Fisher Information for a Partially-Observable Simple Birth Process

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Definition and Notation

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- It is **Markovian** with infinitesimal conditions

$$\Pr(X_{t+h} = j | X_t = i) = \begin{cases} \lambda ih + \mathcal{O}(h) & \text{for } j = i + 1 \\ 1 - \lambda ih + \mathcal{O}(h) & \text{for } j = i \\ \mathcal{O}(h) & \text{otherwise} \end{cases} .$$

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- **Transition probability** $\Pr(X_{s+t} = j | X_s = i) \; := \; p_{ij}(t)$:

$$p_{ij}(t) \;=\; \binom{j-1}{i-1} e^{-\lambda t i}(1 - e^{-\lambda t})^{j-i} .$$

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- A good tool to measure the volume of information gained from a sample is the **Fisher Information**.
- It can be shown that

$$\mathcal{FI}_{(X_{t_1}, X_{t_2}, \cdots, X_{t_n})}(\lambda) = E_{\mathcal{L}}\left[\left(\frac{d}{d\lambda}\ln(\mathcal{L}(X_{t_1}, X_{t_2}, \ldots, X_{t_n}; \lambda))\right)^2\right].$$

The Fisher Information for the Simple Birth Process

Proposition (Becker and Kersting, 1983)

The **Fisher Information** *for the simple birth process with the parameter* λ*, the initial value of* x_0 *and the sampling times of* $(t_1, t_2, \ldots, t_n)$ *is as follows:*

$$\mathcal{FI}_{(x_{t_1}, x_{t_2}, \cdots, x_{t_n})}(\lambda) = x_0 \sum_{i=1}^{n} \frac{(t_i - t_{i-1})^2}{e^{-\lambda t_{i-1}} - e^{-\lambda t_i}} .$$

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- We call the stochastic process $\{Y_t : t \in \mathrm{R}_0^+\}$ the **partially-observable simple birth process (POSBP)** with parameters (λ, p).
- $\text{POSBP}(\lambda, 1) \equiv \text{SBP}(\lambda)$.

Markovian or non-Markovian?

Theorem (Bean, Elliott, Eshragh and Ross; 2013)

The POSBP $\{Y_t : t \in \mathrm{R}_0^+\}$ *with parameters* (λ, p) *is* **not Markovian**.

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- However,

$$\Pr(Y_{t_1} = y_{t_1}, Y_{t_2} = y_{t_2}, \ldots, Y_{t_n} = y_{t_n} | X_{t_1} = x_{t_1}, X_{t_2} = x_{t_2}, \ldots, X_{t_n} = x_{t_n})$$

$$= \prod_{i=1}^{n} \Pr(Y_{t_i} = y_{t_i} | X_{t_i} = x_{t_i}) .$$

The Fisher Information for the POSBP

- The Fisher Information:

$$\mathcal{FI}_{(Y_{t_1},Y_{t_2},\cdots,Y_{t_n})}(\lambda) = \sum_{y_{t_1},y_{t_2},\ldots,y_{t_n}} \frac{\left(\frac{d\mathcal{L}(y_{t_1},y_{t_2},\ldots,y_{t_n};\,\lambda)}{d\lambda}\right)^2}{\mathcal{L}(y_{t_1},y_{t_2},\ldots,y_{t_n};\,\lambda)}.$$

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- Here, the likelihood function $\mathcal{L}(y_{t_1},y_{t_2},\ldots,y_{t_n};\,\lambda)$ is equal to

$$\sum_{x_{t_1},\ldots,x_{t_n}} \prod_{i=1}^{n} \binom{x_{t_i}}{y_{t_i}} p^{y_i} q^{x_{t_i}-y_{t_i}} \binom{x_{t_i}-1}{x_{t_{i-1}}-1} \upsilon_{i-1,i}^{x_{t_{i-1}}} (1-\upsilon_{i-1,i})^{x_{t_i}-x_{t_{i-1}}},$$

where $\upsilon_{i-1,i} := e^{-\lambda(t_i-t_{i-1})}$.

