1)$ with $j_i - 1 > j_{i-1}$ then

$$S_l T = S_l \begin{array}{|c|}\hline j_1 \\ \hline \vdots \\ \hline j_{i-1} \\ \hline j_i \\ \hline \vdots \\ \hline j_k \\ \hline \end{array}_{\,s} = \begin{array}{|c|}\hline j_1 \\ \hline \vdots \\ \hline j_{i-1} \\ \hline j_i - 1 \\ \hline \vdots \\ \hline j_k \\ \hline \end{array}_{\,s} \in \mathrm{Tab}_{\mathrm{A}}.$$

Thus we have verified $S_l \mathrm{Tab}_A \subset \mathrm{Tab}_A$. □

Proof

For $s \in \mathbb{Z}_{\geq 1}$ and $k \in I$, we can verify

$$x_{s,k} = \sum_{i=1}^{k} (x_{s-P(k)+k-i+P(i),i} - x_{s-P(k)+k-i+1+P(i-1),i-1})$$

$$= \boxed{\begin{array}{c} 1 \\ \hline 2 \\ \hline \vdots \\ \hline k \end{array}}_{s-P(k)} \in \mathrm{Tab}_{\mathrm{A}}.$$

Hence by the previous lemma $S_l \mathrm{Tab}_{\mathrm{A}} \subset \mathrm{Tab}_{\mathrm{A}}$, we obtain

$$\Xi_\iota = \{S_{j_m} \cdots S_{j_1} x_{s,k} | m \geq 0, j_1, \cdots, j_m \geq 1, \ s \in \mathbb{Z}_{\geq 1}, \ k \in I\} \subset \mathrm{Tab}_{\mathrm{A}}.$$

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Corollay

ι satisfies the positivity condition.

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[Proof.]

Recall

ι satisfies positivity condition $\Leftrightarrow$ the coefficient of $x_{1,j}$ $(j \in I)$ in each $\varphi \in \Xi_\iota \subset \mathrm{Tab}_{\mathrm{A}}$ is not negative.

Proof

Corollay

ι satisfies the positivity condition.

[Proof.]

Recall

ι satisfies positivity condition $\Leftrightarrow$ the coefficient of $x_{1,j}$ $(j \in I)$ in each $\varphi \in \Xi_\iota \subset \mathrm{Tab}_{\mathrm{A}}$ is not negative.

Recall

Each tableau in $\mathrm{Tab}_{\mathrm{A}}$ is in the following form:

$$T = \sum_{i=1}^{k} (x_{s+k-i+P(j_i),j_i} - x_{s+k-i+1+P(j_i-1),j_i-1})$$

with $1 \leq j_1 < \cdots < j_k \leq r+1$, $k \in I$ and $1 - P(k) \leq s$.

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Note that $j_i \geq i$ $(1 \leq i \leq k)$. If $j_i > i$ $(\Leftrightarrow j_i - 1 \geq i)$ then by $p_{l,l-1} = 1$ or 0 $(l = 2, 3, \cdots, r)$,

$$\begin{aligned} & k - i + 1 + P(j_i - 1) = k - i + 1 + p_{2,1} + p_{3,2} + \cdots + p_{j_i-1,j_i-2} \\ \geq\ & k - i + 1 + p_{2,1} + p_{3,2} + \cdots + p_{i,i-1} \\ \geq\ & p_{2,1} + p_{3,2} + \cdots + p_{i,i-1} + p_{i+1,i} + \cdots + p_{k,k-1} + 1 \\ =\ & P(k) + 1 \geq 2 - s.\ \ (\because 1 - P(k) \leq s) \end{aligned}$$

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Thus we have $s + k - i + 1 + P(j_i - 1) \geq 2$.

Proof

$$T = \sum_{i=1}^{k} (x_{s+k-i+P(j_i),j_i} - x_{s+k-i+1+P(j_i-1),j_i-1}).$$

$(1 \leq j_1 < \cdots < j_k \leq r+1,\ k \in I$ and $1 - P(k) \leq s)$.

