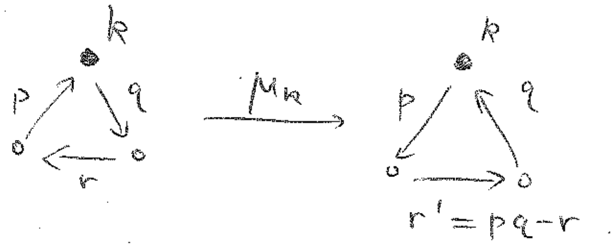
Cluster Algebras Geometric Type I: Quivers

Wan

Definition: Quivers : Directed graphs.

Will only consider quivers without loops or 2-cycles by convention. Such quivers are isomorphic to some skew-symmetric matrices..

Mutation:



p, q, r etc: number of arrows

Examples: see Chelsea's part ...

Two quivers are mutation equivalent if they are linked by a finite sequence of mutations. E.g. Any two orientations of a tree are mutation equivalent.

(no cycles)

The mutation class of Q is the set of all quivers that can be reached from Q by iterated mutations

EXAMPLES:

$Q = 1 \circ \underset{b \text{ arrows}}{\rightrightarrows} \circ 2$: Acyclic quiver with two vertices.

The mutation class is a singleton, since Q, $\mu_1(Q)$, $\mu_2(Q)$ are all isomorphic to each other where isomorphism ~~follows~~ from obvious definition.

(To be precise, ②

Let $Q=(Q_0, Q_1, s, t)$, $Q'=(Q_0', Q_1', s', t')$ be two quivers. Q and Q' are <u>isomorphic</u> if there is a pair (f_0, f_1) of bijective maps $f_0: Q_0 \to Q_0'$ and $f_1: Q_1 \to Q_1'$ such that the following diagrams hold.

$$\begin{tikzcd} Q_1 \arrow[r, "s"] \arrow[d, "f_1"'] & Q_0 \arrow[d, "f_0"] \\ Q_1' \arrow[r, "s'"] & Q_0' \end{tikzcd} \qquad \begin{tikzcd} Q_1 \arrow[r, "t"] \arrow[d, "f_1"'] & Q_0 \arrow[d, "f_0"] \\ Q_1' \arrow[r, "t'"] & Q_0' \end{tikzcd}$$

that is, for all arrows $\alpha \in Q_1$, we have $f_0(s(\alpha)) = s'(f_1(\alpha))$ and $f_0(t(\alpha)) = t'(f_1(\alpha))$. (f_0, f_1) is an isomorphism between Q, Q'. We write $Q \cong Q'$.

$\Rightarrow$ Two quivers are isomorphic iff they have the same structure in the sense that we can obtain one quiver from the other by renaming the vertices (via the map f_0) and edges (via f_1).)

<u>More examples</u>:

Orientation of A_3 $\circ\!-\!\circ\!-\!\circ$

$$\begin{tikzcd} \overset{1}{\circ} \arrow[r] & \overset{2}{\circ} \arrow[r] & \overset{3}{\circ} \end{tikzcd}$$

$$\begin{tikzcd} \overset{1}{\circ} & \overset{2}{\circ} \arrow[l] \arrow[r] & \overset{3}{\circ} \end{tikzcd} \quad \mu_2$$

$$\begin{tikzcd} \overset{1}{\circ} \arrow[r] & \overset{2}{\circ} \arrow[r] & \overset{3}{\circ} \end{tikzcd} \quad \mu_1$$

$$\begin{tikzcd} & \overset{1}{\circ} \arrow[dr] & \\ \overset{3}{\circ} \arrow[ur] & & \overset{2}{\circ} \arrow[ll] \end{tikzcd}$$

μ_2 up to isomorphism

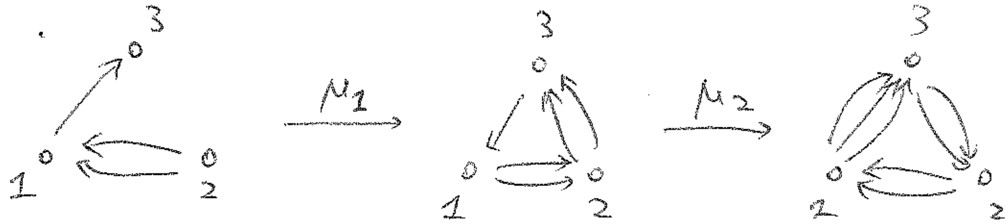
These four quivers form a mutation class of size 4.