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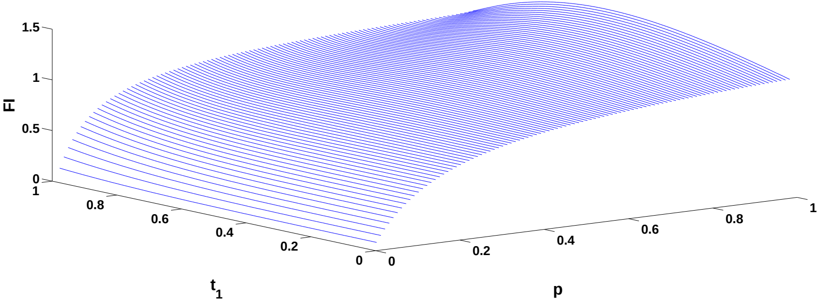
where $\upsilon_{i-1,i} := e^{-\lambda(t_i-t_{i-1})}$.

- By exploiting **Chebyshev's inequality**, we have

$$\Pr\left(E[Z]-12\sqrt{Var(Z)} \leq Z \leq E[Z]+12\sqrt{Var(Z)}\right) \geq 1-\frac{1}{12^2} = 99.3\%.$$

Results for $x_0 = 1$, $\lambda = 2$, $n = 2$ and $t_2 = 1$

- The Fisher Information vs. t_1 and p



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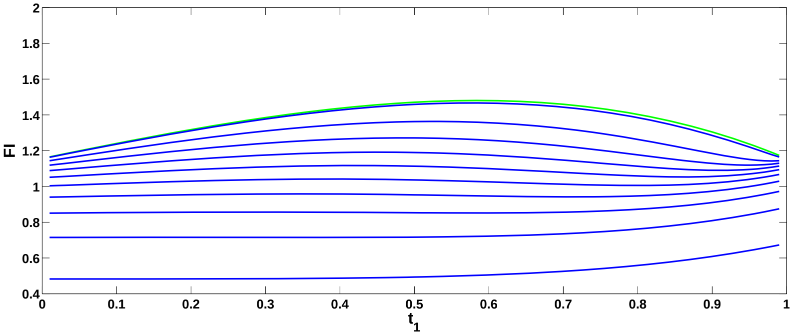

The Chain Rule

- The likelihood function

$$\mathcal{L}(y_{t_1}, y_{t_2}|\lambda) = \Pr(Y_{t_2} = y_{t_2}|Y_{t_1} = y_{t_1}, \lambda)\Pr(Y_{t_1} = y_{t_1}|\lambda).$$

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- Accordingly,

$$\log\left(\mathcal{L}(y_{t_1}, y_{t_2}|\lambda)\right) = \log\left(\Pr(Y_{t_2} = y_{t_2}|Y_{t_1} = y_{t_1}, \lambda)\right) + \log\left(\Pr(Y_{t_1} = y_{t_1}|\lambda)\right).$$

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- The Fisher Information:

$$\mathcal{FI}_{(Y_{t_1}, Y_{t_2})}(\lambda) = \mathcal{FI}_{(Y_{t_2}|Y_{t_1})}(\lambda) + \mathcal{FI}_{(Y_{t_1})}(\lambda).$$

Delayed Geometric Distribution

Definition

A discrete random variable V has the "**Delayed Geometric**" distribution with parameters $\alpha \in [0,1)$ and $\beta \in (0,1)$, denoted by **DG**(α,β), if its **probability mass function** (**p.m.f.**) is

$$P_V(v) = \begin{cases} \alpha & \text{for } v = 0 \\ (1-\alpha)\beta(1-\beta)^{v-1} & \text{for } v = 1, 2, \ldots . \end{cases}$$

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Remark

The $DG(\beta,\beta)$ and $DG(0,\beta)$ distributions reduce, respectively, to the **Geometric distribution-failure model** *and* **-success model** *both with parameter β.*

Delayed Negative Binomial Distribution

Definition

Suppose $V_1, \cdots, V_r$ are **i.i.d.** random variables with common $DG(\alpha, \beta)$ distribution. If $\mathbf{W} := \sum_{i=1}^{r} \mathbf{V}_i$, then W has "**Delayed Negative Binomial**" distribution with parameters $\mathbf{r}$, α and β, denoted by $\mathbf{DNB}(\mathbf{r}, \alpha, \beta)$.