If $j_i = i$ then we have $j_1 = 1$, $j_2 = 2, \cdots, j_{i-1} = i-1$. Recall that if $j_i = j_{i-1} + 1$ then $x_{s+k-i+1+P(j_i-1),j_i-1}$ are cancelled. □

Proof

[Proof of $\Xi_\iota = \mathrm{Tab}_{\mathrm{A}}$]

We have already proved $\Xi_\iota \subset \mathrm{Tab}_{\mathrm{A}}$.
Let us take $T \in \mathrm{Tab}_{\mathrm{A}}$,

$$T = \begin{array}{|c|}\hline j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \\ \hline \end{array}_{\,s} = \sum_{i=1}^{k} (x_{s+k-i+P(j_i),j_i} - x_{s+k-i+1+P(j_i-1),j_i-1}),$$

and prove $T \in \Xi_\iota$.

Proof

Recall that we have

$$\boxed{i}_{s+k-i} \xrightarrow{S_{s+k-i+P(i),i}} \boxed{i+1}_{s+k-i} \xrightarrow{S_{s+k-i+P(i+1),i+1}} \cdots \xrightarrow{S_{s+k-i+P(j_i-1),j_i-1}} \boxed{j_i}_{s+k-i}$$

for $i \in \{1, 2, \cdots, k\}$.

Proof

Since

$$x_{s,k} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline k-1 \\ \hline k \\ \hline \end{array}_{\,s-P(k)} , \quad T = \begin{array}{|c|}\hline j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \\ \hline \end{array}_{\,s}$$

we obtain

$$T = (S_{*,j_1-1} \cdots S_{*,2}S_{*,1}) \cdots (S_{*,j_{k-1}-1} \cdots S_{*,k}S_{*,k-1})(S_{*,j_k-1} \cdots S_{*,k+1}S_{*,k})x_{s,k} \in \Xi_\iota.$$

□

Conclusion and remarks

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$$\iota \text{ is adapted to the Cartan matrix of } \mathfrak{g} \Rightarrow \iota \text{ satisfies the positivity condition.}$$

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- We didn't try the exceptional cases or Kac-Moody cases, and the crystal bases $B(\lambda)$.
- 'ι satisfies the positivity condition $\Rightarrow \iota$ is adapted' does **not** hold.

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- In the case $\mathfrak{g}$ is simple, we can construct a Kashiwara embedding
$$\Psi_{\mathbf{i}} : B(\infty) \hookrightarrow \mathbb{Z}^{l(w_0)}$$
by using a reduced word $\mathbf{i}$ of the longest element instead of an infinite sequence ι.

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- If $\mathbf{i}$ is adapted, we can extend it to an infinite sequence ι adapted to the Cartan matrix of $\mathfrak{g}$.
$$\mathrm{Ex)}\ \mathbf{i} = (3,1,2,3,1,2) \Rightarrow \iota = (\cdots, 3,1,2,3,1,2,3,1,2).$$

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$$\mathrm{Ex)}\ \mathbf{i} = (3,1,2,3,1,2) \Rightarrow \iota = (\cdots, 3,1,2,3,1,2,3,1,2).$$
Then we obtain
$$\begin{aligned}\mathrm{Im}(\Psi_\iota) &= \{\mathbf{x} \in \mathbb{Z}^\infty | x_k = 0 \ \ (k > l(w_0)),\ \varphi(\mathbf{x}) \geq 0\ \forall \varphi \in \Xi_\iota\} \\ &= \mathrm{Im}(\Psi_{\mathbf{i}}).\end{aligned}$$

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Open problem

- The following is true or not? :

  $\mathbf{i}$ is **not** adapted to the Cartan matrix of $\mathfrak{g} \Rightarrow$ we can **not** extend $\mathbf{i}$ to ι satisfying the positivity condition.