Solution for Quiz #1

Consider a discrete time SISO, LTI system, \( O(\cdot) \), with impulse response,

\[
h(\ell) = \begin{cases} 
\frac{1}{3}, & \ell \in \{-1, 0, 1\}, \\
0, & \text{otherwise.}
\end{cases}
\]

1. Is the system causal? Explain/prove your answer.

**Answer:**
No. The system is not causal because \( h(-1) \neq 0 \). Consider the two inputs:
\( u_1(\ell) = \delta[\ell] \) (yields output \( y_1(\ell) = h(\ell) \)) and \( u_2(\ell) \equiv 0 \) (yields output \( y_2(\ell) = 0 \)).
The inputs are the same for \( \ell \leq -1 \), but over that time segment (specifically at \( \ell = -1 \)) they yield a different output.

2. What is the output of \( O(h(\cdot)) \)? (I.e. the output when the impulse response is fed as input).

**Answer:**
The output is the convolution of the impulse response with itself:

\[
(h \ast h)(\ell) = \sum_{k=-\infty}^{\infty} h(k)h(\ell-k) = \sum_{k=-1}^{1} h(k)h(\ell-k)
\]

\[
\begin{cases} 
0, & \ell < -2, \\
h(-1)h(-1), & \ell = -2, \\
h(-1)h(0) + h(0)h(-1), & \ell = -1, \\
h(-1)h(1) + h(0)h(0) + h(1)h(-1), & \ell = 0, \\
h(0)h(1) + h(1)h(0), & \ell = 1, \\
h(1)h(1), & \ell = 2, \\
0, & \ell > 2.
\end{cases}
\]

\[
= \begin{cases} 
\frac{1}{9}, & \ell = -2, \\
\frac{2}{9}, & \ell = -1, \\
\frac{3}{9}, & \ell = 0, \\
\frac{2}{9}, & \ell = 1, \\
\frac{1}{9}, & \ell = 2, \\
0, & \text{otherwise.}
\end{cases}
\]
3. Give probabilistic reasoning for your answer to 2.

**Answer:**
Consider discrete i.i.d. random variables $X_1$ and $X_2$ each distributed uniformly on $\{-1, 0, 1\}$. The convolution $h \times h$ is the PMF (probability mass function) of $Z = X_1 + X_2$. Now from first principles (using independence and multiplying probabilities):

$$
P(Z = \ell) = \begin{cases} 
\frac{1}{5}, & \ell = -2, \text{ (need two -1 values)}, \\
\frac{2}{5}, & \ell = -1, \text{ (need a -1 value and a 0 value)}, \\
\frac{3}{5}, & \ell = 0, \text{ (need two 0 values or a -1 value together with a 1 value)}, \\
\frac{2}{5}, & \ell = 1, \text{ (need a 1 value and a 0 value)}, \\
\frac{1}{5}, & \ell = 2, \text{ (need two 1 values)}, \\
0, & \text{ otherwise.}
\end{cases}
$$