Grassmannian k-dim in $\mathbb{C}^n$ points given by full rank $k\times n$ matrices (non uniquely) ①

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \end{bmatrix} \in Gr(2,n)$ Plücker coords $\Delta_{ij} = a_{1i}a_{2j} - a_{1j}a_{2i}$

This defined up to simultaneous rescaling, so embedding in $\mathbb{P}^{\binom{n}{2}-1}$

Want to uniquely id points in $Gr(k,n)$

$(a_{11}, \cdots, a_{1n})$
$(a_{21}, \cdots, a_{2n})$ lines that define the plane ~~Normals of~~

$1 \le i < j < k < l \le n$

But other relations $\Delta_{ik}\Delta_{jl} = \Delta_{ij}\Delta_{kl} + \Delta_{il}\Delta_{jk}$

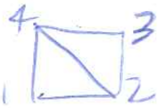
Consider triangulation T n-gon w/ chords $\overline{ij}$ and boundary $\overline{i\,i{+}1}$ and $\overline{1n}$ ~~(mod n)~~

$\Delta_T = \{\Delta_{ij} \mid \overline{ij} \in T\}$ Prop/ If $\Delta_{ij} = x_{ij}$, then Δ_{ij} for $\overline{ij} \notin T$ is subtraction free ~~Laurent poly~~ rational funcs in x_{ij}

For T fixed

Ex/ $A = \begin{bmatrix} 1 & x_{23}/x_{13} & 0 & -x_{34}/x_{13} \\ 0 & x_{12} & x_{13} & x_{14} \end{bmatrix}$

$\Delta_{24} = \dfrac{x_{12}x_{34} + x_{14}x_{23}}{x_{13}}$ comes from exchange by Plücker rel's



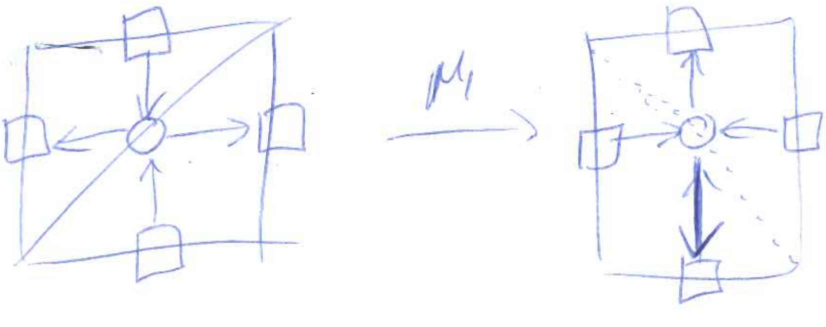
To better model this: Quiver := directed graph no directed 2-cycles
Partition into frozen & mutable vertices

Mutation: μ_k

1) $\forall$ path $i \to k \to j$ add $i \to j$ unless i,j both frozen
2) flip all arrows incident to k
3) remove 2-cycles

For n-gon triangulation. frozen verts for boundry, mutable chords w/ cycles ↻ in triangles

$x_k' = \dfrac{1}{x_k}\left(\prod_{i \to k} x_i + \prod_{k \to j} x_j\right)$



Ex/ $\overset{0}{0} \rightrightarrows \overset{1}{1} \longmapsto 2 \leftleftarrows 1$

$\cancel{x_1 = 0}\quad \cancel{x_2 = 1}\quad \cancel{x_3 = \frac{2^2+0}{1} = 2}\quad \cancel{x_4 = \frac{2^2}{1} = 4}$

$x_0 = 1 \quad x_1 = 1 \quad x_2 = \frac{1^2+1}{1} = 2 \quad x_3 = \frac{2^2+1}{1} = 5$

$x_4 = \frac{5^2+1}{2} = \frac{26}{2} = 13 \quad x_5 = 34 \quad x_6 = 89 \quad x_7 = 233$

$$x_2 = \frac{x_0^2+1}{x_1} \qquad x_{n+1} = \frac{x_{n-1}^2+1}{x_n}$$

Take incidence matrix $b_{ij} = \begin{cases} \ell & \text{if } i \to j \ (\ell \text{ copies}) \\ -\ell & \text{if } j \to i \end{cases}$

No 2-cycles $\Rightarrow B = (b_{ij})$ skew symmetric

Mutation μ_k becomes B': $b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0, \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0 \\ b_{ij} & \text{otherwise} \end{cases}$

We can generalize this for B being skew-symmetrizable

$\exists\, d_1, \dots, d_n \in \mathbb{Z}_{>0}$ st $d_i b_{ij} = -d_j b_{ji}$

Can extend this to frozen variables by augmenting bottom w/ rows (w/o conditions)

