Multitype branching processes in a random environment
Would they survive forever?

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Multitype Markovian branching processes

- Describe the evolution of a population of $m$ types of individuals over time
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\Omega := [\Omega_{ij}] \quad \text{where } \Omega_{ij} \text{ is the total rate of } i \to j, \ (1 \leq i, j \leq m).
\]
Multitype Markovian branching processes

- Describe the evolution of a population of \( m \) types of individuals over time

\[ \Omega := [\Omega_{ij}] \text{ where } \Omega_{ij} \text{ is the total rate of } i \to j, \ (1 \leq i, j \leq m). \]

\[ e^{\Omega t} : \text{ the expected population size matrix at time } t \]
Extinction!

\( \{Z_t, t \in \mathbb{R}^+\}, \ Z_t := (Z_{t1}, Z_{t2}, \ldots, Z_{tm}) \)

\( Z_{ti} \): the number of individuals of type \( i \) alive at time \( t \)
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- Extinction probability vector \( q := (q_1, q_2, \ldots, q_m) \), with
  \[
  q_i := P[\exists T < \infty : Z_T = 0 | Z_0 = e_i]
  \]

- Growth rate \( \lambda \): dominant eigenvalue of \( \Omega \)
  \[
  q = 1 \iff \lambda \leq 0
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Random environment

\begin{align*}
\Omega^{(1)} & \quad \Omega^{(2)} & \quad \Omega^{(3)} \\
\end{align*}
Markovian random environment

- $\{X(t) : t \in \mathbb{R}^+\}$: ergodic continuous-time Markov chain, s.t.
  \[ \Omega = \Omega^{(i)} \text{ when } X(t) = i, \quad i \in \{1, 2, \ldots, r\}. \]
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- \( \{\hat{X}_n : n \in \mathbb{N}\} \): jump chain associated with \( \{X(t)\} \),
- \( \{\xi_n : n \in \mathbb{N}\} \): sequence of intervals between two transitions in \( \{X(t)\} \).

Theorem (Tanny, 1981)
There exists a constant \( \omega \) such that
\[
\lim_{n \to \infty} \frac{1}{n} \log \prod_{i=0}^{n} \mathbb{E}_{\Omega_{\hat{X}_0}} \mathbb{E}_{\Omega_{\hat{X}_1}} \cdots \mathbb{E}_{\Omega_{\hat{X}_{n-1}}} = \omega \quad \text{a.s.},
\]
independently of \( i \) and \( j \), and
\[ q = 1 \iff \omega \leq 0. \]
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**Theorem (Tanny, 1981)**

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\]

independently of \( i \) and \( j \), and \( q = 1 \Leftrightarrow \omega \leq 0 \).
Motivation

The limit

$$\omega = \lim_{n \to \infty} \frac{1}{n} \log \left\{ e^{\Omega(\hat{\mathbf{X}}_0)\xi_0} e^{\Omega(\hat{\mathbf{X}}_1)\xi_1} \ldots e^{\Omega(\hat{\mathbf{X}}_{n-1})\xi_{n-1}} \right\}$$

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With Guy Latouche and Giang Nguyen, we have worked on a similar problem:


We have constructed lower and upper bounds for \( \omega \).
Catastrophes

- Follow a **Poisson process** with rate $\beta = 1/E[\xi]$

- At each catastrophe epoch: type $i$ survives with probability $\delta_i$, or dies with probability $1 - \delta_i$
Extinction criteria

- Survival probability matrix $\Delta_{\delta} := \text{diag}(\delta_1, \ldots, \delta_m)$
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- $\{\xi_n, n \geq 1\}$: sequence of time intervals between catastrophes
  
  Here: $\xi_n$ i.i.d and $\sim \text{Exp}(\beta)$
  
  But our results hold for any stationary ergodic sequence
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  But our results hold for any stationary ergodic sequence

  Tanny’s Theorem implies that there exists a constant $\omega$ such that

  $$\lim_{n \to \infty} \frac{1}{n} \log \{ e^{\Omega \xi_1} \Delta_\delta e^{\Omega \xi_2} \Delta_\delta \cdots e^{\Omega \xi_n} \Delta_\delta \}_{ij} = \omega$$

  a.s., independently of $i$ and $j$, and

  $$q = 1 \iff \omega \leq 0$$
Looking for bounds

\[ \omega = \lim_{n \to \infty} \frac{1}{n} \log \left\{ e^{\Omega \xi_1 \Delta_\delta} e^{\Omega \xi_2 \Delta_\delta} \cdots e^{\Omega \xi_n \Delta_\delta} \right\}_{ij} \]
Looking for bounds

