Question 1

(a) The caterers will be serving a ‘surprise drink’ at the upcoming AMSI BBQ on Friday. To make this concoction, they have to mix a special liquid, which is soluble in water, with large amounts of water. Specifically, they pump these two different liquids via two separate hoses into a large cylindrical container and these liquids are mixed together with an electric stirrer situated inside the container. At the bottom of the container is a nozzle to dispense the drink.

In this system, we may be interested in modelling mathematically the rate of outward flow from the output nozzle over time or the distribution of the concentration of the liquid in the tank or the proportion of liquid in the tank over time. These quantities can vary and can be controlled via the flow-in rate of the two liquids from the hoses, the concentration of the special liquid, and the speed of the stirrer. Optimally, we would like the concentration of the drink to be spread uniformly (otherwise the drink would taste yukky!), and that the amount of flow-in and flow-out of the liquids are balanced such that there will be no overflows or an empty system at any point in time. A natural model to consider when modelling the flow rate out of the tank is differential equations as we consider the system to continually evolve over time. After developing a suitable mathematical model to describe our situation, we may solve the differential equation for a trajectory solution and derive quantities of interest, such as the fixed points and their stability, which would tell us the long term or equilibrium behaviour of our system. We may also extract insight of our system by graphing our model, for example plotting the vector field of the differential equation and its trajectories. Lastly, we may want to control and optimize the rate of flow-in from the two inputs and the speed of the mixer to get the best tasting refreshment, so everyone gets to have an enjoyable evening!

Question 2

We have the function $f(t) = e^{at} + e^{bt}$ with $a, b \in \mathbb{R}$ and $t \in \mathbb{R}^+$. 
(a) Let \( f_1(t) = e^{at} \) and \( f_2(t) = e^{bt} \). Then
\[
\hat{f}_1(s) = \int_0^\infty e^{-(s-a)t} dt = \frac{e^{-(s-a)t}}{-(s-a)} \bigg|_0^\infty = \frac{1}{s-a}, \text{ for } \Re(s) > \Re(a) = a.
\]
Similarly, \( \hat{f}_2(s) = \frac{1}{s-b} \), for \( \Re(s) > b \). Thus, by the linearity property of Laplace transforms,
\[
\hat{f}(s) = \hat{f}_1(s) + \hat{f}_2(s) = \frac{1}{s-a} + \frac{1}{s-b}, \text{ for } \Re(s) > b, \Re(s) > a.
\]

(b) \( \mathcal{L}(g_1(t)) = \mathcal{L}(f'(t)) \)
\[
= -f(0) + sf(s) = -2 + s \left( \frac{1}{s-a} + \frac{1}{s-b} \right), \text{ for } \Re(s) > b, \Re(s) > a.
\]

(c) \( \mathcal{L}(g_2(t)) = \mathcal{L} \left( \int_0^t f(\tau)d\tau \right) \)
\[
= \frac{1}{s} \hat{f}(s) = \frac{1}{s} \left( \frac{1}{s-a} + \frac{1}{s-b} \right) \text{ for } \Re(s) > b, \Re(s) > a.
\]

**Question 3**

Let \( f \) and \( g \) be two functions with non-negative support. Its Laplace transforms are \( \hat{f}(s) = \int_0^\infty e^{-st}f(t)dt \) and \( \hat{g}(s) = \int_0^\infty e^{-st}g(t)dt \).
\[
\mathcal{L}(f \ast g)(s) = \int_0^\infty e^{-st}(f \ast g)(t)dt
= \int_0^\infty e^{-st} \int_\infty^\infty f(\tau)g(t-\tau)d\tau dt
= \int_\infty^\infty \int_0^\infty e^{-s(t-\tau)} e^{-st} f(\tau)g(t-\tau)d\tau dt
= \int_\infty^\infty \int_0^\infty e^{-s(t-\tau)} e^{-st} f(\tau)g(t-\tau)dtd\tau
\]
Let \( x = t - \tau \), then \( dx = dt \); Old integration bounds (w.r.t. \( t \)): \( \tau \) to \( \infty \), so new integration bounds (w.r.t. \( x \)): 0 to \( \infty \). So

\[
\mathcal{L}(f \ast g)(s) = \int_0^\infty \int_0^\infty e^{-sx} e^{-st} f(\tau) g(x) dx d\tau
\
= \int_0^\infty e^{-sx} g(x) dx \int_0^\infty e^{-st} f(\tau) d\tau
\
= \hat{f}(s) \hat{g}(s).
\]

**Question 4**

**Theorem 0.1.** An LTI system with impulse response \( h(\cdot) \) and complex input signal \( u(\cdot) \) is BIBO stable if and only if \( ||h||_1 < \infty \). Further, if this holds then,

\[
||y||_\infty \leq ||h||_1 ||u||_\infty,
\]

for every bounded input.