Ex/ $B = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$ $\quad d_1 = 1 \quad d_2 = 2$

$$\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ +1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{jk} < 0} x_j^{-b_{jk}}$$

③ $\mu_1(B)=\begin{bmatrix}0&2\\-1&0\end{bmatrix}$ $\quad\mu_2(B)=\begin{bmatrix}0&2\\-1&0\end{bmatrix}$

$\cancel{b_{22}=b_{21}b_{12}} = b'_{22} = b_{22}$

since $b_{21}>0$ $b_{12}<0$

$x_2' = \frac{1}{x_2}(x_1^2+1)$

$x_1' = \frac{1}{x_1}(1+x_2)$

$\mathring{x}_1 = \frac{1}{x_1}(x_2'+1) = \frac{x_1^2+1+x_2}{x_1x_2}$

$\mathring{x}_2 = \frac{1}{x_2}((x_1')^2+1) = \frac{1+2x_2+x_2^2+x_1^2}{x_1^2x_2}$

$\mathring{x}_2' = \frac{1}{x_2'}(\mathring{x}_1^2+1)$

$= \mathring{x}_2$

$\mathring{x}_1' = \frac{1}{x_1'}(1+\mathring{x}_2) = \mathring{x}_1$

$= \frac{x_1}{(1+x_2)}\left(1+\frac{1+2x_2+x_2^2+x_1^2}{x_1^2x_2}\right) = \frac{(1+x_2)(x_1^2+1+x_2)}{(1+x_2)\,x_1x_2}$

So 2 mutation types, but 6 seeds: (x_1,x_2)

(x_1',x_2) (x_1,x_2') $(\mathring{x}_1,x_2')$ $(x_1',\mathring{x}_2)$ $(\mathring{x}_1,\mathring{x}_2)$

w/ matrix B, $\mu_1(B)$

Def'n

Cluster alg

Ext. Seed $((\tilde{x},\tilde{y}),\tilde{B})$

ext cluster, ext exchange matrix

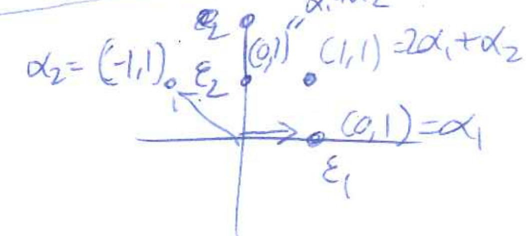
Cluster algebra is alg generated by all possible (ext) seeds

Fact/ Changing initial seed gives ~~...~~ isomorphic cluster algebra.

Denominator vectors

x_1	$(-1,0)$	$(0,-1)$	x_2
x_1'	$(1,0)$	$(0,1)$	x_2'
$\mathring{x}_1$	$(1,1)$	$(2,1)$	$\mathring{x}_2$

B_2 positive roots



④ Thm/ Cluster ~~va~~ variables are all (positive) Laurent polynomials in the initial seed.

Thm/ The CA w/ finite # clusters is in bijection w/ ~~Dynkin diag~~ finite dim simple Lie algs.

For those from quivers, choose any orientation of Dynkin diagram.

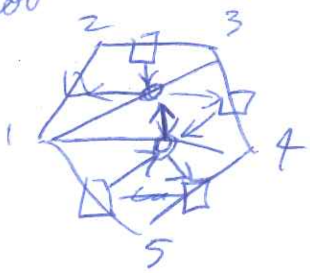
A_n:  D_n: 

$E_{6,7,8}$ 

Dynkin Quivers

Thm/ $Gr_2(2, \cancel{10}\, n+3) \cong$ type A_n w/ frozen variables

Plücker coords



$1 \longrightarrow 2$ $\quad (x_1, x_2)$

$1 \longleftarrow 2$ $\quad \left(\frac{1}{x_1}(1+x_2),\ x_2\right)$

$1 \longrightarrow 2$ $\quad \left(\frac{1}{x_1}(1+x_2),\ \frac{1}{x_1 x_2}(x_1+x_2+1)\right)$

$\left(\frac{1}{x_2}(x_1+1),\ \frac{1}{x_1 x_2}(x_1+x_2+1)\right)$

$\left(\frac{1}{x_2}(1+x_1);\ x_1\right) \to (x_2, x_1)$

Cluster mutation corresp. to flips

Rem/ If we consider $TNN_{2,m} \subseteq Gr(2,m)$ w/ all minors positive it is sufficient to check on any fixed cluster for those Plücker relations (so $2m-3$ minors)

