Exercise 1

This exercise is concerned with robustness. Loosely speaking, we call a controlled system robust if small errors in the model or in the controller have small effects on the controlled behavior. In this exercise, we consider robustness both with respect to measurement errors and with respect to parameter uncertainty. Consider the input-output system

\[ 6\dot{y}(t) - 5\ddot{y}(t) + \dddot{y}(t) = u(t). \]  

1. Show that this system is open loop \((u(t) = 0)\) unstable. I.e. show that even for the input \(u(t) = 0\), the output diverges.

Assume that we want to stabilize the system using feedback control (as is in Section 2.5 in the course reader). Our first attempt is

\[ u(t) = -5\dot{y}(t) + \dddot{y}(t). \]  

It appears that this yields an extremely fast and accurate controller, since the system output is

\[ y(t) = 0. \]

We now investigate whether the proposed controller is indeed such a superior controller. If we were able to implement the controller with infinite precision, then, there seems to be no problem. Suppose, however, that this controller is implemented by means of a sensor that does not measure \(y(t)\) exactly. Assume that the sensor output is \(y(t) + v(t)\), where \(v(t)\) is a (deterministic) noise term (also known as a disturbance). The controller is then given by

\[ u(t) = -5(\dot{y}(t) + \dot{v}(t)) + \dddot{y}(t) + \dddot{v}(t). \]
2. Determine the output $y(t)$ for the case that $v(t) = \epsilon \sin(2\pi ft)$, $\epsilon > 0$, $f \in \mathbb{R}$. Conclude that an arbitrarily small disturbance can have a significant impact if $f$ is sufficiently large. Thus, the controller (2) is not robust with respect to measurement noise.

3. Determine the controller canonical form for the system (1). I.e. propose a state representation and describe the system as an $(A, B, C, D)$ system in controller canonical form.

4. Prove that the system (1) can not be stabilized by static output feedback, that is by a feedback of the form $u(t) = -ky(t)$.

5. Determine now a state feedback that assigns the closed-loop poles to $-1; -2$.

6. Design an observer with observer poles equal to $-3; -4$.

7. Combine the controller and the observer to obtain a feedback compensator with poles at $-1; -2; -3; -4$.

8. Suppose that this observer has the noisy sensor output as input. The observer equation then becomes

$$\dot{\hat{x}}(t) = A\hat{x} + b u(t) - K_o c' \hat{x}(t) + K_o \left( (y(t) + v(t)) \right)$$

Does this observer lead to an acceptable controlled system? Compare your conclusion with the one obtained in part 2.

**Exercise 2**

Consider the nonlinear dynamical system

$$y(t)^3 + \dot{y}(t) + \sin y(t) = u(t)(u(t) - 1).$$

1. Determine the linearization around the equilibrium point $\bar{y} = 0$, $\bar{u} = 0$.

2. Determine a linear output feedback controller that locally stabilizes the equilibrium point $\bar{y} = 0$, $\bar{u} = 0$.

3. Determine the region of attraction of the system with this linear output feedback (or a non-empty subset of it). That is, determine a region around the origin where you can guarantee that if the system starts in that region it will stay in that region and furthermore that solutions will converge to the origin.