Approximation of the conditional number of exceedances

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July 9, 2013

Joint work with Aihua Xia
Conditional exceedances

Given a sequence of identical but not independent random variables \( X_1, \ldots, X_n \), define

\[
N_s, n = \sum_{i=1}^{n} \mathbb{1}\{X_i > s\}.
\]

The fragility index of order \( m \), denoted \( FI_n(m) \) is defined as

\[
FI_n(m) = \lim_{s \to \infty} E(N_{s, n} | N_{s, n} \geq m).
\]

Applications: Insurance.
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Applications: Insurance.
Compound Poisson limit

Under certain conditions, Tsing et al. (1988) showed that the exceedance process converges to a Compound Poisson limit.
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- Uniform convergence.
- Uniform integrability.

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Approximation of the conditional number of exceedances
In contrast to limit theorems, we can use approximation theory. Our tool of choice will be Stein’s method.
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For the next section of the talk, we shall stick to conditioning on just one exceedance, Poisson approximation, and total variation distance.
A simple example

Let $X_1, \ldots, X_n$ be a sequence of i.i.d. exponential random variables, $p = p_s = \mathbb{P}(X_1 > s)$ and $Z_\lambda \sim Po(\lambda)$. 
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$$d_{TV}(\mathcal{L}(N^1), \text{Po}^1(\lambda)) = \frac{1}{2} \sum_{j=1}^{\infty} \left| \mathbb{P}(N^1 = j) - \mathbb{P}(Z_\lambda^1 = j) \right|.$$
A simple example (2)

Set $\lambda$ such that $e^{-\lambda} = (1 - p)^n$. 
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So now we have

$$d_{TV}(\mathcal{L}(N^1), Po^1(\lambda)) = \frac{1}{2(1 - \mathbb{P}(N = 0))} \sum_{j=1}^{\infty} |\mathbb{P}(N = j) - \mathbb{P}(Z_\lambda = j)|.$$
A simple example (3)

Set $\lambda^* = np$, and we can reduce the problem to known results.

$$d_{TV}(\mathcal{L}(N^1), Po^1(\lambda)) = \frac{1}{2(1 - P(N = 0))} \sum_{j=1}^{\infty} \left[ |P(N = j) - P(Z_{\lambda^*} = j)| + |P(Z_{\lambda^*} = j) - P(Z_{\lambda} = j)| \right].$$

Using results from Barbour, Holst and Janson (1992), it can be shown that:

$$d_{TV}(\mathcal{L}(N^1), Po^1(\lambda)) = p + o(p).$$
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Set $\lambda^* = np$, and we can reduce the problem to known results.

$$d_{TV}(\mathcal{L}(N^1), Po^1(\lambda)) = \frac{1}{2(1 - \mathbb{P}(N = 0))} \sum_{j=1}^{\infty} [\left| \mathbb{P}(N = j) - \mathbb{P}(Z_{\lambda^*} = j) \right| + \left| \mathbb{P}(Z_{\lambda^*} = j) - \mathbb{P}(Z_\lambda = j) \right|].$$

Using results from Barbour, Holst and Janson (1992), it can be shown that:

$$d_{TV}(\mathcal{L}(N^1), Po^1(\lambda)) = p + o(p).$$
The conditional Poisson Stein Identity

**Lemma**

\[ W \sim \text{Po}^1(\lambda) \text{ if and only if for all bounded functions } g : \mathbb{Z}^+ \to \mathbb{R}, \]

\[ \mathbb{E} \left[ \lambda g(W + 1) - Wg(W) \cdot 1_{\{W \geq 2\}} \right] = 0. \]
Stein’s method in one slide

We construct a function $g$ for any set $A$, that satisfies:

$$1_{\{j \in A\}} - Po^1(\lambda)\{A\} = \lambda g(j + 1) - jg(j) \cdot 1_{\{j \geq 2\}}.$$
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$$\mathbb{P}(W \in A) - Po(\lambda)\{A\} = \mathbb{E} \left[ \lambda g(W + 1) - Wg(W) \cdot 1_{\{W \geq 2\}} \right].$$
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It now follows that,

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P(W \in A) - Po(\lambda)\{A\} = \mathbb{E} \left[ \lambda g(W + 1) - Wg(W) \cdot 1_{\{W \geq 2\}} \right].
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We then only need to bound the right hand side.
Generator interpretation

If we set $g(w) = h(w) - h(w - 1)$, the equation becomes:
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This looks like the generator for an immigration death process.
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This looks like the generator for a immigration death process.

Moreover, the stationary distribution for this generator is \( Po^1(\lambda) \).
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Theorem

The solution $g$ that satisfies the Stein Equation for the total variation distance satisfies:

$$||\Delta g|| \leq \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{\lambda(1 - e^{-\lambda})}.$$
The example revisited

Recall that from before:

\[ d_{TV}\left(\mathcal{L}(N_{1}), \text{Po}(\lambda)\right) = p + o(p). \]

Using Stein's method directly for conditional Poisson approximation, it can be shown that:

\[ d_{TV}\left(\mathcal{L}(N_{1}), \text{Po}(\lambda^*)\right) = p^2 + o(p). \]
The example revisited

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Using Stein’s method directly for conditional Poisson approximation, it can be shown that:

\[ d_{TV}(\mathcal{L}(N^1), Po^1(\lambda^*)) = \frac{p}{2} + o(p). \]
Generalisations

The conditional Poisson Stein Identity generalises for conditioning on multiple exceedances:

\[
W \sim \text{Po}(\lambda) \quad \text{if and only if for all bounded functions } g: \{a, a+1, \ldots\} \to \mathbb{R},
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E\left[\lambda g(W+1) - Wg(W) \cdot 1\{W > a\}\right] = 0.
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Generalisations

The conditional Poisson Stein Identity generalises for conditioning on multiple exceedances:

**Lemma**

\[ W \sim Po^a(\lambda) \text{ if and only if for all bounded functions } g : \{a, a + 1, \ldots\} \to \mathbb{R}, \]

\[ \mathbb{E} \left[ \lambda g(W + 1) - Wg(W) \cdot 1_{\{W > a\}} \right] = 0. \]
Negative Binomial Approximation

We can do exactly the same for negative binomial random variables.

Lemma

\[ W \sim \text{Nb}^a(r, p) \text{ if and only if for all bounded functions } \]

\[ g : \{a, a + 1, \ldots\} \to \mathbb{R}, \]

\[ \mathbb{E} \left[ (1 - p)(r + W)g(W) - Wg(W)) \cdot 1_{\{W > a\}} \right] = 0. \]
Different metrics

Have we solved our original problem? Can we calculate errors when estimating fragility indices?
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No, we still have the uniform integrability problem.
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No, we still have the uniform integrability problem.

We need to use a stronger metric, such as the Wasserstein distance.