Chapter 1: Introduction
Preface

This booklet contains lecture notes and exercises for a 2016 AMSI Summer School Course: “Linear Control Theory and Structured Markov Chains” taught at RMIT in Melbourne by Yoni Nazarathy. The notes are based on a subset of a draft book about a similar subject by Sophie Hautphenne, Erjen Lefeber, Yoni Nazarathy and Peter Taylor. The course includes 28 lecture hours spread over 3.5 weeks. The course includes assignments, short in-class quizzes and a take-home exam. These assessment items are to appear in the notes as well.

The associated book is designed to teach readers, elements of linear control theory and structured Markov chains. These two fields rarely receive a unified treatment as is given here. It is assumed that the readers have a minimal knowledge of calculus, linear algebra and probability, yet most of the needed facts are summarized in the appendix, with the exception of basic calculus. Nevertheless, the level of mathematical maturity assumed is that of a person who has covered 2-4 years of applied mathematics, computer science and/or analytic engineering courses.

Linear control theory is all about mathematical models of systems that abstract dynamic behavior governed by actuators and sensed by sensors. By designing state feedback controllers, one is often able to modify the behavior of a system which otherwise would operate in an undesirable manner. The underlying mathematical models are inherently deterministic, as is suited for many real life systems governed by elementary physical laws. The general constructs are system models, feedback control, observers and optimal control under quadratic costs. The basic theory covered in this book has reached relative maturity nearly half a century ago: the 1960’s, following some of the contributions by Kalman and others. The working mathematics needed to master basic linear control theory is centered around linear algebra and basic integral transforms. The theory relies heavily on eigenvalues, eigenvectors and others aspects related to the spectral decomposition of matrices.

Markov chains are naturally related to linear dynamical systems and hence linear control theory, since the state transition probabilities of Markov chains evolve as a linear dynamical system. In addition the use of spectral decompositions of matrices, the matrix exponential and other related features also resembles linear dynamical systems. The field of structured Markov chains, also referred to as Matrix Analytic Methods, goes back to the mid 1970’s, yet has gained popularity in the teletraffic, operations research
and applied probability community only in the past two decades. It is unarguably a more esoteric branch of applied mathematics in comparison to linear control theory and it is currently not applied as abundantly as the former field.

A few books at a similar level to this one focus on dynamical systems and show that the probabilistic evolution of Markov chains over finite state spaces behaves as linear dynamical systems. This appears most notably in [Lue79]. Yet, structured Markov chains are more specialized and posses more miracles. In certain cases, one is able to analyze the behavior of Markov chains on infinite state spaces, by using their structure. E.g. underlying matrices may be of block diagonal form. This field of research often focuses on finding effective algorithms for solutions of the underlying performance analysis problems. In this book we simply illustrate the basic ideas and methods of the field. It should be noted that structured Markov chains (as Markov chains in general) often make heavy use of non-negative matrix theory (e.g. the celebrated Perron-Frobenius Theorem). This aspect of linear algebra does not play a role in the classic linear control theory that we present here, yet appears in the more specialized study of control of non-negative systems.

Besides the mathematical relation between linear control theory and structured Markov chains, there is also a much more practical relation which we stress in this book. Both fields, together with their underlying methods, are geared for improving the way we understand and operate dynamical systems. Such systems may be physical, chemical, biological, electronic or human. With its styled models, the field of linear control theory allows us to find good ways to actually control such systems, on-line. With its ability to capture truly random behavior, the field of structured Markov chains allows us to both describe some significant behaviors governed by randomness, as well as to efficiently quantify (solve) their behaviors. But control does not really play a role.

With the exception of a few places around the world (e.g. the Mechanical Engineering Department at Eindhoven University of Technology), these two fields are rarely taught simultaneously. Our goal is to facilitate such action through this book. Such a unified treatment will allow applied mathematicians and systems engineers to understand the underlying concepts of both fields in parallel, building on the connections between the two.

Below is a detailed outline of the structure of the book. Our choice of material to cover was such as to demonstrate most of the basic features of both linear control theory and structured Markov chains, in a treatment that is as unified as possible.

