On the mixing advantage

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Joint work with Aidan Sudbury, Peter Jagers & Daniel Tokarev

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- We wish to compare $\mathbb{E}[\max(X_1, X_2, \ldots, X_n)]$ to $M_i = \mathbb{E}[\max(X_i^1, \ldots, X_i^n)]$, $i = 1, \ldots, n$.

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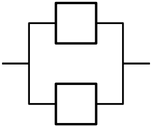

Reliability – Warm Duplication Method

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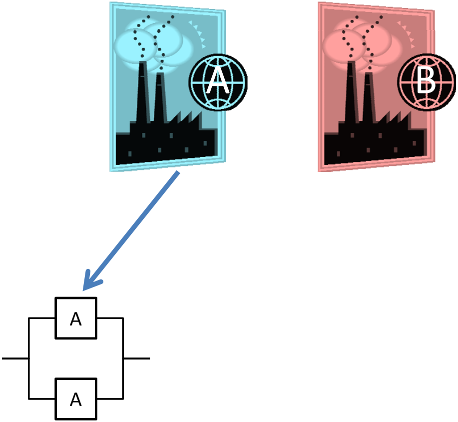

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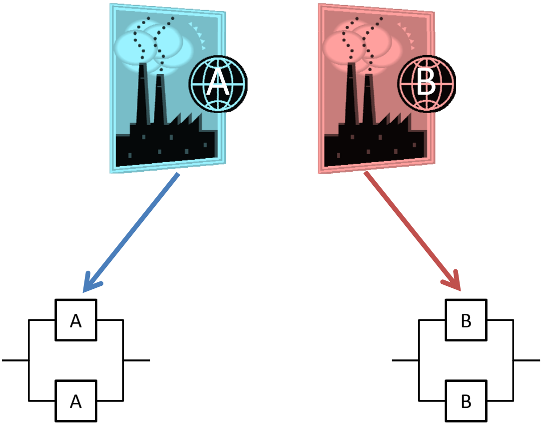

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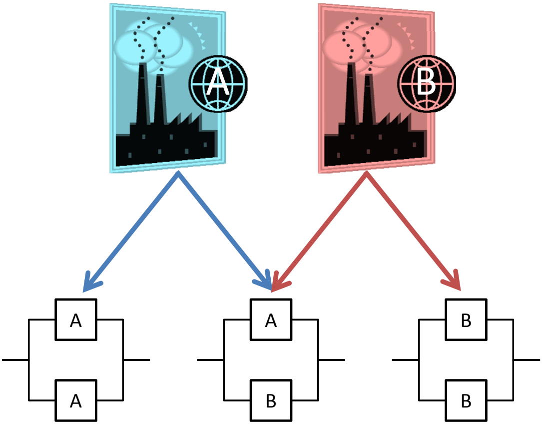

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- Question: What if all types are similar (no dominant type); i.e.

$$\mathbb{E}[\max(X_1^1, \ldots, X_1^n)] = \ldots = \mathbb{E}[\max(X_n^1, \ldots, X_n^n)]?$$

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In fact, the same arithmetic-geometric mean inequality shows that

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  In other words, mixing is advantageous.

- If $M_i = \mathbb{E}[\max(X_i^1, \dots, X_i^n)]$, $i = 1, \dots, n$, we call mixing factor

$$\theta = \frac{\mathbb{E}[\max(X_1, \dots, X_n)]}{\max(M_1, \dots, M_n)}.$$

  We show that when $M_i = M$, $\theta \leq 2 - 1/n < 2$.

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- Arnold and Groeneveld (1979) obtain upper and lower bounds on $\mathbb{E}[\max(X_1, \ldots, X_n)]$ even when $X_1, \ldots, X_n$ are not independent and not identically distributed, but in terms of $\mathbb{E}[X_1]$ and $\text{var}(X_i)$, not $M_1, \ldots, M_n$.
  This generalises Hartley and David (1954) and Gumbel (1954) who deal with the iid case.

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- Sen (1970) shows that $\max(X_1, \ldots, X_n)$ stochastically dominates $\max(Y^1, \ldots, Y^n)$, where $Y^1, \ldots, Y^n$ are iid equally-weighted probability mixtures of $X_1, \ldots, X_n$:

$$\mathbb{P}(\max(X_1, \ldots, X_n) \leq z) \leq \mathbb{P}(\max(Y^1, \ldots, Y^n) \leq z).$$

In particular

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[\max(X_i^1, \ldots, X_i^n)] \\ &\leq \; \mathbb{E}[\max(Y^1, \ldots, Y^n)] \; \leq \; \mathbb{E}[\max(X_1, \ldots, X_n)]. \end{aligned}$$

However, $\mathbb{E}[\max(Y^1, \ldots, Y^n)]$ cannot be expressed in terms of $M_1, \ldots, M_n$.