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- If killing is \textit{uniform}, that is, \( \delta := \delta_1 = \delta_2 = \cdots = \delta_m \), then

\[ \omega = \lambda E[\xi] + \log \delta, \]

where \( \lambda \) is the dominant eigenvalue of \( \Omega \).
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- If killing is **uniform**, that is, \( \delta := \delta_1 = \delta_2 = \cdots = \delta_m \), then
  \[ \omega = \lambda E[\xi] + \log \delta, \]
  where \( \lambda \) is the dominant eigenvalue of \( \Omega \)

- If killing is **not uniform**, \( \Delta_\delta \) modifies the eigenvectors of \( e^{\Omega \xi} \) in different ways for different values of \( \xi \)
A duality approach

(1) \( \Omega^* := \Omega - \lambda I \): has one eigenvalue 0, and all others have strictly negative real part

\[
\omega = \lim_{n \to \infty} \frac{1}{n} \log \{ e^{\Omega \xi_1 \Delta \delta} \cdots e^{\Omega \xi_n \Delta \delta} \}_{ij}
\]

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= \lambda E[\xi] + \lim_{n \to \infty} \frac{1}{n} \log \{ e^{\Omega^* \xi_1 \Delta \delta} \cdots e^{\Omega^* \xi_n \Delta \delta} \}_{ij}
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$$= \lambda E[\xi] + \lim_{n \to \infty} \frac{1}{n} \log \{ e^{\Omega^* \xi_1 \Delta \delta} \cdots e^{\Omega^* \xi_n \Delta \delta} \}_{ij}$$

(2) Let $\mathbf{v}$ be the left eigenvector of $\Omega^*$ corresponding to 0. Define $\Theta := \Delta_v^{-1} \Omega^* \Delta_v$, with $\Delta_v = \text{diag}(\mathbf{v})$.

$\Theta$ is a generator!

$$\rightarrow \omega = \lambda E[\xi] + \lim_{n \to \infty} \frac{1}{n} \log \{ e^{\Theta \xi_1 \Delta \delta} \cdots e^{\Theta \xi_n \Delta \delta} \}_{ij}$$
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\[ \omega = \lim_{n \to \infty} \frac{1}{n} \log \{ e^{\Omega_1 \Delta_\delta} \cdots e^{\Omega_n \Delta_\delta} \} \]

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- Random matrices \( e^{\Omega \xi} \Rightarrow \) random **stochastic** matrices \( e^{\Theta \xi} \)
A duality approach

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- Random matrices \( e^{\Omega \xi} \Rightarrow \text{random stochastic matrices} \ e^{\Theta \xi} \)
- The whole population of a branching process \( \Rightarrow \text{one single particle} \) which evolves according to the Markov dual process \( \{ \varphi_t \} \) with generator \( \Theta \)
A duality approach

- \( \{\theta_n, n \geq 1\} \): successive epochs of catastrophes
- \( S \): first epoch when the single particle does not survive
- \( \varphi_n \): the state of the single particle at catastrophe epoch \( \theta_n \)
A duality approach

- \{\theta_n, n \geq 1\}: successive epochs of catastrophes
- \(S\): first epoch when the single particle does not survive
- \(\varphi_n\): the state of the single particle at catastrophe epoch \(\theta_n\)

\[
\omega = \lambda E[\xi] + \lim_{n \to \infty} \frac{1}{n} \log \{ e^{\Theta \xi_1 \Delta \delta \cdots e^{\Theta \xi_n \Delta \delta}} \}_{ij}
\]

\[
= \lambda E[\xi] + \lim_{n \to \infty} \frac{1}{n} \log P[S > \theta_n, \varphi_n = j | \varphi_0 = i, \theta_1, \ldots, \theta_n]
\]
An upper bound for $\omega$

\[\omega = \lambda E[\xi] + \lim_{n \to \infty} \frac{1}{n} \log P[S > \theta_n, \varphi_n = j|\varphi_0 = i, \theta_1, \ldots, \theta_n]\]

\[\leq \lambda E[\xi] + \lim_{n \to \infty} \frac{1}{n} \log E[P[S > \theta_n, \varphi_n = j|\varphi_0 = i, \theta_1, \ldots, \theta_n]]\]

\[= \lambda E[\xi] + \lim_{n \to \infty} \frac{1}{n} \log P[S > \theta_n, \varphi_n = j|\varphi_0 = i]\]

\[= \lambda E[\xi] + \log \text{sp}\{\beta(\beta I - \Theta)^{-1}\Delta_\delta}\}

