Final 36 hour (Take Home) Exam

- Please e-mail y.nazarathy@uq.edu.au a confirmation e-mail once you start the exam (around 8am Thursday, February 4). This is to know you got the exam paper and have started.

- Answers should be sent to y.nazarathy@uq.edu.au by no later than 8pm (Melbourne time), Friday February 5.

- A scan/photo of the answers should be sent in. Students may also wish to typeset their answers (but this is not mandatory). Answers should clear and legible. Please make sure that the content of the answers e-mail does not exceed 3MB (process the images).

- Students taking the exam are not allowed to communicate with peer students or other academic experts about the exam questions until they have handed in their solution.

- Yoni Nazarathy will be available through e-mail/phone-text (0499-028-705) to answer questions about the exam until 12PM (noon), Friday February 5 with a response time of up to 4 hours (a bit more during night time).

- During the week following the exam, students may be contacted (phone/e-mail) and be questioned about their solution. This is mostly to see that the work is solely theirs.

- Any material and computing resource can be used during the exam. But no other human may be of aid.

The exam consists of 2 parts: **Part 1** has a weighting of 75% and consists of 3 questions, 25% each. **Part 2** has a weighting of 25% and is open ended. It contains a tougher question. Answers to this question will be ranked with the “top answer” getting full marks and the “weakest answer” getting only half of the marks. Empty answers, or fully faulty answers will get no marks.

**Good Luck.**
Part 1:

**Question 1:** Consider a birth death process \( \{X(t)\} \) on the state space \( S = \{0, 1, 2\} \) with birth rates \( \lambda_0, \lambda_1 \) and death rates \( \mu_1, \mu_2 \) all non-negative.

1. Take \( \lambda_1 = \mu_1 = \mu_2 = 13 \) and set \( \lambda_0 \) to equal the month in which you were born \((\in \{1, 2, \ldots, 12\}) \). Find
   \[
   \pi_i := \lim_{t \to \infty} \Pr(X(t) = i) \quad \text{for} \quad i = 0, 1, 2. 
   \]

2. Find
   \[
   \lim_{t \to \infty} \mathbb{E}[X(t)].
   \]

3. Assume \( \Pr(X(0) = 0) = 1 \) and set \( \tilde{\tau} = \inf\{t \geq 0 : X(t) = 1\} \) and,
   \[
   \tau = \inf\{t \geq \tilde{\tau} : X(t) = 0\}. 
   \]
   Use your answer to 1 to find \( \mathbb{E}[\tau] \).

4. Represent \( \tau \) as a PH random variable with 3 phases (transient states). Use this representation to find \( \mathbb{E}[\tau] \) and ensure your answer agrees with (3).

5. Find the squared coefficient of variation of \( \tau \): \( \text{var}(\tau) / \mathbb{E}[\tau]^2 \).
**Question 2:** Consider the SISO system in continuous time \( y(\cdot) = \mathcal{O}(u(\cdot)) \) with,

\[
y(t) = \int_{t-1}^{t} u(s) \, ds.
\]


2. Find the impulse response of the system, the step response and the transfer function.

3. Consider now \( y(\cdot) = \mathcal{O}(\mathcal{O}(u(\cdot))) \) and find the impulse response, step response the transfer function.

4. Assume \( U_1 \) and \( U_2 \) are two i.i.d. uniform(0,1) random variables. Find the CDF of \( Z := U_1 + U_2 \). Relate your answer to your answer of the previous item (3).

5. Rely on the central limit theorem to prescribe an explicit approximation for the impulse response of the system \( y(\cdot) = \mathcal{O}(\mathcal{O}(\ldots \mathcal{O}(u(\cdot)) \ldots)) \) where \( \mathcal{O} \) is applied to the input \( n \) times where \( n = 10 + MM \) and \( MM \) is your month of birth. Write an explicit expression for your approximation of the impulse response and justify why it is a sensible approximation.
**Question 3:** Consider the \((A, B, C, D)\) system:

\[
\dot{x}(t) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-10 & 0 & 10 & 0
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix}
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} x(t)
\]

1. Is the system controllable?

2. Is the system observable?

3. Assume a state feedback controller of the form \(u(t) = k_f' \hat{x}(t)\) where \(\hat{x}(\cdot)\) results from an observer with \(K_o\). Represent the 8 dimensional system as an \((A, B, C, D)\) system involving both \(x\) and \(\hat{x}\).

4. Explain why in a controller/observer design setting, the parameters of \(k_f'\) and \(K_o\) may be chosen separately to achieve any desired behaviour. Use equations (and basic calculations) in your explanation.
Part 2:

Generalize the state-space of the birth death process of Question (1) to be $S = \{0, 1, \ldots, K\}$ for some finite integer $K \geq 1$. Now the birth and death parameters are $\lambda_0, \ldots, \lambda_{K-1}$ and $\mu_1, \ldots, \mu_K$ respectively (all assumed to be strictly positive). The random variable $\tau$ (as defined in Question 1.3) is sometimes referred to as the “busy-cycle” in the context of queueing models.

Consider now the squared coefficient of variation of $\tau$, namely

$$c^2 := \frac{\text{var}(\tau)}{E[\tau]^2}.$$ 

1. Show that if $K = 1$ it holds that $c^2 < 1$.

2. Write an expression (or a few expressions) for $c^2$ involving matrices built using the birth and death parameters, and/or any other form.

3. Can you find any case where $c^2 = 1$ (or prove such a case does not exist)?

4. Can you find any case where $c^2 > 1$ (or prove such a case does not exist)?

5. In general, try to find general conditions (or cases) that determine if $c^2 < 1$, $c^2 = 1$ and $c^2 > 1$.

Note: Some of the above may be difficult. You may wish to initially consider special cases of birth-death parameters (e.g. $\lambda_i$ constant for all $i$).