1. Do Exercise 3.24 from the lecture notes.

2. Do Exercise 3.26 from the lecture notes.

3. Do Exercise 3.31 from the lecture notes.

4. Do Exercise 3.32 from the lecture notes.

A manufacturing machine at a factory is required in the production process non-stop (24 hours a day and 7 days a week). Nevertheless, the machine experiences both “off periods” and “on periods”, where in the former it is not operating due to maintenance or malfunction and in the later it is operating as needed.

In analyzing the performance of the factory, an elementary model for the machine is that of an alternating sequence of independent random variables,

\[ X_1, Y_1, X_2, Y_2, X_3, Y_3, \ldots, \]

where \( X_i \sim F_X(\cdot) \) represents an “on period” and \( Y_i \sim F_Y(0) \) represents “off periods”. It is known that at time \( t = 0 \) the machine has just changed from “off” to “on”. In such a case, the state of the machine at time \( t \) is represented by \( X(t) \) (where say 0 implies “off” and 1 implies “on”).

As a first step it is assumed that, \( F_X(t) = 1 - e^{-\mu t} \) and \( F_Y(t) = 1 - e^{-\lambda t} \). In this case:

5. Argue why \( X(t) \) is a CTMC. What is the generator matrix?

6. Simulate a random sample path of \( \{X(t), t \in [0,20]\} \) with \( \mu = 2 \) and \( \lambda = 1 \). Plot the trajectory that you have simulated.
7. Calculate the long term proportion of time (i.e. over \( t \in [0, \infty) \)) during which the machine is “on” (respectively “off”). State your result in terms of the symbols \( \mu \) and \( \lambda \).

8. Simulate a long trajectory and use your simulation result to verify your answer to the question above.

9. Let \( q(t) = P(\text{“on” at time } t) \). Estimate \( \{q(t), t \in [0, 10]\} \) by means of simulation. Plot your result.

10. Now calculate \( \{q(t), t \in [0, 10]\} \) numerically (without simulation). You may compare to the result above.

11. Now try to find a precise analytic expression for \( q(t) \) (in terms of \( \lambda \) and \( \mu \)), compare your expression to the result above.

12. Is the information that a change occurred “exactly at time 0” important or are the results the same if it were simply stated that the machine is “on” at time 0? Explain your result.

Assume further that after the machine is in “off” state it needs to be in “warmup” state before moving to “on”. Thus the operation of the machine is determined by the sequence,

\[ X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3, Z_3 \ldots, \]

where \( X_i \) and \( Y_i \) are distributed as before and the “warmup” periods \( Z_i \) are as follows: \( Z_i \sim F_Z(\cdot) \) with \( F_Z(t) = 1 - e^{-\gamma t} \).

13. Repeat now questions 5, 7, 10, 11 (assuming \( \gamma = 3 \) for questions 6 and 7).

It was found now that there is a chance of \( p \) that at the end of the warmup period the machine will enter ”off” instead of “on”.

14. Repeat now questions 5, 7. Leave your answer symbolic in terms of \( p \).

The above CTMC model appears restrictive as it assumes that the distribution of “on”, “off” and “warmup” durations is exponential. Comparison to data indicates that it is plausible to assume “on” and ”off” durations are exponentially distributed, yet this is not the case for “warmup”. In that case, it is suggested to use a PH distribution with \( m \) phases, \( PH(c', A) \).

15. Incorporate the assumption about the PH distribution of “warmup” in the CTMC model. You should now have a Markov chain where \(|S| = m + 2\). Write down the generator matrix of this CTMC.
The last exercise illustrated one of the strengths of PH distributions: They allow to incorporate the distribution of arbitrary behavior in a CTMC. Often the construction of the PH distribution constitutes modeling in its own right:

16. Assume that the “warmup duration” is either “long” or “short”. In the “long” case it is exponentially distributed with $\gamma_1$. In the “short” case it is exponentially distributed with $\gamma_2$. We have $\gamma_1 < \gamma_2$. There is a chance of $r \in (0, 1)$ that it is long, and a chance of $1 - r$ that it is short. This is an hyper-exponential distribution. Show how it is a special case of the PH distribution and incorporate it in the CTMC model.

17. Assume now that it is measured that “warm up periods” have a mean of $m$ and a squared coefficient of variation of $c^2 > 1$ (the squared coefficient of variation of a random variable is the variance divided by the mean squared). Show how to incorporate this in the CTMC by means of a PH distribution of order 2 yielding arbitrary mean and arbitrary squared coefficient of variation $> 1$.

18. Why is the restriction of $c^2 > 1$ important? Can you answer the case of $c^2 \in (0, 1)$ with only 2 phases? If not argue why not. As a bonus you may try to find a PH distribution of higher order for this.

Note: For hints on the last 2 exercises, see for example Section 3 of [1].

References