**Outline of the contents:**

The notes contains a few chapters and some appendices. The chapters are best read sequentially. Notation is introduced sequentially. The chapters contain embedded short exercises. These are meant to help the reader as she progresses through the book, yet at the same time may serve as mini-theorems. That is, these exercises are both deductive
and informative. They often contain statements that are useful in their own right. The end of each chapter contains a few additional exercises. Some of these exercises are often more demanding, either requiring computer computation or deeper thought. We do not refer to computer commands related to the methods and algorithms in the book explicitly. Nevertheless, in several selected places, we have illustrated example MATLAB code that can be used.

For the 2016 AMSI summer school, we have indicated besides each chapter the in-class duration that this chapter will receive in hours.

**Chapter 1 (2h)** is an elementary introduction to systems modeling and processes. In this chapter we introduce the types of mathematical objects that are analyzed, give a feel for some applications, and describe the various use-cases in which such an analysis can be carried out. By a use-case we mean an activity carried out by a person analyzing such processes. Such use cases include “performance evaluation”, “controller design”, “optimization” as well as more refined tasks such as stability analysis, pole placement or evaluation of hitting time distributions.

**Chapter 2 (7h)** deals with two elementary concepts: Linear Time Invariant (LTI) Systems and Probability Distributions. LTI systems are presented from the viewpoint of an engineering-based “signals and systems” course. A signal is essentially a time function and system is an operator on functional space. Operators that have the linearity and time-invariance property are LTI and are described neatly by either their impulse response, step response, or integral transforms of one of these (the transfer function). It is here that the convolution of two signals plays a key role. Signals can also be used to describe probability distributions. A probability distribution is essentially an integrable non-negative signal. Basic relations between signals, systems and probability distributions are introduced. In passing we also describe an input–output form of stability: BIBO stability, standing for “bounded input results in bounded output”. We also present feedback configurations of LTI systems, showing the usefulness of the frequency domain (s-plane) representation of such systems.

**Chapter 3 (11h)** moves onto dynamical models. It is here that the notion of state is introduced. The chapter begins by introducing linear (deterministic) dynamical systems. These are basically solutions to systems of linear differential equations where the free variable represents time. Solutions are characterized by matrix powers in discrete time and matrix exponentials in continuous time. Evaluation of matrix powers and matrix exponentials is a subject of its right as it has to do with the spectral properties of matrices, this is surveyed as well. The chapter then moves onto systems with discrete countable (finite or infinite) state spaces evolving stochastically: Markov chains. The basics of discrete time and continuous time Markov chains are surveyed. In doing this a
few example systems are presented. We then move onto presenting input–state–output systems, which we refer to as \((A, B, C, D)\) systems. These again are deterministic objects. This notation is often used in control theory and we adopt it throughout the book. The matrices \(A\) and \(B\) describe the effect on input on state. The matrices \(C\) and \(D\) are used to describe the effect on state and input on the output. After describing \((A, B, C, D)\) systems we move onto distributions that are commonly called Matrix Exponential distributions. These can be shown to be directly related to \((A, B, C, D)\) systems. We then move onto the special case of phase type (PH) distributions that are matrix exponential distributions that have a probabilistic interpretation related to absorbing Markov chains. In presenting PH distributions we also show parameterized special cases.

Chapter 4 (0h) is not taught as part of the course. This chapter dives into the heart of Matrix Analytic Modeling and analysis, describing quasi birth and deaths processes, Markovian arrival processes and Markovian Binary trees, together with the algorithms for such models. The chapter begins by describing QBDs both in discrete and continuous time. Then moves onto Matrix Geometric Solutions for the stationary distribution showing the importance of the matrices \(G\) and \(R\). The chapter then shows elementary algorithms to solve for \(G\) and \(R\) focusing on the probabilistic interpretation of iterations of the algorithms. State of the art methods are summarized but are not described in detail. Markovian Arrival Point Processes and their various sub-classes are also surveyed. As examples, the chapter considers the M/PH/1 queue, PH/M/1 queue as well as the PH/PH/1 generalization. The idea is to illustrate the power of algorithmic analysis of stochastic systems.

Chapter 5 (4h) focuses on \((A, B, C, D)\) systems as used in control theory. Two main concepts are introduced and analyzed: state feedback control and observers. These are cast in the theoretical framework of basic linear control theory, showing the notions of controllability and observability. The chapter begins by introducing two physical examples of \((A, B, C, D)\) systems. The chapter also introduces canonical forms of \((A, B, C, D)\) systems.