$\beta(\beta I - \Theta)^{-1}\Delta_\delta$: transition matrix for $\{\varphi_t\}$ embedded immediately after catastrophe epochs
**A lower bound for \( \omega \)**

\[
\omega = \lambda E[\xi] + \lim_{n \to \infty} \frac{1}{n} \log P[S > \theta_n, \varphi_n = j | \varphi_0 = i, \theta_1, \ldots, \theta_n]
\]

\[
\geq \lambda E[\xi] + \lim_{n \to \infty} \frac{1}{n} \log(P[S > \theta_n, \varphi_n = j | \varphi_0 = i, \varphi_1, \ldots, \varphi_{n-1}, \theta_1, \ldots, \theta_n])
\]

\[
= \lambda E[\xi] + \lim_{n \to \infty} \log \delta_1^{n_1/n} \delta_2^{n_2/n} \cdots \delta_m^{n_m/n} [(e^{\Theta n \Delta \delta})_{\varphi_{n-1,j}}]^{1/n}
\]

\[
= \lambda E[\xi] + \sum_{1 \leq i \leq m} \pi_i \log \delta_i,
\]

\(\pi\): stationary distribution of \(\{\varphi_t\}\), \(\pi = u^T \Delta v\)

\(u\) and \(v\): left and right eigenvectors of \(\Omega^*\) corresponding to 0
Bounds for Poisson catastrophes

In summary,

**Theorem**

\[
\frac{\lambda}{\beta} + \sum_{1 \leq i \leq n} u_i v_i \log \delta_i \leq \omega \leq \frac{\lambda}{\beta} + \log \text{sp} \left[ \beta (\beta I - \Theta)^{-1} \Delta \right]
\]

where

\[
\begin{align*}
\lambda & \quad = \text{dominant eigenvalue of } \Omega, \\
u, v & \quad = \text{left & right eigenvectors of } \Omega \text{ corresp. to } \lambda, \text{ with } u 1 = 1, uv = 1, \\
\Theta & \quad = \Delta v^{-1} (\Omega - \lambda I) \Delta v.
\end{align*}
\]
The bounds are tight

Recall that when killing is uniform:

$$
\omega = \lambda E[\xi] + \log \delta
$$

In this case,

$$
\lambda E[\xi] + \sum_{1 \leq i \leq m} \pi_i \log \delta_i = \omega = \lambda E[\xi] + \log sp\{\beta(\beta I - \Theta)^{-1} \Delta \delta}\}$$
North Atlantic right whales
North Atlantic right whales: The model

1=calf, 2=immature, 3=mature, 4=reproducing, 5=post-breeding

Transition rates

Birth rate
North Atlantic right whales: The effect of survival probabilities

Catastrophes follow a Poisson process with $E(\xi) = 25$ years
Back to random environments

We have high hopes that the same type of duality approach may be used to find bounds for

$$\omega = \lim_{n \to \infty} \frac{1}{n} \log \left\{ e^{\Omega(\hat{X}_0)\xi_0} e^{\Omega(\hat{X}_1)\xi_1} \ldots e^{\Omega(\hat{X}_{n-1})\xi_{n-1}} \right\}_{ij}$$

for more general random environments.
Markovian random environment with **two states**:

**Theorem**

\[ \omega_l \leq \omega \leq \omega_u \]  

with

\[
\begin{align*}
\omega_l &= \frac{1}{2} \left[ (\lambda_1/c_1 + \lambda_2/c_2) + (\pi_1 - \pi_2) \log(\Delta_{v_1}^{-1} v_2) \right] \\
\omega_u &= \frac{1}{2} \left[ (\lambda_1/c_1 + \lambda_2/c_2) + \log \text{sp}(\tilde{M}) \right]
\end{align*}
\]

where

\[
\begin{align*}
\tilde{M} &= c_1 c_2 [(c_1 + \lambda_1)I - \Omega^{(1)}]^{-1} [(c_2 + \lambda_2)I - \Omega^{(2)}]^{-1} \\
c_i &= \text{parameter of the exponential sojour time in environment } i \\
\lambda_i &= \text{max. eigenvalue of } \Omega^{(i)} \\
\mathbf{u}_i, \mathbf{v}_i &= \text{left & right eigenvectors corresp. to } \lambda_i, \text{ s.t. } \mathbf{u}_i^T \mathbf{1} = 1, \mathbf{u}_i^T \mathbf{v}_i = 1 \\
\pi_i &= \mathbf{u}_i^T \Delta_{v_i}
\end{align*}
\]
Thank you for your attention.