A_2 root system

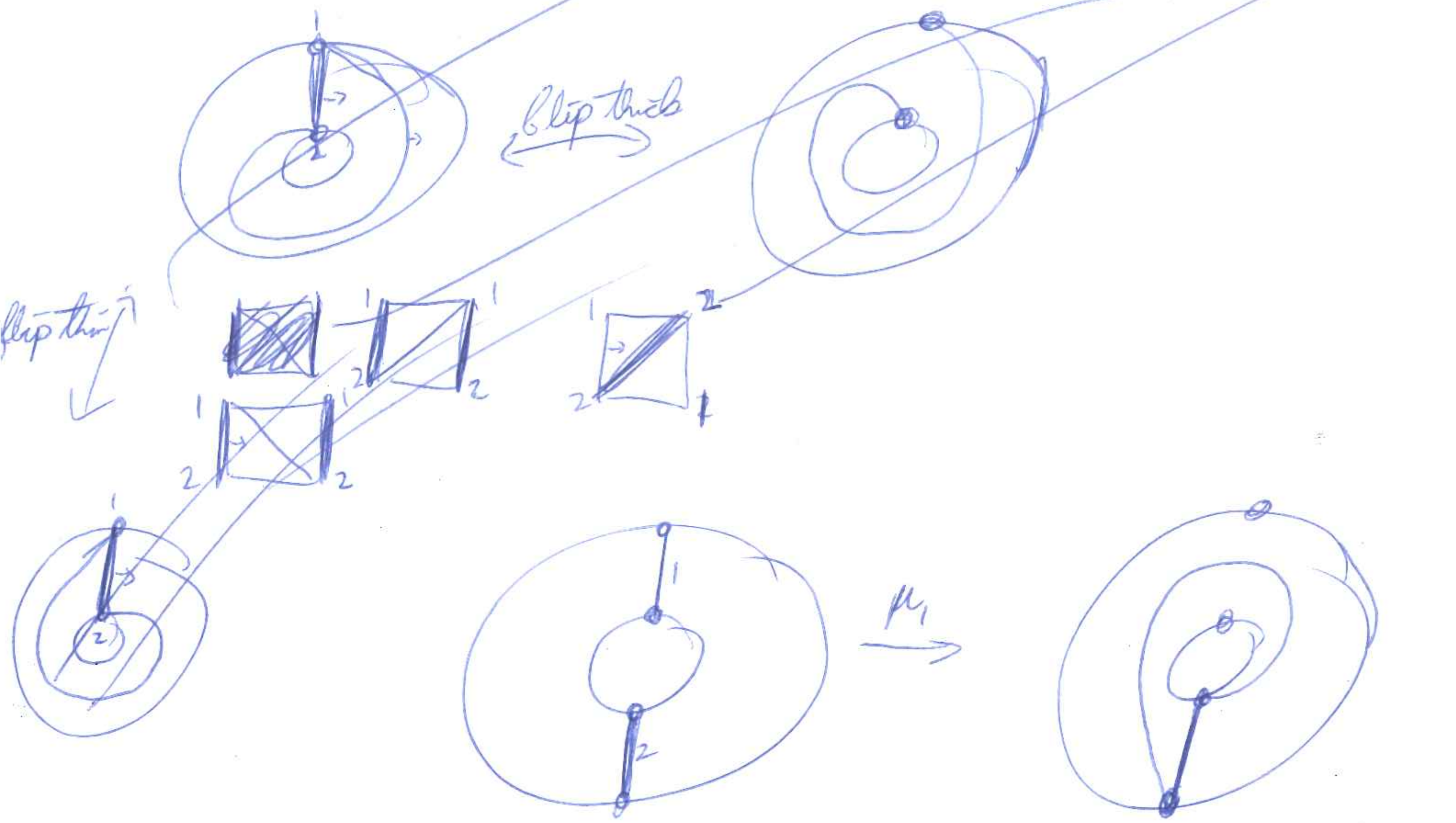
Denominator vectors of cluster vars

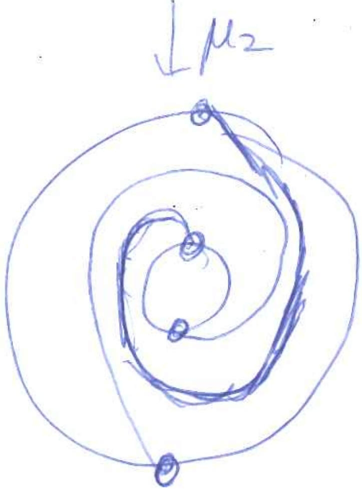
$(-1,0)$ $(0,-1)$

$(1,0)$ $(1,1)$

$(0,1)$

Cluster algebra on a surface





Questions become

What is a basis (have answer rank 2)? Dehn twists

~~What have finite # mutation types?~~ ok

What are other ~~(tame type~~ cluster algebra structures?