Chapter 6 (3h) deals with stability of both deterministic and stochastic systems. Notions and conditions for stability were alluded to in previous chapters, yet this chapter gives a comprehensive treatment. At first stability conditions for general deterministic dynamical systems are presented. The concept of a Lyapounov function is introduced. This is the applied to linear systems and after that stability of arbitrary systems by means of linearization is introduced. Following this, examples of setting stabilizing feedback control rules are given. We then move onto stability of stochastic systems (essentially positive recurrence). The concept of a Foster-Lyapounov function is given for showing positive recurrence of Markov chains. We then apply it to quasi-birth-death processes.
proving some of the stability conditions given in Chapter 4 hold. Further stability conditions of QBD's are also given. The chapter also contains the Routh-Hourwitz and Jury criterions.

Chapter 7 (0h) is not taught as part of the course. is about optimal linear quadratic control. At first Bellman’s dynamic programming principle is introduced in generality, and then it is formulated for systems with linear dynamics and quadratic costs of state and control efforts. The linear quadratic regulator (LQR) is introduced together with its state feedback control mechanism, obtained by solving Riccati equations. Relations to stability are overviewed. The chapter then moves onto Model-predictive control and constrained LQR.

Chapter 8 (0h) is not taught as part of the course. This chapter deals with fluid buffers. The chapter involves both results from applied probability (and MAM), as well as a few optimal control examples for deterministic fluid systems controlled by a switching server. The chapter begins with an account of the classic fluid model of Anick, Mitra and Sondhi. It then moves onto additional models including deterministic switching models.

Chapter 9 (0h) is not taught as part of the course. This chapter introduces methods for dealing with deterministic models with additive noise. As opposed to Markov chain models, such models behave according to deterministic laws, e.g. 
\[(A, B, C, D)\]
systems, but are subject to (relatively small) stochastic disturbances as well as to measurement errors that are stochastic. After introducing basic concepts of estimation, the chapter introduces the celebrated Kalman filter. There is also brief mention of linear quadratic Gaussian control (LQG).

The notes also contains an extensive appendix which the students are required to cover by themselves as demand arises. The appendix contains proofs of results in cases where we believe that understanding the proof is instructive to understanding the general development in the text. In other cases, proofs are omitted.

Appendix A touches on a variety of basics: Sets, Counting, Number Systems (including complex numbers), Polynomials and basic operations on vectors and matrices.

Appendix B covers the basic results of linear algebra, dealing with vector spaces, linear transformations and their associated spaces, linear independence, bases, determinants and basics of characteristic polynomials, eigenvalues and eigenvectors including the Jordan Canonical Form.
Appendix C covers additional needed results of linear algebra.

Appendix D contains probabilistic background.

Appendix E contains further Markov chain results, complementing the results presented in the book.

Appendix F deals with integral transforms, convolutions and generalized functions. At first convolutions are presented, motivated by the need to know the distribution of the sum of two independent random variables. Then generalized functions (e.g. the delta function) are introduced in an informal manner, related to convolutions. We then present the Laplace transform (one sided) and the Laplace-Stiltijes Transform. Also dealing with the region of convergence (ROC). In here we also present an elementary treatment of partial fraction expansions, a method often used for inverting rational Laplace transforms. The special case of the Fourier transform is briefly surveyed, together with a discussion of the characteristic function of a probability distribution and the moment generating function. We then briefly outline results of the z-transform and of probability generating functions.

Besides thanking Sophie, Erjen and Peter, my co-authors for the book on which these notes are based, I would also like to thank (on their behalf) to several colleagues and students for valuable input that helped improve the book. Mark Fackrell and Nigel Bean’s analysis of Matrix Exponential Distributions has motivated us to treat the subjects of this book in a unified treatment. Guy Latouche was helpful with comments dealing with MAM. Giang Nugyen taught jointly with Sophie Hautphenene a course in Vietnam covering some of the subjects. A Master’s student from Eindhoven, Kay Peeters, visiting Brisbane and Melbourne for 3 months and prepared a variety of numerical examples and illustrations, on which some of the current illustrations are based. Also thanks to Azam Asanjarani and to Darcy Bermingham. The backbone of the book originated while the authors were teaching an AMSI summer school course, in Melbourne during January 2013. Comments from a few students such as Jessica Yue Ze Chan were helpful.