Unbounded independent case

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Theorem (H., Jagers, Sudbury & Tokarev, 2009)

If $X_1, \ldots, X_n$ are independent random variables with the property that $\mathbb{E}[\max(X_i^1, \ldots, X_i^n)] = M_i$, $i = 1, 2, ..., n$, then

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} M_i &\leq \mathbb{E}[\max(X_1, \ldots, X_n)] \\ &\leq \frac{1}{n}\sum_{i=1}^{n} M_i + \frac{n-1}{n}\max(M_1, \ldots, M_n). \end{aligned}$$

In particular, if $M_i = M$, $i = 1, ..., n$,

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In particular, if $M_i = M$, $i = 1, ..., n$,

$$M \leq \mathbb{E}[\max(X_1, \ldots, X_n)] \leq (2 - 1/n)M.$$

The upper bound is obtained by letting some of the random variables be concentrated on 0 and x and letting $x \to \infty$.

Bounded independent case

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Theorem (H. & Sudbury, 2011)

If a set of random variables $X_1, \dots, X_n$ are independent, concentrated on $[0, b]$ and s.t.

$$\mathbb{E}[\max(X_i^1, \dots, X_i^n)] = M_i, i = 1, \dots, n,$$

then, putting $M_n = \max(M_1, \dots, M_n)$,

$$\begin{aligned} b - \prod_{i=1}^{n}(b - M_i)^{1/n} &\leq \mathbb{E}[\max(X_1, \dots, X_n)] \\ &\leq b - (b - M_n)\prod_{i=1}^{n-1}(1 - M_i/b)^{1/n}. \end{aligned}$$

Bounded independent case

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Corollary

In the case $M_i = M, i = 1, \dots, n$ we have

$$M \leq \mathbb{E}[\max(X_1, \dots, X_n)] \leq b - b(1 - M/b)^{2-1/n}$$

where the latter expression approaches $(2-1/n)M$ as $b \to +\infty$ and $M(2 - M/b)$ as $n \to +\infty$.

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The equivalent result for the minima, with $m_1 = \min(m_1, \ldots, m_n)$, is

$$m_1 \prod_{i=2}^{n} (m_i/b)^{1/n} \leq \mathbb{E}[\min(X_1, \ldots, X_n)] \leq \prod_{i=1}^{n} m_i^{1/n}.$$

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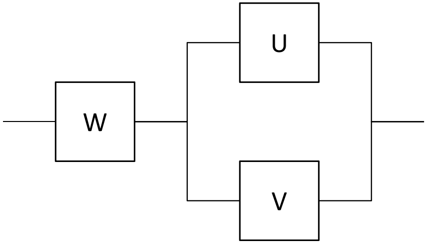

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  - $K(s, t) = s \wedge t - \psi(s \wedge t) + (s \vee t)\psi(s \wedge t)$ – $(U \wedge W, V \wedge W)$.

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- $M_i = \mathbb{E}[\max(X_i^1, X_i^2)] = C(p_i, p_i)a_i + (1 - C(p_i, p_i))x_i$, $i = 1, 2$.

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Assumption

We assume that for any (s,t),

$$C(s,t) - sC(t,t) \geq 0 \text{ and } C(s,t) - tC(s,s) \geq 0. \qquad (\star)$$

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- Π, M and K satisfy this condition; W does not.
- If (U, V) are uniform $(0, 1)$ and have copula C, then $(\star)$ translates to

$$\mathbb{P}(U \leq s | \max(U,V) \leq t) \geq \mathbb{P}(U \leq s), \quad s < t.$$

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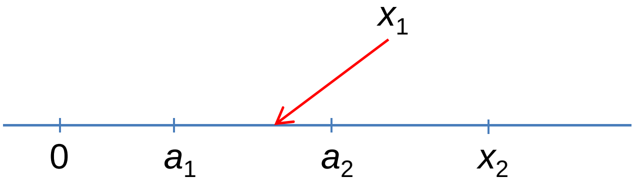

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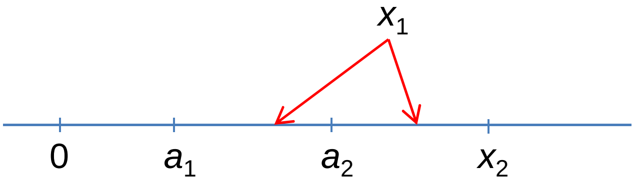

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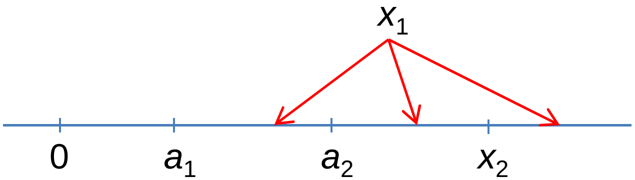

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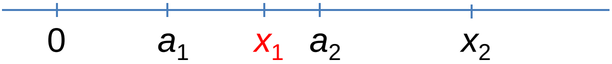

The case $a_1 \leq x_1 \leq a_2 \leq x_2$ is trivial since in this case X_2 dominates X_1.

