RSK correspondence of type D and affine crystals [1]

Jae-Hoon Kwon
(joint work with Il-Seung Jang)

Seoul National University

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Motivation

- $\mathfrak{g}$: a classical Lie algebra with $\mathfrak{b}$ a Borel subalgebra
- $\mathfrak{l}$: proper maximal Levi subalgebra of (sum of) type A
- $\mathfrak{p} = \mathfrak{l} + \mathfrak{b}$: the parabolic subalgebra
- $\mathfrak{u}^-$: the negative nilradical of $\mathfrak{p}$ with $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}^-$
- $U(\mathfrak{u}^-)$ has a multiplicity-free decomposition as $\mathfrak{l}$-module
- The expansion into irreducible $\mathfrak{l}$-characters of

$$\mathrm{ch} U(\mathfrak{u}^-) = \prod_{\alpha \in \Phi(\mathfrak{u}^-)} (1 - e^{\alpha})^{-1}$$

gives the well-known Cauchy identity and Littlewood identity

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- This decomposition has a rich combinatorial structure
- A bijective proof of the character identity is given by

  RSK correspondence and its variation

- It also has a connection with quantum affine algebra since

$$\mathrm{ch}\, U(\mathfrak{u}^-) = \lim_{s \to \infty} e^{-s\omega_r} \mathrm{ch}\, W_s^{(r)}$$

  where $W_s^{(r)}$ is a KR module which is "classically irreducible"

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PBW basis and crystal

- $U_q(\mathfrak{g}) = \langle e_i, f_i, t_i \,|\, i \in I \,\rangle$: the quantum group of $\mathfrak{g}$ over $\mathbb{Q}(q)$
- $U_q^- = \langle\, f_i \,|\, i \in I \,\rangle$: the negative part of $U_q(\mathfrak{g})$
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- w_0 : the longest element of length N in W
- $R(w_0)$: the set of reduced expression $(i_1, \ldots, i_N)$ of w_0

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- For $\mathbf{i} = (i_1, \ldots, i_N) \in R(w_0)$ and $\mathbf{c} = (c_1, \ldots, c_N) \in \mathbb{Z}_+^N$,

$$b_{\mathbf{i}}(\mathbf{c}) = f_{i_1}^{(c_1)} T_{i_1}(f_{i_2}^{(c_2)}) \cdots T_{i_1} T_{i_2} \cdots T_{i_{N-1}}(f_{i_N}^{(c_N)}) \in U_q^-$$

  where T_i : an automorphism of $U_q(\mathfrak{g})$ ($T_i = T''_{i,1}$)

- $B_{\mathbf{i}} = \{ b_{\mathbf{i}}(\mathbf{c}) \,|\, \mathbf{c} \in \mathbb{Z}_+^N \}$: a basis of U_q^-

- $L(\infty) = \bigoplus_{v \in B_{\mathbf{i}}} A_0 v$ and $\pi : L(\infty) \to L(\infty)/qL(\infty)$

  $B(\infty) := \pi(B_{\mathbf{i}})$: the crystal associated to U_q^-

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- Recall that for $\mathbf{c} = (c_1, c_2, \ldots, c_N) \in \mathbf{B_i}$,

$$\widetilde{f}_i \mathbf{c} = (c_1 + 1, c_2, \ldots, c_N), \qquad \text{when } \beta_1 = \alpha_i,$$
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- In general, it is not easy to describe $\widetilde{f}_i$ and $\widetilde{f}_i^*$ for any i

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PBW crystal of type A

- $\mathfrak{g} = A_{n-1}$ with $I = \{1, \ldots, n-1\}$

$\mathfrak{l} = A_{r-1} \times A_{n-r-1}$ with $J = I \setminus \{r\}$ $(r \in I)$

- Φ^+ : the positive roots of $\mathfrak{g}$

Φ^+_J : the positive roots of $\mathfrak{l}$, $\quad \Phi^+(J) = \Phi^+ \setminus \Phi^+_J$

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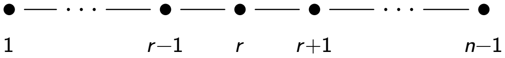