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Proposition (Bean, Eshragh and Ross; 2013)

If W follows the $DNB(r, \alpha, \beta)$ distribution, then its **p.m.f.** *is*

$$P_W(w) = \begin{cases} \alpha^r & \text{for } w = 0 \\ \displaystyle\sum_{\xi=1}^{\min\{r,w\}} \binom{w-1}{\xi-1} \beta^{\xi} (1-\beta)^{w-\xi} \binom{r}{\xi} (1-\alpha)^{\xi} \alpha^{r-\xi} & \text{for } w \geq 1 \end{cases}$$

The Distribution of Y_t

Theorem (Bean, Eshragh and Ross; 2013)

Consider the **POSBP** $\{Y_t,\ t \geq 0\}$ *with* **parameters** (λ, p) *and the* **initial population size** $x_0 \geq 1$. *For any real value* $t > 0$, *the random variable* Y_t *follows the* $\mathbf{DNB(x_0, (1-p)\beta_t, \beta_t)}$ *distribution where*

$$\beta_{\mathbf{t}} := \frac{\mathbf{e}^{-\lambda \mathbf{t}}}{\mathbf{p} + (\mathbf{1} - \mathbf{p})\mathbf{e}^{-\lambda \mathbf{t}}}.$$

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Corollary (Bean, Eshragh and Ross; 2013)

Consider the **POSBP** $\{Y_t,\ t \geq 0\}$ *with* **parameters** (λ, p) *and the* **initial population size** $x_0 = 1$. *For any real value* $t > 0$, *the random variable* Y_t *follows the* $\mathbf{DG}((\mathbf{1}-\mathbf{p})\boldsymbol{\beta}_\mathbf{t}, \boldsymbol{\beta}_\mathbf{t})$ *distribution.*

The Fisher Information for a Single Observation

Proposition (Bean, Eshragh and Ross; 2013)

Consider the **POSBP** $\{Y_t,\ t \geq 0\}$ *with* **parameters** (λ, p) *and the* **initial population size** $x_0 = 1$. *The Fisher Information of a single observation* Y_{t_1} *for parameter* λ *is equal to*

$$\mathcal{FI}_{\mathbf{Y}_1}(\lambda) = \frac{p t_1^2 \left(p + (1-p)(1-e^{-\lambda t_1})e^{-\lambda t_1}\right)}{(1-e^{-\lambda t_1})(p + (1-p)e^{-\lambda t_1})^2}.$$

The Distribution of $(Y2|Y1 = y_{t_1})$

Theorem (Bean, Eshragh and Ross; 2013)

Consider the **POSBP** $\{Y_t,\ t \geq 0\}$ *with* **parameters** (λ, p) *and the* **initial population size** $x_0 = 1$. *Then*

$$\mathbf{W_1 \stackrel{d}{=} (Y_{t_2}|Y_{t_1} = y_{t_1}) + V_1}$$

where $(Y_{t_2}|Y_{t_1} = y_{t_1})$ *and* V_1 *are mutually independent and* $W_1 \sim DNB(y_{t_1} + 1, (1-p)\beta^\circ, \beta^\circ)$ *and* $V_1 \sim DG((1-p)\beta_{t2-t1}, \beta_{t2-t1})$.

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$$(\mathbf{Y_{t_2}} | \mathbf{Y_{t_1}} = \mathbf{y_{t_1}}) \overset{\mathbf{d}}{=} \mathbf{W_2} + \mathbf{V_2}$$

where $W_2 \sim DNB(y_{t_1}, (1-p)\beta^\circ, \beta^\circ)$ *and* $V_2 \sim DG\left(\left(pe^{\lambda(t_2-t_1)} + 1 - p\right)\beta^\circ, \beta^\circ\right)$ *are two independent random variables.*

Bounds for the General Form of the Fisher Information

Theorem

If $Z_1, \cdots, Z_n$ are independent random variables from distributions with common unknown parameter γ and $\mathbf{g} : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}$ is a real-value function, then

$$\mathcal{FI}_{\mathbf{g}(\mathbf{Z}_1,\cdots,\mathbf{Z}_\mathbf{n})}(\gamma) \leq \sum_{\mathbf{i}=1}^{\mathbf{n}} \mathcal{FI}_{\mathbf{Z}_\mathbf{i}}(\gamma).$$