I hope you find these notes useful,
Yoni.
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Chapter 1

Introduction (2h)

A process is a function of time describing the behavior of some system. In this book we deal with several types of processes. Our aim is to essentially cover processes coming from two fields of research:

1. Deterministic linear systems and control.

2. Markovian stochastic systems with a structured state-space.

The first field is sometimes termed systems and control theory. Today it lies on the intersection of engineering and applied mathematics. The second field is called Matrix Analytic Methods (MAM), it is a sub-field of Applied Probability (which is sometimes viewed as a branch of Operations Research). MAM mostly deals with the analysis of specific types of structured Markov models.

Control and systems theory advanced greatly in the 1960’s due to the American and Soviet space programs. Matrix Analytic Methods is a newer area of research. It became a “recognized” subfield of applied probability sometime in the past 25 years. Thousands of researchers (and many more practitioners including control engineers) are aware and knowledgeable of systems and control theory. As opposed to that, MAM still remains a rather specialized area. At the basis of systems and control theory, lies the study of linear control theory (LCT). In this book we teach MAM and LCT together, presenting a unified exposition of the two fields where possible.

Our motivation for this unification is that both LCT and MAM use similar mathematical structures, patterns and results from linear algebra to describe models, methods and their properties. Further, both fields can sometimes be used to approach the same type of application, yet from different viewpoints. LCT yields efficient methods for designing automatic feedback controllers to systems. MAM yields efficient computational methods for performance analysis of a rich class of stochastic models.

In this introductory chapter informally introduce a variety of basic terms. In doing so, we do not describe LCT nor MAM further. We also motivate the study of dynamical
models, namely models that describe the evolution of processes over time. Further, we
survey the remainder of the book as well as the mathematical background appendix.

1.1 Types of Processes

The dynamical processes arising in LCT and MAM can essentially be classified into four
types. These types differ based on the time-index (continuous or discrete) and their
values (uncountable or countable). We generally use the following notation:

\begin{itemize}
  \item \( \{x(t)\} \) with \( t \in \mathbb{R} \) and \( x(t) \in \mathbb{R}^n \).
  \item \( \{X(t)\} \) with \( t \in \mathbb{R} \) and \( X(t) \in S \), where \( S \) is some countable (finite or infinite set).
  \item \( \{x(\ell)\} \) with \( \ell \in \mathbb{Z} \) and \( x(\ell) \in \mathbb{R}^n \).
  \item \( \{X(\ell)\} \) with \( \ell \in \mathbb{Z} \) and \( X(\ell) \in S \), where \( S \) is some countable (finite or infinite set).
\end{itemize}

The processes \( \{x(t)\} \) and \( \{X(t)\} \) are continuous time while the processes \( \{x(\ell)\} \) and
\( \{X(\ell)\} \) are discrete time. Considering the values that the processes take, \( \{x(t)\} \) and
\( \{x(\ell)\} \) take on values in some Euclidean vector space (uncountable), as opposed to that,
\( \{X(t)\} \) and \( \{X(\ell)\} \) take on values in some countable set.

In some instances the processes are viewed as deterministic. By this we mean their
trajectory is fixed and does not involve randomness. Alternatively they are modelled as
stochastic. This implies that their evolution involves some chance behaviour that can be
formally specified through a probability space. This means that there is not one unique
possible trajectory (also known as sample path in the stochastic case) of the process but
rather a collection (typically infinite collection) of possible realizations:

\[ \{X_\omega(\cdot), \ \omega \in \Omega \} \]

It is then a matter of the probability law of the process to indicate which specific real-

ization is taking place in practice.

Most of the LCT models that we cover in this book are of a deterministic nature. As
opposed to that, all of the MAM models that we cover are stochastic. The basic MAM
models that we introduce are based on Markov chains on countable state space (with
the exception of Chapter 8 on fluid queues). Hence we consider the processes \( X(\cdot) \) as
stochastic. Similarly the processes \( x(\cdot) \) are considered deterministic.

1.1.1 Representations of Countable State Spaces

Since the state space, \( S \) of the discrete-state stochastic processes, \( X(\cdot) \), is countable, we
can often treat it as \( \{1, \ldots, N\} \) for some finite \( N \) or \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \) depending on if
1.1. TYPES OF PROCESSES

1.1.1 Types of Processes

Figure 1.1: Illustration of realizations of different types of processes

it is finite or infinite. Nevertheless, for many of the stochastic processes that we shall consider it will be useful to represent $S$ as $\mathbb{Z}_+^2$ or some subset of it. In that case we shall call one coordinate of $s \in S$ as the level and the other coordinate as the phase. Further, since the process is now vector valued we will denote it by $\{X(t)\}$ in the continuous time case and $\{X(\ell)\}$ in the discrete time case.