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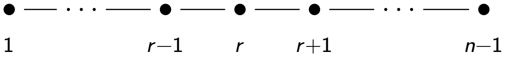

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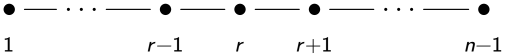

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- Choose $\mathbf{i} \in R(w_0)$ such that $\mathbf{i}$ is adapted to the quiver Ω

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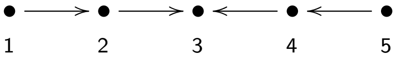

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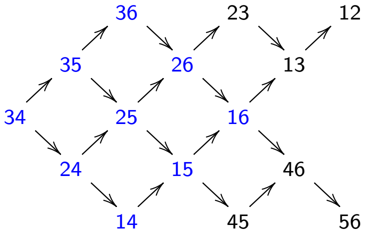

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- Let $\mathbf{B} = \mathbf{B}_{\mathbf{i}}$ and write $\mathbf{c} = (c_{ij})_{1\leq i<j\leq n} \in \mathbf{B}$

  where c_{ij} : the multiplicity of the root vector for $\epsilon_i - \epsilon_j$

- The crystal structure of $\mathbf{B}$ can be described explicitly

  (due to Reineke 97, Salisbury-Schultze-Tingley 18)

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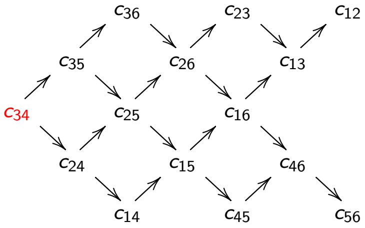

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- For $i \neq r$, $\tilde{f}_i$ can be described in terms of "signature rule"

When $i = 4$, apply signature rule to the sequence below

$$\underbrace{-\cdots-}_{c_{35}}\ \underbrace{+\cdots+}_{c_{34}}\ \underbrace{-\cdots-}_{c_{25}}\ \underbrace{+\cdots+}_{c_{24}}\ \underbrace{-\cdots-}_{c_{15}}\ \underbrace{+\cdots+}_{c_{14}}\ \underbrace{-\cdots-}_{c_{45}}$$

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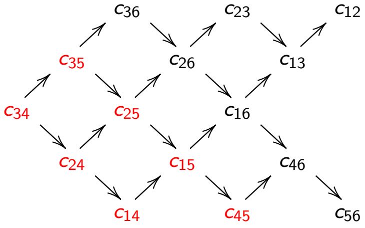

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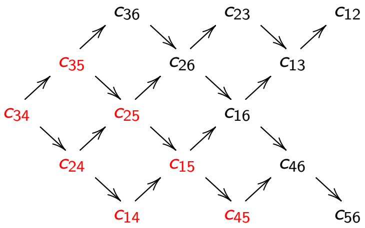

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Crystal for $U_q(\mathfrak{u}^-)$

- Consider subscrystals of $\mathbf{B}$;

$$\mathbf{B}^J := \left\{ \mathbf{c} = (c_{ij}) \in \mathbf{B} \,\middle|\, c_{ij} = 0 \text{ for } \epsilon_i - \epsilon_j \in \Phi_J^+ \right\},$$
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- ω_r : the r-th fundamental weight
- For $s \geq 1$, $B(s\omega_r) = \{\, \mathbf{c} \in \mathbf{B}^J \,|\, \varepsilon_r^*(\mathbf{c}) \leq s \,\} \subset \mathbf{B}^J$
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$$\varepsilon_r^*(\mathbf{c}) = \max_{\mathbf{p}} \left\{ \sum_{ij \in \mathbf{p}} c_{ij} \right\}$$

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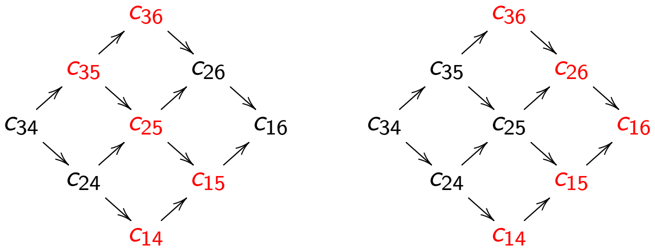