Proof. Let \( u(t) \) be a complex valued signal. That is, \( u(t) = \sigma(t) + iw(t) \), where \( \sigma(t) = Re(u(t)) \) and \( w(t) = Im(u(t)) \). Note that \( |a + bi| = \sqrt{a^2 + b^2} \), the triangle inequality: \( |(a+bi) + (c+di)| \leq |a+bi| + |c+di| \) and \( |(a+bi)(c+di)| = |(a+bi)||(c+di)| \).

\[
||y||_\infty = \max_l |y(l)|, ||u||_\infty = \max_k |u(k)| \text{ and } ||h||_1 = \sum_{k=-\infty}^{\infty} |h(k)|.
\]

(\(\Leftarrow\)) Suppose \( ||h||_1 < \infty \) and \( ||u||_\infty < \infty \). Then for any \( l \in \mathbb{Z} \),

\[
|y(l)| = \left| \sum_{k=-\infty}^{\infty} h(l-k)u(k) \right|
\leq \sum_{k=-\infty}^{\infty} |h(l-k)u(k)|, \text{ by triangle ineq.}
\leq \sum_{k=-\infty}^{\infty} |h(l-k)||u(k)|,
\leq ||u||_\infty \sum_{k=-\infty}^{\infty} |h(l-k)|
= ||u||_\infty ||h||_1.
\]

Hence, \( ||y||_\infty \leq ||u||_\infty ||h||_1 < \infty \), and Equation 0.1 is satisfied.

(\(\Rightarrow\)) Suppose the system is BIBO stable, then want to show that \( ||h||_1 < \infty \). Equivalently, want to show that if \( ||h||_1 = \infty \) then the system is not BIBO.

Define

\[
u(k) = \begin{cases} \frac{\bar{h}(-k)}{|\bar{h}(-k)|} & \text{if } h(-k) \neq 0, \\
0 & \text{if } h(-k) = 0,
\end{cases}
\]

where \( \bar{h} \) denotes the complex conjugate of \( h \). Note that \( \frac{\bar{h}(-k)}{|\bar{h}(-k)|} \) is bounded within the
unit circle of the complex plane, hence \( u(\cdot) \) is bounded. But,
\[
g(0) = \sum_{k=-\infty}^{\infty} h(-k)u(k) \\
= \sum_{k=-\infty}^{\infty} |h(-k)| \\
= ||h||_1 = \infty,
\]
so the output is not bounded for the specific bounded input signal, hence system is not BIBO. □

**Question 5**

Let \( U \sim \text{unif}(0,1) \). Set

\[
U = 1 - \exp(-\lambda X) \\
1 - U = \exp(-\lambda X) \\
X = -\frac{1}{\lambda} \log(1 - U) \\
X = -\frac{1}{\lambda} \log(U),
\]

since \( U \sim \text{unif}(0,1) \), then \((1 - U) \sim \text{unif}(0,1)\). So that \( X \sim \exp(\lambda) \).

Matlab code:

```matlab
N=10^5; %# iterations
lambda=2; %exp param

U=rand(1,N); % A 1xN vector of unif(0,1)
X=(-1/lambda)*log(U);

sammean=mean(X);
samvar=var(X);
thmean=1/lambda;
thvar=1/(lambda^2);
disp(['The theoretical mean is ',num2str(thmean)]);
disp(['The theoretical variance is ',num2str(thvar)]);
disp(['The sample mean is ',num2str(sammean)]);
disp(['The sample variance is ',num2str(samvar)]);
```

One instance of output is

The theoretical mean is 0.5
The theoretical variance is 0.25
The sample mean is 0.49911
The sample variance is 0.24798
Question 6

The system is

\[
\frac{d}{dt} y(t) + ay(t) = u(t), \text{ with } y(0) = 0, \tag{0.2}
\]

assuming that \(a\) is constant.

(a)

First show time invariance. That is, if the input signal \(u(t)\) produces an output \(y(t)\) then any time shifted input, \(u(t - \tau)\), results in a time-shifted output \(y(t - \tau)\).

The output, denoted \(y_d(t)\), of time delayed input, \(u(t - \tau)\), is

\[
\frac{d}{dt} y_d(t) + ay_d(t) = u(t - \tau), \text{ with } y_d(0) = 0.
\]

Now we delay the output by \(\tau\), so we make a change of variable \(t = t' - \tau\) in Equation 0.2. \(\tau\) is constant so \(dt = dt'\). So time delayed output satisfies

\[
\frac{d}{dt'} y(t' - \tau) + ay(t' - \tau) = u(t' - \tau), \text{ with } y(\tau) = 0,
\]

or rewriting,

\[
\frac{d}{dt} y(t - \tau) + ay(t - \tau) = u(t - \tau), \text{ with } y(\tau) = 0.
\]

Thus,

\[
\frac{d}{dt} y(t - \tau) + ay(t - \tau) = \frac{d}{dt} y_d(t) + ay_d(t),
\]

so \(y(t - \tau) = y_d(t)\) and time invariance property holds.

Next, we look at linearity. That is, if the input \(u_1(t)\) produces response \(y_1(t)\) and input \(u_2(t)\) produces response \(y_2(t)\), then the scaled and summed input \(a_1u_1(t) + a_2u_2(t)\) produces the scaled and summed response \(a_1y_1(t) + a_2y_2(t)\) where \(a_1\) and \(a_2\) are real scalars.