1.1.2 Other Variations of Processes (omitted from course)

We shall also touch variations of the types of process, 1–4, detailed above. Which we informally discuss now. One such variation is taking a process with inherently deterministic dynamics, $x(\cdot)$, and adding stochastic “perturbations” to it. In discrete time this is typically done by adding “noise terms” at each of the steps of the process. In continuous time it is typically done by means of a stochastic differential equation. Both of these cases are important, yet they are out of the scope of this book.

Another variation is a continuous time, uncountable state (referred to as "continuous"
state) stochastic process that has *piece-wise linear* trajectories taking values in $\mathbb{R}$. In that case, one way to describe a trajectory of the process is based on a sequence of time points,

$$T_0 < T_1 < T_2, \ldots,$$

where the values of $X(t)$ for $t = T_\ell$, $\ell = 0, 1, 2, \ldots$ is given. Then for time points,

$$t \notin \{T_0, T_1, \ldots\},$$

we have,

$$X(t) = X(T_\ell) + (t - T_\ell) \frac{X(T_{\ell+1}) - X(T_\ell)}{T_{\ell+1} - T_\ell} \quad \text{if} \quad t \in (T_\ell, T_{\ell+1}).$$

### 1.1.3 Behaviours

We shall informally refer to the behavior of $x(\cdot)$ or $X(\cdot)$ as a description of the possible trajectories that these processes take. Some researchers have tried to formalize this in what is called the *behavioral approach* to systems. We do not discuss this further. The next section describes what we aim to do with respect to the behaviors of processes.

### 1.2 Use-cases: Modeling, Simulation, Computation, Analysis, Optimization and Control

What do we do with these processes, $x(\cdot)$ or $X(\cdot)$ in their various forms? Well, they typically arise as models of true physical situations. Concrete non-trivial examples are in the section below.

We now describe use-cases of models. i.e. the actions that we (as applied mathematicians) do with respect to models of processes. Each of these use-cases has an ultimate purpose of helping reach some goal (typically in applications).

#### 1.2.1 Modelling

We shall refer to the action of modeling as taking a true physical situation and setting up a deterministic process $x(\cdot)$ or a stochastic process $X(\cdot)$ to describe it. Note that "physical" should be interpreted in the general sense, i.e. it can be monetary, social or related to bits on digital computers. The result of the modeling process is a model which is essentially $x(\cdot)$ or $X(\cdot)$ or a family of such processes parameterized in some manner.

**Example 1.2.1.** Assume a population of individuals where it is observed (or believed):

Every year the population doubles.
Assume that at onset there are 10 individuals.

Here are some suggested models:

1. \( x(0) = 10 \) and \( x(\ell + 1) = 2x(\ell) \).

2. \( x(0) = 10 \) and \( \dot{x}(t) = (\log 2)x(t) \),

   where we use the notation \( \dot{x}(t) := \frac{dx}{dt} \) and \( \log \) is with the natural base.

3. \( \mathbb{P}(X(0) = 10) = 1 \) and

   \[
   X(\ell + 1) = \sum_{k=1}^{X(\ell)} \xi_{\ell,k},
   \]

   with \( \xi_{\ell,k} \) i.i.d. non-negative random variables with a specified distribution satisfying \( \mathbb{E}[\xi_{1,1}] = 2 \).

4. A continuous time branching process model with a behavior similar to 3 in the same way that the behavior of 2 is similar to 1. We do not specify this model further now.

As can be seen from the example above we have 4 different models that can be used to describe the same physical situation. The logical reasoning of which model is best is part of the action of modeling.
Exercise 1.2.2. Suggest another model that can describe the same situation. There is obviously not one correct answer.

1.2.2 Simulation

The action of simulation is the action of generating numeric realizations of a given model. For deterministic models it implies plotting $x(\cdot)$ in some manner or generating an array that represents a sample of its values. For stochastic models there is not one single realization, so it implies generating one or more realizations of $X(\cdot)$ by means of Monte-Carlo. That is, by using pseudo-random number generation and methods of stochastic simulation.

Simulation is useful for visualization but also for computation and analysis as we describe below.

