Solution for Quiz #6

Let $X_1(t)$ and $X_2(t)$ be two CTMCs operating independently with respective state spaces $S_1 = \{1, 2\}$ and $S_2 = \{3, 4\}$ and generator matrices,

$$
Q_1 = \begin{bmatrix} -1 & 1 \\ \mu_1 & -\mu_1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -1 & 1 \\ \mu_2 & -\mu_2 \end{bmatrix}.
$$

Assume further that $\mathbb{P}(X_1(0) = 1) = 1$ and $\mathbb{P}(X_2(0) = 3) = 1$. Denote,

$$
\tau_1 = \inf\{t \geq 0 : X_1(t) = 2\} \quad \text{and} \quad \tau_2 = \inf\{t \geq \tau_1 : X_2(t) = 3\} - \tau_1.
$$

1) Argue why $\tau_1$ is an exponentially distributed random variable.

Answer: The time that $X_1(\cdot)$ spends in state 1 is exponentially distributed and afterwards it can only move to state 2.

2) What is $\mathbb{E}[\tau_1]$?

Answer: 1.

3) Is the distribution of $\tau_1$ phase-type?

Answer: Yes. The exponential distribution is a special case of PH.

4) Denote $\eta(t) = \mathbb{P}(X_2(t) = 3)$ and $\alpha = \mathbb{P}(\tau_2 = 0)$. Represent $\alpha$ in terms of $\eta(t)$.

Answer:

$$
\alpha = \mathbb{P}(\tau_2 = 0) = \mathbb{P}(X_2(\tau_1) = 3) = \int_0^{\infty} r(t)\mu_2 e^{-\mu_2 t} dt.
$$

5) Is the distribution of $\tau_2$ phase-type? If yes, give parameters of it in terms of $\alpha$ and $Q_2$.

Answer: PH$(1 - \alpha, -\mu_2)$. This is an exponential distribution with rate $\mu_2$ with probability $1 - \alpha$ and a chance for 0 with probability $\alpha$. 