Let \(a_1\) and \(a_2\) be real scalars, and \(u_1(t)\) and \(u_2(t)\) inputs. The output, denoted \(y_1(t)\), of the input \(u_1(t)\) is

\[
\frac{d}{dt} y_1(t) + ay_1(t) = u_1(t), \text{ with } y_1(0) = 0. \tag{0.3}
\]

The output, denoted \(y_2(t)\), of the input \(u_2(t)\) is

\[
\frac{d}{dt} y_2(t) + ay_2(t) = u_2(t), \text{ with } y_2(0) = 0. \tag{0.4}
\]

The output, denoted \(y(t)\), of the input \(a_1u_1(t) + a_2u_2(t)\) is

\[
\frac{d}{dt} y(t) + ay(t) = a_1u_1(t) + a_2u_2(t), \text{ with } y(0) = 0. \tag{0.5}
\]

Multiplying Equation 0.3 by \(a_1\) and multiplying Equation 0.4 by \(a_2\) and summing them together, we obtain

\[
\left( a_1 \frac{d}{dt} y_1(t) + a_2 \frac{d}{dt} y_2(t) \right) + a (a_1y_1(t) + a_2y_2(t)) = a_1u_1(t) + a_2u_2(t).
\]

Comparing the above equation with Equation 0.5, it is immediate that \(y(t) = a_1y_1(t) + a_2y_2(t)\). Thus, this system is linear. Hence, this system is LTI.
(b) Taking Laplace transform of both sides of the (ODE) system, gives

\[
\mathcal{L} \left( \frac{d}{dt} y(t) + ay(t) - u(t) \right) = 0
\]

\[-y(0) + s\hat{y}(s) + a\hat{y}(s) - \hat{u}(s) = 0 \]

\[\Rightarrow \hat{h}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{1}{s + a} \]

is the transfer function, since \(y(0) = 0\).

(c) We have the relationship

\[\hat{h}(s) = \frac{\hat{g}_1(s)\hat{p}(s)}{1 + \hat{g}_1(s)\hat{g}_2(s)\hat{p}(s)}\]

where \(\hat{g}_1(s) = 1\), \(\hat{g}_2(s) = K\) and \(\hat{p}(s) = \frac{1}{s + a}\). Hence,

\[\hat{h}(s) = \frac{1}{s + a + K}\]

and so from Laplace transform tables,

\[h(t) = e^{-(K+a)t}, \text{ for } t \geq 0,\]

and 0 otherwise.

The step response is

\[H(t) = \int_{-\infty}^{t} e^{-(K+a)\tau} d\tau = \int_{0}^{t} e^{-(K+a)\tau} d\tau = \frac{1}{K + a} - \frac{e^{-(K+a)t}}{K + a}, \text{ for } t \geq 0,\]

0 otherwise.

**Question 7**

(a) Have a sequence of LTI systems in tandem with

\[u(t) = h(t) = \begin{cases} e^{-t} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases} \]

So

\[\hat{y}_1(s) = \mathcal{L}(\mathcal{O}(u(t))) = \hat{u}(s)\hat{h}(s) = \hat{h}(s)^2\]

\[\hat{y}_2(s) = \mathcal{L}(\mathcal{O}(\mathcal{O}(u(t)))) = \hat{u}(s)\hat{h}(s)\hat{h}(s) = \hat{h}(s)^3\]

\[\vdots\]

\[\hat{y}_n(s) = \hat{h}(s)^{n+1}\]
Figure 1: Plot of the step response function. The blue plot is $K = 0.1$, the red plot is $K = 1.5$, and the green plot is $K = 5$.

Since $h(t)$ is in the form of the pdf of exponential RV with parameter $\lambda = 1$, so $\hat{h}(s) = \frac{1}{s+1}$. So $\hat{y}(s) = \hat{y}_n(s) = \frac{1}{(n+1)s+1}$, which is Laplace transform of Gamma($k, \theta$) pdf with $k = n + 1$ and $\theta = 1$. So

$$y(t) = t^{k-1} \frac{e^{-t/\theta}}{\theta^k \Gamma(k)} = t^n \frac{e^{-t}}{\Gamma(n+1)} = t^n \frac{e^{-t}}{n!}, \quad t \geq 0.$$ 

(b)

The output of the sequence of systems has a Gamma($n + 1, 1$) pdf, which, with the exercise, is the sum of $n+1$ i.i.d. exponential random variables with parameter 1, which is the expression for the impulse response. So putting a sequence of LTI systems in tandem allows for an additive effect of the input.

(c)

$n!$ grows faster than $t^n$, so as $n \to \infty$, we expect $y(t) \to 0$. This is in agreement with the central limit theorem which says that as $n \to \infty$, $\sqrt{n}(y(t) - 1) \xrightarrow{d} N(0, 1)$, where $y(t)$ denotes the output of the system.