Exercise 1.2.3. Simulate the trajectories of models (1) and (2) from Example 1.2.1. For model (3), simulate 4 sample trajectories. Plot all 6 realizations on one graph.

1.2.3 Computation and Analysis

The action of computation is all about finding descriptors related to the underlying models (or the underlying processes). Computation may be done by generating closed formulas for descriptors, by running algorithms, or by conducting deterministic or stochastic simulations of $x(\cdot)$ or $X(\cdot)$ respectively.

For example. A computation associated with model (1) of Example 1.2.1 is solving the difference equation to get,

$$x(\ell) = 10 \cdot 2^\ell.$$ (1.1)

In this case, the computation results in an analytical solution.

Exercise 1.2.4. What is the solution of model (2) of Example 1.2.1? How does it compare to (1.1)?

Getting explicit analytical solutions to differential equations is not always possible. Hence the difference between analysis and computation.

The action of analyzing is all about understanding the behaviors of the processes resulting from the model. In a concrete numerical setting it may mean comparing values for different parameters. For example, assume the parameter “twice” in Example 1.2.1 was replaced by $\alpha$. Alternatively it may mean proving theorems about the behaviors. This is perhaps the difference between practice and research, although the distinction is vague.

A synonymous term that encompasses both computation and analysis is performance analysis. Associated with the behaviors of $x(\cdot)$ or $X(\cdot)$ we often have performance measures. Here are some typical performance measures that may be of interest. Some of these are qualitative and some are quantitative:
1.2. USE CASES: MODELING, SIMULATION, COMPUTATION, ANALYSIS, OPTIMIZATION AND CONTROL

1. Stability
2. Fixed point
3. Mean
4. Variance
5. Distribution
6. Hitting times

Computation and analysis is typically done with respect to performance measures such as the ones above or others.

1.2.4 Optimization

Making models is often so that we can optimize the underlying physical process. The idea is that trying the underlying process for all possible combinations is typically not possible, so optimizing the model may be preferred. In a sense optimization may be viewed as a decoupled step from the above, since one can often formulate some optimization problem in terms of objects that come out of performance measures of the process.

1.2.5 Control

Optimization is typically considered to be something that we do over a slow time scale, while control implies intervening with the physical process continuously with a hope of making the behavior more suitable to requirements. The modeling type of action done here is the design of the control law. This in fact, yields a modified model, with modified behaviors.

**Example 1.2.5.** We continue with the simple population growth example. Assume that culling is applied whenever the population reaches a certain level, d. In that case, individuals are removed bringing the population down to level c where c < d.

This is a control policy. Here the aim of the control is obviously to keep the “population at bay”. The values c and d are parameters of the control policy (also called the “control” or the “controller”).

**Exercise 1.2.6.** Repeat Exercise 1.2.3 with this policy where c = 10 and d = 300.

**Exercise 1.2.7.** Formulate some non-trivial optimization problem on the parameters of the control policy. For this you need to “make up some story” of costs etc...
1.2.6 Our Scope

In this book we focus on quite specific processes. For the stochastic ones we carry out analysis (and show methods to do so) - but do not deal with control. For the deterministic ones we do both analysis and control. The reason for “getting more” out of the deterministic models is that they are in fact simpler. So why use stochastic models if we do not talk about control? Using them for performance measures can be quite fruitful and can perhaps give better models of the physical reality than the deterministic models (in some situations).

1.3 Application Examples

Moving away from the population growth example of the previous section, we now introduce four general examples that we will vaguely follow throughout the book. We discuss the underlying “physics” of these examples and will continue to refer to them in the chapters that follow.

1.3.1 An Inverted Pendulum on a Cart

Consider a cart fixed on train tracks on which there is a tall vertical rod above the cart, connected to the cart on a joint. The cart can move forward and backwards on the train tracks. The rod tends to fall to one of the sides – it has 180 degrees of movement.

For simplicity we assume that there is no friction for the cart on the train tracks and that there is no friction for the rod. That is there is no friction on the joint between the rod and the cart and there is no air friction when the rod falls down.

We assume there are two controlled motors in the system. The first can be used to apply force on the cart pushing it forward or backwards on the train tracks. The second can be used to apply a torque on the rod at the joint.

This idealized physical description is already a physical model. It is a matter of physical modeling to associate this model (perhaps after mild modifications or generalizations) to certain applications. Such applications may be a “Segway Machine” or the firing of a missile vertically up to the sky.