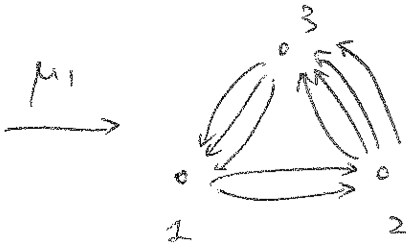
<u>EX3</u>

$$Q = \begin{tikzcd} & \overset{3}{\circ} \arrow[dl, shift left] \arrow[dl, shift right] & \\ \underset{1}{\circ} \arrow[rr, shift left] \arrow[rr, shift right] & & \underset{2}{\circ} \arrow[ul, shift left] \arrow[ul, shift right] \end{tikzcd} \quad \xrightarrow{\mu_1} \text{ or } \xrightarrow{\mu_2} \text{ or } \xrightarrow{\mu_3} \quad \begin{tikzcd} & \overset{3}{\circ} \arrow[dr, shift left] \arrow[dr, shift right] & \\ \underset{1}{\circ} \arrow[ur, shift left] \arrow[ur, shift right] & & \underset{2}{\circ} \arrow[ll, shift left] \arrow[ll, shift right] \end{tikzcd}$$

$Q \cong \mu_1(Q) \cong \mu_2(Q) \cong \mu(Q_3)$. The mutation class of Q is again a singlet

EX4



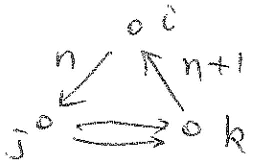


infinite number of mutation - equivalent quivers!

<u>Def</u>: A quiver Q is <u>mutation finite</u> if its mutation class is finite. Otherwise it is <u>mutation infinite</u>

Not many quivers are of finite mutation type. And it is in general difficult to determine if a quiver is mutation finite.

(Examples: this generalises EX4 :



, $n \in \mathbb{N}$.

Denote such quivers as $T(n+1, n, 2)$

Then $\mu_k(T(n+1, n, 2)) \cong T(n+2, n+1, 2)$. So that $T(n+1, n, 2)_{n \in \mathbb{N}}$ is an infinite family of mutation equivalent, pairwise nonisomorphic quivers.

Further example: if Q connected, number of vertices $|Q| \geq 3$, and Q contains arrow $\xrightarrow{p}$ w/ $p > 2$. $\Rightarrow$ Q is mutation infinite.)

Seed:

A pair $(Q, \vec{u})$ where:

- Q is a quiver with $n := |Q|$ vertices.
- $\vec{u} = (u_1, \cdots u_n)$ is a ~~set~~ tuple of rational functions in n variables. $(x_1 \cdots x_n) \equiv \vec{u_0}$ ($\leftarrow$ the "initial cluster") ($\vec{u}$ is ordered set)

For example. $u_1 \longrightarrow u_2 \longrightarrow u_3$ is a seed $(Q, \vec{u})$ where

$$Q = \underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \underset{3}{\circ}$$

$$\vec{u} = (u_1, u_2, u_3)$$

We basically assign a variable to each node.

Def^n:

$u_1 \cdots u_n$ are called <u>cluster variables</u>, and $\vec{u} = (u_1, \cdots u_n)$ is an example of a <u>cluster</u>.

These are not the only cluster variables / culster we can have. If we are given a <u>initial seed</u> $(Q_0, \vec{u_0})$ where $\vec{u_0} = (x_1, \cdots x_n)$, we can generate many more clusters and therefore cluster variables through a process called <u>seed mutation</u>. $\vec{u_0} = (x_1 \cdots x_n)$ is the "initial cluster" we encountered earlier. All other cluster variables can be written as rational functions of $(x_1 \cdots x_n)$.

Seed Mutation

$$\mu_k(Q, (u_1 \cdots u_n)) = (\mu_k(Q), (u_1, \cdots, u_k' \cdots u_n))$$

where $u_k' = \dfrac{1}{u_k}\Big(\underbrace{\prod_{i \to k} u_i}_{\text{incoming arrows}} + \underbrace{\prod_{k \to j} u_j}_{\text{outgoing arrows}} \Big)$.