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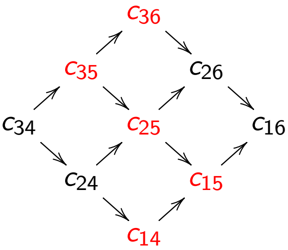

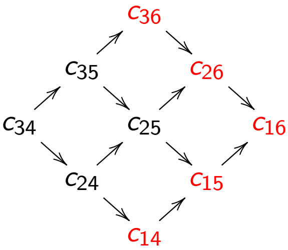

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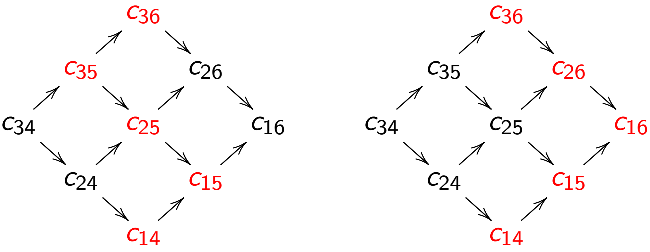

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Affine crystal structure and KR crystals

- Define $\tilde{e}_0, \tilde{f}_0 : \mathbf{B}^J \longrightarrow \mathbf{B}^J \cup \{\mathbf{0}\}$ by

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($\mathbf{1}_\theta$ corresponds to the longest root vector of A_{n-1})

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(a) $\mathbf{B}^J$ becomes a $U_q'(A_{n-1}^{(1)})$-crystal with respect to $\tilde{e}_0, \tilde{f}_0$

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(1) We have a polytope realization of $B^{r,s}$

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$\mathfrak{l} = A_{n-1}$ with $J = I \setminus \{n\}$

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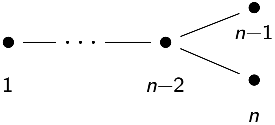

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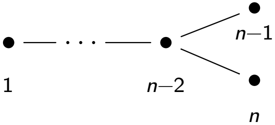

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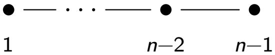

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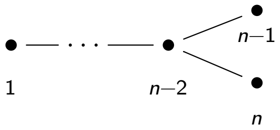

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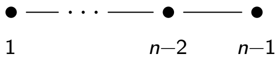

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The crystal structure of $\mathbf{B}$ can be described explicitly

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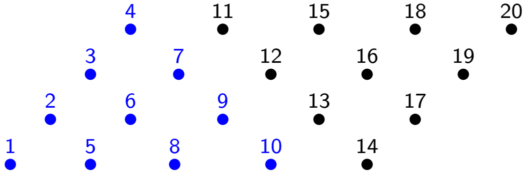

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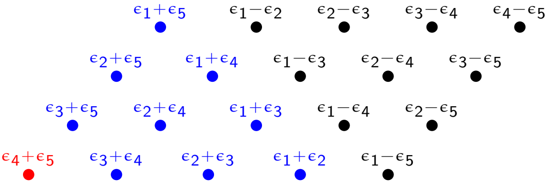

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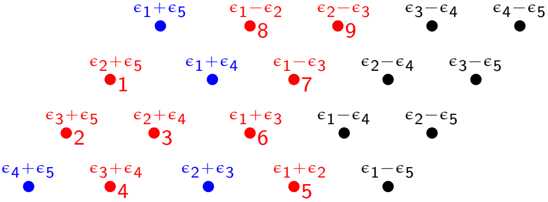

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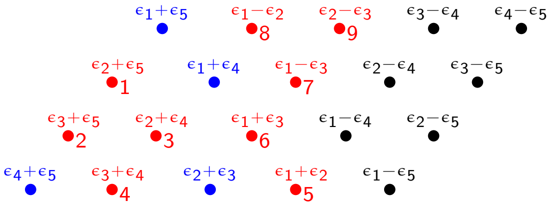