This physical model can be described by differential equations based on Newton’s laws (we will do so later on). Such a mathematical model describes the physical system well and can then be used for simulation, computation, analysis, optimization and control.

It is with respect to this last use-case (control) that the inverted pendulum on a cart is so interesting. Indeed if forces are not applied through the motor and if the rod is not at rest in an angle of either 0, 90 or 180 degrees, then it will tend to fall down to the angles of 0 or 180. That is, it is unstable. Yet with proper “balancing” through the motors, the rod may be stabilized at 90 degrees. As we discuss control theory, we will
see how to do this and analyze this system further.

### 1.3.2 A Chemical Engineering Processes

Consider a cylindrical fluid tank containing water and a dissolved chemical in the water. Assume that there is a stirring propeller inside the tank that is stirring it well. The tank is fed by two input flows. One of pure water and one of water with the chemical dissolved in it. There is output flow from the tank at the bottom. It is known that the output flow rate is proportional to the square root of the height of the water level in the tank.

The system operator may control the incoming flow of pure water, the incoming flow of water with dissolved chemical, and the concentration of dissolved chemical coming in. Two goals that the operator wants to achieve are:

1. Keep the fluid level in tank within bounds. I.e. not to let it underflow and not to let it overflow.
2. Maintain a constant (or almost constant) concentration of the chemical in the outgoing flow.

Here also we will see how such a model can be described and controlled well by means of linear control theory. Further, this model has some flavor of a queueing model. Queueing models play a central role in MAM.

### 1.3.3 A Manufacturing Line

Consider a manufacturing process in which items move from one operating station to the next until completion. Think of the items as cars in a car manufacturing plant. Frames arrive to the line from outside and then cars pass through stations one by one until they pass the last station and are fully assembled and ready. At each station assume there is one operator which serves the items that have arrived to it sequentially - one after the other. Thus, each station in isolation is in fact a queue of items waiting to be served. In practice there are often room limitations: most stations may only accommodate a finite number of items. If a station is full, the station “upstream to it” can not pass completed items down, etc.

Industrial engineers managing, optimizing and controlling such processes often try to minimize randomness and uncertainty in such processes, yet this is not always possible:

- Service stations break down occasionally, often at random durations.
- The arrivals of raw materials is not always controlled.
• There is variability in the service times of items at individual stations. Thus the output from one station to the next is a variable process also.

Besides the fact that variability plays a key role, this application example is further different from the previous two in that items are discrete. Compare this to the previous two applications where momentum, speed, concentration, fluid flows and volume are all purely continuous quantities.

A mathematical model based on MAM can be applied to this application example. Especially to each of the individual stations in isolation (aggregating the whole model using an approximation). Yet, if item processing durations are short enough and there is generally a non-negligible amount of items, then the process may also be amenable to control design based on LCT.

1.3.4 A Communication Router

A Communication router receives packets from $n$ incoming sources and passes each to $m$ outgoing destinations. Upon arrival of a packet it is known to which output port (destination) it should go, yet if that port is busy (because another packet is being transmitted on it) then the incoming packets needs to be queued in memory. In practice such systems sometimes work in discrete time enforced by the design of the router. Here packet arrivals are random and bursty and it is often important to make models that capture the essential statistics of such arrival processes. This is handled well by MAM. Further, the queueing phenomena that occur are often different than those of the manufacturing line due to the high level of variability in packet arrivals.

Bibliographic Remarks

There are a few books focusing primarily on MAM. The first of these was [Neu94] which was followed by [Neu89]. A newer manuscript which gives a comprehensive treatment of methods and algorithms is [LR99]. Certain chapters of [Asm03] also deal with MAM. Other MAM books are [BB05].

Exercises

1. Choose one of the four application examples appearing in Section 1.3 (Inverted Pendulum, Chemical Plant, Manufacturing Line, Communication Router). For this example do the following:
(a) Describe the application in your own words while stating the importance of having a mathematical model for this application. Use a figure if necessary. Your description should be half a page to two pages long.

(b) Suggest the flavor of the type of mathematical model (or models) that you would use to analyze, optimize and control this example. Justify your choice.

(c) Refer to the use cases appearing in Section 1.2. Suggest how each of these applies to the application example and to the model.

(d) Consider the performance analysis measures described under the use case “computation and Analysis” in Section 1.2. How does each of these use cases apply to the application example and model that you selected?
Bibliography