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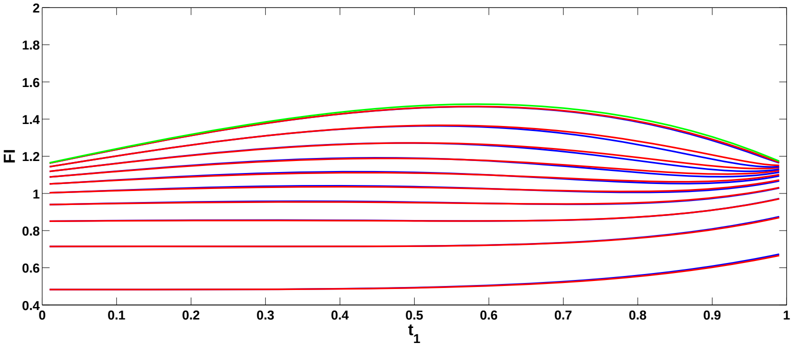
- Also, the **Carmer-Rao lower bound** implies that

$$\mathcal{FI}_{\mathbf{g}(\mathbf{Z}_1, \cdots, \mathbf{Z}_n)}(\gamma) \geq \frac{\left(\frac{\partial \mathrm{E}\left[\mathbf{g}(\mathbf{Z}_1, \cdots, \mathbf{Z}_n)\right]}{\partial \gamma}\right)^2}{\mathrm{Var}\left(\mathbf{g}(\mathbf{Z}_1, \cdots, \mathbf{Z}_n)\right)}.$$

Results for $x_0 = 1$, $\lambda = 2$, $n = 2$ and $t_2 = 1$

- The Fisher Information (blue) and its Approximation (red) vs. t_1



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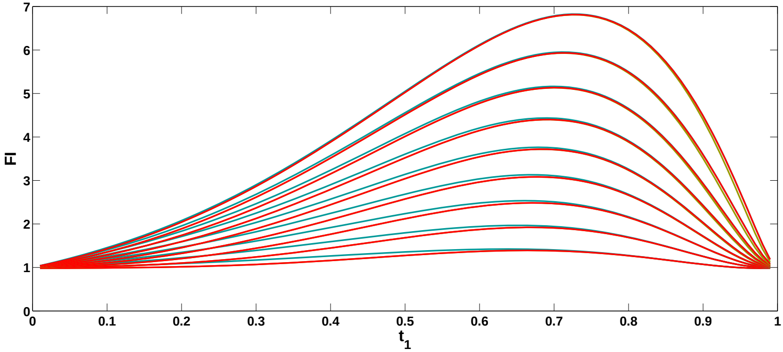
Theorem (Bean, Eshragh and Ross; 2013)

The lower and upper bounds for the Fisher Information **approach together** *as λ tends to infinity.*

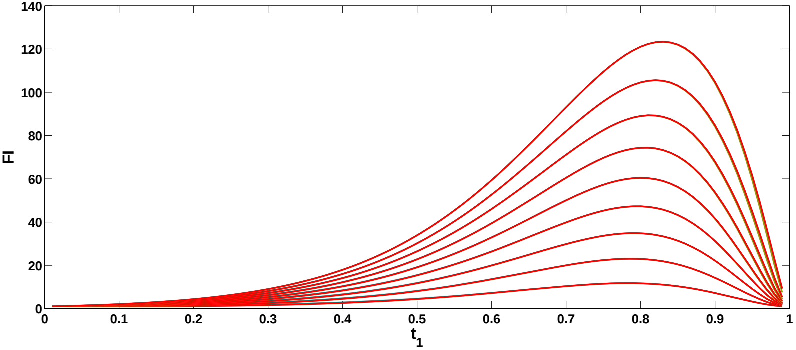
Results for $x_0 = 1$, $\lambda = 6$, $n = 2$ and $t_2 = 1$

- Lower (brown) and Upper (green) Bounds for The Fisher Information and its Approximation (red) vs. t_1



Results for $x_0 = 1$, $\lambda = 10$, $n = 2$ and $t_2 = 1$

- Lower (brown) and Upper (green) Bounds for The Fisher Information and its Approximation (red) vs. t_1



Further Developments

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- Finding the Fisher Information to estimate parameter **p** along with λ, both together.

End

Thank you $\cdots$ Questions?