If no incoming/outgoing arrows exist, the product is understood to be 1

So the proper def^n of clusters : they are the $\vec{u}$'s appeared in $(Q, \vec{u})$ obtained from $(Q_0, \vec{u}_0)$ by iterated mutation.

Cluster Algebras $\mathcal{A}_Q$:

the subalgebra of $\mathbb{Q}(x_1 \cdots x_n)$ generated by all cluster variables

* A cluster algebra is of finite type if it contains finitely many cluster variables

EXAMPLES of seed mutation

$$\begin{tikzcd}[column sep=small]
& \boxed{x_1 \to x_2 \leftarrow x_3} \arrow[dl, dash, "\mu_1"'] \arrow[d, dash, "\mu_2"] \arrow[dr, dash, "\mu_3"] & \\
\boxed{\frac{1+x_2}{x_1} \leftarrow x_2 \leftarrow x_3} \arrow[d, dash, "\mu_2"'] \arrow[d, dash, shift left=4, "\mu_3"] & \boxed{x_1 \leftarrow \frac{1+x_1x_3}{x_2} \leftarrow x_3} \arrow[d, dash, "\mu_1"'] \arrow[d, dash, shift left=4, "\mu_3"] & \boxed{x_1 \to x_2 \to \frac{1+x_2}{x_3}} \arrow[d, dash, "\mu_1"'] \arrow[d, dash, shift left=4, "\mu_2"] \\
\cdots & \cdots & \cdots
\end{tikzcd}$$

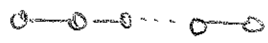
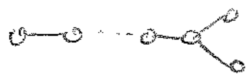
. finitely many clusters / cluster variables.

EX2 : $\underset{x_1}{\circ} \rightrightarrows \underset{x_2}{\circ}$: infinitely many cluster variables.
(even though the quiver mutation class is finite)

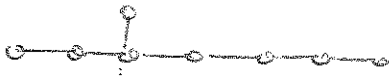
More examples : Chelsea's part.

Some theorems :

- Cluster variables are positive Laurent polynomials.
- The number of cluster variables is finite iff Q is mutation equivalent to an orientation of a Dynkin diagram of type A_n, D_n, E_6, E_7, E_8

Recall: A_n :  D_n : 

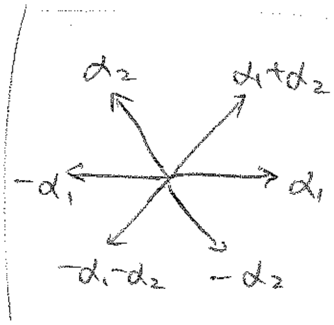
E_6 :  E_7 : 

E_8 : 

- Each cluster $\vec{u}$ occurs in a unique seed $(Q, \vec{u})$

Interesting connections to root system :

Consider A_2. the root vectors are $\underbrace{\underbrace{\alpha_1, \alpha_2}_{\text{simple roots}}, \alpha_1+\alpha_2}_{\text{positive roots}}$



the cluster variables generated by $\underset{x_1}{\circ} \longrightarrow \underset{x_2}{\circ}$ are

$$\left\{ x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1 x_2} \right\}.$$

The number of initial variables x_1, x_2 = number of simple roots.

" " " non-initial variables = number of positive roots.

The denominator of — " — : $x_1^{d_1} x_2^{d_2}$ correspond to the root $d_1\alpha_1 + d_2\alpha_2$

↑ generalises to arbitrary dynkin diagrams

Consider A_3 : The roots are

$\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \alpha_2+\alpha_3$ — 3 simple roots, 6 positive roots.

Cluster variables generated by $\underset{x_1}{\circ} \longrightarrow \underset{x_2}{\circ} \longrightarrow \underset{x_3}{\circ}$:

$$\left\{ x_1, x_2, x_3, \frac{1+x_2}{x_1}, \frac{x_1+x_3}{x_2}, \frac{1+x_2}{x_3}, \frac{1+x_3+x_2x_3}{x_1x_2}, \frac{x_1+x_1x_2+x_3+x_2x_3}{x_1x_2x_3}, \frac{x_1+x_1x_2+x_3}{x_2x_3} \right\}$$

Cluster algebra generated by $\underbrace{\circ \to \circ \to \cdots \to \circ}_{n}$

It is only for A_n that the number of cluster variables = rank + number of +ve roots