Approximation of the conditional number of exceedances

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Joint work with Aihua Xia

Conditional exceedances

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Applications: Insurance.

Compound Poisson limit

Under certain conditions, Tsing et. al. (1988) showed that the exceedance process converges to a Compound Poisson limit.

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- Uniform integrability.

Alternatives

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For the next section of the talk, we shall stick to conditioning on just one exceedance, Poisson approximation, and total variation distance.

A simple example

Let $X_1, \ldots, X_n$ be a sequence of i.i.d. exponential random variables, $p = p_s = \mathbb{P}(X_1 > s)$ and $Z_\lambda \sim Po(\lambda)$.

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$$d_{TV}(\mathcal{L}(N^1), Po^1(\lambda)) = \frac{1}{2} \sum_{j=1}^{\infty} \left| \mathbb{P}(N^1 = j) - \mathbb{P}(Z^1_\lambda = j) \right| .$$

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Set λ such that $e^{-\lambda} = (1 - p)^n$.

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So now we have

$$d_{TV}(\mathcal{L}(N^1), Po^1(\lambda)) = \frac{1}{2(1-\mathbb{P}(N=0))} \sum_{j=1}^{\infty} \left| \mathbb{P}(N=j) - \mathbb{P}(Z_\lambda = j) \right|.$$

A simple example (3)

Set $\lambda^* = np$, and we can reduce the problem to known results.

$$d_{TV}(\mathcal{L}(N^1), Po^1(\lambda)) = \frac{1}{2(1 - \mathbb{P}(N = 0))} \sum_{j=1}^{\infty} [|\mathbb{P}(N = j) - \mathbb{P}(Z_{\lambda^*} = j)| + |\mathbb{P}(Z_{\lambda^*} = j) - \mathbb{P}(Z_{\lambda} = j)|].$$

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Using results from Barbour, Holst and Janson (1992), it can be shown that:

$$d_{TV}(\mathcal{L}(N^1), Po^1(\lambda)) = p + o(p).$$

The conditional Poisson Stein Identity

Lemma

$W \sim Po^1(\lambda)$ if and only if for all bounded functions $g : \mathbb{Z}^+ \to \mathbb{R}$,

$$\mathbb{E}\left[\lambda g(W+1) - Wg(W) \cdot \mathbf{1}_{\{W \geq 2\}}\right] = 0.$$

Stein's method in one slide

We construct a function g for any set A, that satisfies:

$$\mathbf{1}_{\{j \in A\}} - Po^1(\lambda)\{A\} = \lambda g(j+1) - jg(j) \cdot \mathbf{1}_{\{j \geq 2\}}.$$

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We then only need to bound the right hand side.

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Moreover, the stationary distribution for this generator is $Po^1(\lambda)$.

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Theorem

The solution g that satisfies the Stein Equation for the total variation distance satisfies:

$$||\Delta g|| \leq \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{\lambda(1 - e^{-\lambda})}.$$

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Using Stein's method directly for conditional Poisson approximation, it can be shown that:

$$d_{TV}(\mathcal{L}(N^1), Po^1(\lambda^*)) = \frac{p}{2} + o(p).$$

Generalisations

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$W \sim Po^a(\lambda)$ if and only if for all bounded functions $g : \{a, a+1, \dots\} \to \mathbb{R}$,

$$\mathbb{E}\left[\lambda g(W+1) - Wg(W) \cdot \mathbf{1}_{\{W>a\}}\right] = 0.$$

Negative Binomial Approximation

We can do exactly the same for negative binomial random variables.

Lemma

$W \sim Nb^a(r, p)$ if and only if for all bounded functions $g : \{a, a+1, \ldots\} \to \mathbb{R}$,

$$\mathbb{E}\left[(1-p)(r+W)g(W) - Wg(W)) \cdot \mathbf{1}_{\{W>a\}}\right] = 0.$$

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We need to use a stronger metric, such as the Wasserstein distance.