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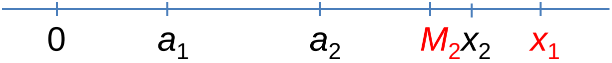

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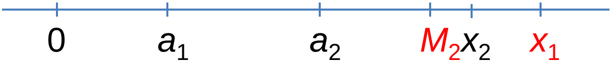

Assume $a_1 \leq a_2 \leq x_2 \leq x_1$.

$$\begin{aligned}
&\mathbb{E}[\max(X_1, M_2)] - \mathbb{E}[\max(X_1, X_2)] \\
&\quad = \quad p_1 M_2 - C(p_1, p_2) a_2 - (p_1 - C(p_1, p_2)) x_2 \\
&\quad = \quad \Big(C(p_1, p_2) - p_1 C(p_2, p_2) \Big) (x_2 - a_2) \ \geq \ 0.
\end{aligned}$$

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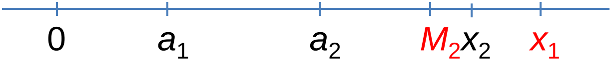

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$$
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&\mathbb{E}[\max(X_1, M_2)] - \mathbb{E}[\max(X_1, X_2)] \\
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\end{aligned}
$$

Therefore we may replace X_2 with M_2.

Dependent case – A toy example

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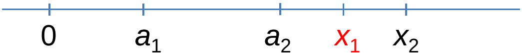

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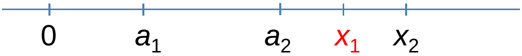
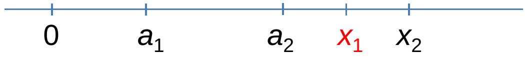
Dependent case – A toy example



Assume $a_1 \leq a_2 \leq x_1 \leq x_2$.
We vary a_2 and x_2 keeping p_2 (and a_1, x_1, p_1) constant:

$$\mathbb{E}[\max(X_2^1, X_2^2)] = C(p_2, p_2)a_2 + (1 - C(p_2, p_2))x_2 = M_2.$$

Dependent case – A toy example



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$$\mathbb{E}[\max(X_1, X_2)] = C(p_1, p_2)a_2 + (1 - p_2)x_2 + (p_2 - C(p_1, p_2))x_1$$

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- $x_2 = x_1$. In this case we may collapse X_2 into M_2.
- In any case, we may assume that $X_2 = M_2$ and $a_1 \leq M_2 \leq x_1$.

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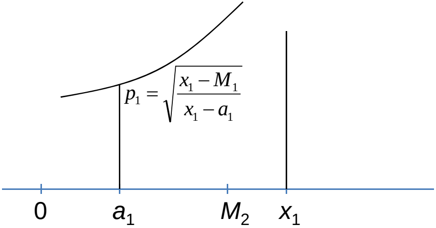

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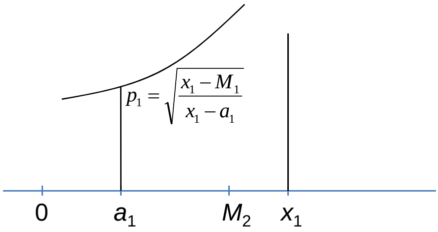

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$$\mathbb{E}[\max(X_1^1, X_1^2)] = C(p_1, p_1)a_1 + (1 - C(p_1, p_1))x_1 = M_1.$$

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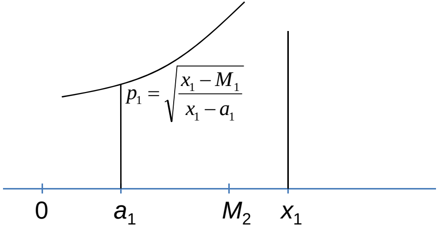

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Then,

$$\mathbb{E}[\max(X_1, M_2)] = p_1 M_2 + (1 - p_1)x_1 = x_1 - (x_1 - M_2)p_1$$

is maximum for p_1 minimum i.e. $a_1 = 0$.

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Therefore we may assume that $X_2 = M_2$, $a_1 = 0$ and $0 \leq M_2 \leq x_1$.

Dependent case – A toy example

Therefore we may assume that $X_2 = M_2$, $a_1 = 0$ and $0 \leq M_2 \leq x_1$. In this case

$$\mathbb{E}[\max(X_1, X_2)] = p_1 M_2 + (1 - p_1) \frac{M_1}{1 - C(p_1, p_1)}$$

Theorem

$$\mathbb{E}[\max(X_1, X_2)] \leq \sup_{0 \leq r < 1} \left(M_2 r + M_1 \frac{1 - r}{1 - C(r, r)} \right).$$

The upper bound for Π, M and K

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Let $\gamma(r) = C(r, r)$ and assume that $M_1 \leq M_2$.

- $C = \Pi$. In this case $\gamma(r) = r^2$, $\gamma'(1) = 2$ and

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- Note that the definition of M_i depends on C and the three bounds cannot be compared.

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