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Crystal for $U_q(\mathfrak{u}^-)$

- Set $\mathbf{B} = \mathbf{B}_{\mathbf{i}}$ and

$$\mathbf{B}^J := \left\{ \mathbf{c} = (c_\beta) \in \mathbf{B} \,\middle|\, c_\beta = 0 \text{ for } \beta \in \Phi_J^+ \right\},$$
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- The crystal structure on $\mathbf{B}^J$ and $\mathbf{B}_J$ is induced from $\mathbf{B}$ and

$$\mathbf{B} \cong \mathbf{B}^J \otimes \mathbf{B}_J$$

- $\mathbf{B}^J$: a crystal of the quantum nilpotent subalgebra $U_q(\mathfrak{u}^-)$
- For $s \geq 1$, $B(s\omega_n) = \{ \mathbf{c} \in \mathbf{B}^J \,|\, \varepsilon_n^*(\mathbf{c}) \leq s \} \subset \mathbf{B}^J$
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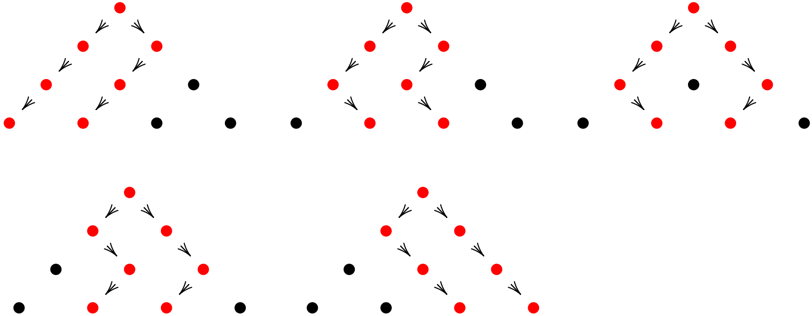

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- For $\mathbf{c} \in \mathbf{B}^J$ and a double path $\mathbf{p}$, let

$$\|\mathbf{c}\|_{\mathbf{p}} = \sum_{\beta \text{ lying on } \mathbf{p}} c_\beta.$$

Theorem (Jang-K 18)

$$\varepsilon_n^*(\mathbf{c}) = \max\left\{ \|\mathbf{c}\|_{\mathbf{p}} \,\middle|\, \mathbf{p} \text{ is a double path in } \Phi^+(J) \right\}$$

Proof) We use the transition map from Lusztig data to Kashiwara string parametrization due to Berenstein-Zelevinsky (01) to get the formula for ε_n^*

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Affine crystal structure and KR crystals

- Define $\tilde{e}_0, \tilde{f}_0 : \mathbf{B}^J \longrightarrow \mathbf{B}^J \cup \{\mathbf{0}\}$ by

$$\tilde{e}_0\mathbf{c} = \mathbf{c} + \mathbf{1}_\theta, \quad \tilde{f}_0\mathbf{c} = \begin{cases} \mathbf{c} - \mathbf{1}_\theta & \text{if } c_\theta > 0, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

($\mathbf{1}_\theta$ corresponds to the root vector of $\theta = \epsilon_1 + \epsilon_2$)

Theorem (Jang-K 18)

(a) $\mathbf{B}^J$ becomes a $U_q'(D_n^{(1)})$-crystal with respect to $\tilde{e}_0, \tilde{f}_0$

(b) For $s \geq 1$, the affine subcrystal

$$\mathbf{B}^{J,s} := \{\, \mathbf{c} \in \mathbf{B}^J \mid \varepsilon_n^*(\mathbf{c}) \leq s \,\} \subset \mathbf{B}^J$$

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- $[\overline{n}] := \{\, \overline{n} < \cdots < \overline{1} \,\}$
- $SST_{\overline{n}}(\lambda/\mu)$: the set of SST of shape λ/μ with letters in $[\overline{n}]$
- Put

$$\mathbf{T}^{\searrow} := \bigsqcup_{\substack{\ell(\lambda)\le n \\ \lambda':\text{even}}} SST_{\overline{n}}(\lambda^{\pi}), \qquad \mathbf{T}^{\nwarrow} := \bigsqcup_{\substack{\ell(\lambda)\le n \\ \lambda':\text{even}}} SST_{\overline{n}}(\lambda),$$

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- We identify $\mathbf{c} \in \mathbf{B}^J$ as a biword with letters in $[\overline{n}]$ where

$$c_{\epsilon_i+\epsilon_j} = \underbrace{\begin{matrix} \overline{j} & \cdots & \overline{j} \\ \overline{i} & \cdots & \overline{i} \end{matrix}}_{c_{\epsilon_i+\epsilon_j}} \qquad (i<j)$$

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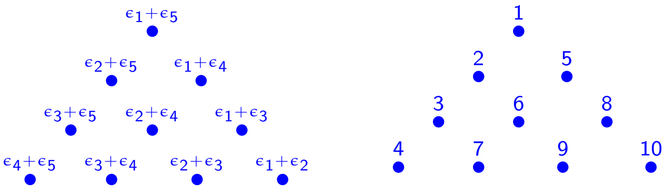

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- (Burge 74) There exist bijections

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Burge correspondence

For example, $n = 4$

$$\mathbf{c} = \begin{array}{ccccc} & & 0 & & \\ & 2 & & 1 & \\ 1 & & 0 & & 1 \end{array} = \begin{pmatrix} \overline{4} & \overline{4} & \overline{4} & \overline{3} & \overline{2} \\ \overline{2} & \overline{2} & \overline{3} & \overline{1} & \overline{1} \end{pmatrix}$$

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- One can define a D_n-crystal structure on $\mathbf{T}^{\searrow}$ where

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- One can also define a D_n-crystal structure on $\mathbf{T}^{\nwarrow}$ where

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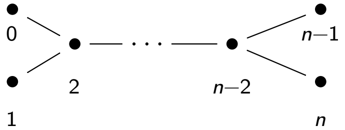

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(a) For $s \geq 1$, we have an isomorphism of $D_n^{(1)}$-crystals

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- For $\mathbf{c} \in \mathbf{B}^J$, let

$$\lambda(\mathbf{c}) := \mathrm{sh}(\kappa^{\nwarrow}(\mathbf{c})) = (\lambda_1(\mathbf{c}) \geq \ldots \geq \lambda_\ell(\mathbf{c}))$$

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$$\lambda_1(\mathbf{c}) + \lambda_3(\mathbf{c}) + \cdots + \lambda_{2l-1}(\mathbf{c}) = \max_{\mathbf{p}_1,\ldots,\mathbf{p}_l} \{ ||\mathbf{c}||_{\mathbf{p}_1} + \cdots + ||\mathbf{c}||_{\mathbf{p}_l} \},$$

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For example, let $n = 6$ and let $\mathbf{c} \in \mathbf{B}^J$ be given by

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 2 & & 3 & & & \\ & & 2 & & 1 & & 1 & & \\ & 1 & & 3 & & 2 & & 1 & \\ 2 & & 3 & & 2 & & 0 & & 3 \end{array}$$

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$\lambda(\mathbf{c})_1 = 19$ with maximal value $||\mathbf{c}||_{\mathbf{p}} = 19$

$\lambda(\mathbf{c})_1 + \lambda(\mathbf{c})_3 = 25$ with maximal value $||\mathbf{c}||_{\mathbf{p}_1} + ||\mathbf{c}||_{\mathbf{p}_2} = 25$

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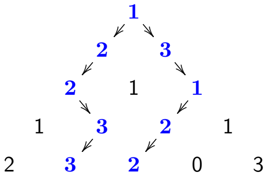

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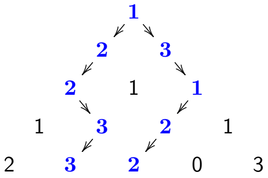

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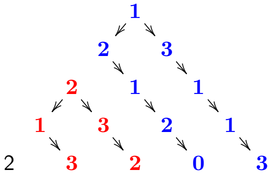

THANK YOU

ありがとうございます。