# Multitype branching processes in a random environment

## Would they survive forever?

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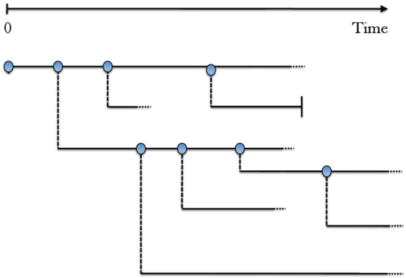
# Multitype Markovian branching processes

- Describe the evolution of a population of $m$ types of individuals over time



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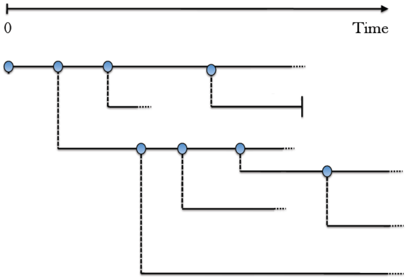


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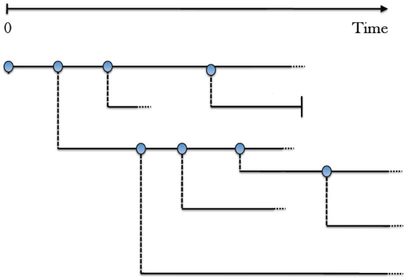


- $\Omega := [\Omega_{ij}]$ where $\Omega_{ij}$ is the total rate of $i \to j$, $(1 \leq i, j \leq m)$.
- $e^{\Omega t}$: the expected population size matrix at time $t$

# Extinction!

- $\{\mathbf{Z}_t, t \in \mathbb{R}^+\}$, $\mathbf{Z}_t := (Z_{t1}, Z_{t2}, \ldots, Z_{tm})$

  $Z_{ti}$: the number of individuals of type $i$ alive at time $t$

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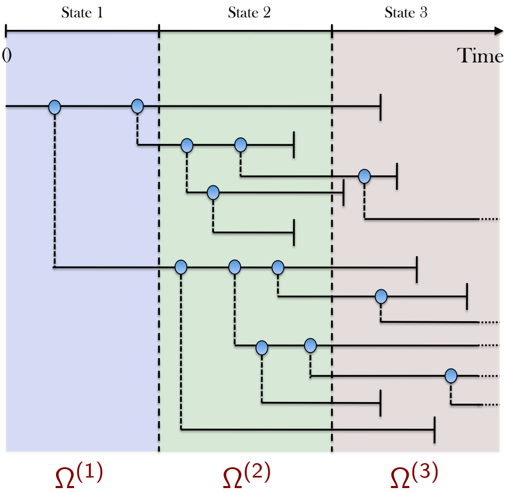
- Growth rate $\lambda$: dominant eigenvalue of $\Omega$

$$\mathbf{q} = \mathbf{1} \quad \Leftrightarrow \quad \lambda \leq 0$$

# Random environment



# Markovian random environment

- $\{X(t) : t \in \mathbb{R}^+\}$: ergodic continuous-time Markov chain, s.t.

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**Theorem (Tanny, 1981)**

*There exists a constant $\omega$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \left\{ e^{\Omega^{(\hat{X}_0)}\xi_0} \, e^{\Omega^{(\hat{X}_1)}\xi_1} \, \dots \, e^{\Omega^{(\hat{X}_{n-1})}\xi_{n-1}} \right\}_{ij} = \omega \quad a.s.,$$

*independently of $i$ and $j$, and* $\quad \mathbf{q} = \mathbf{1} \Leftrightarrow \omega \leq 0.$

# Motivation

The limit

$$\omega = \lim_{n\to\infty} \frac{1}{n} \log \left\{ e^{\Omega^{(\hat{x}_0)}\xi_0}\, e^{\Omega^{(\hat{x}_1)}\xi_1} \,\ldots\, e^{\Omega^{(\hat{x}_{n-1})}\xi_{n-1}} \right\}_{ij}$$

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With Guy Latouche and Giang Nguyen, we have worked on a similar problem:

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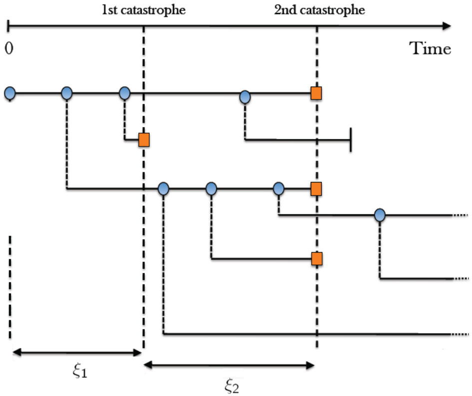
We have constructed lower and upper bounds for $\omega$.

# Catastrophes

- Follow a Poisson process with rate $\beta = 1/E[\xi]$




- At each catastrophe epoch: type $i$ survives with probability $\delta_i$, or dies with probability $1 - \delta_i$

# Extinction criteria

- Survival probability matrix $\Delta_\delta := \mathrm{diag}(\delta_1, \ldots, \delta_m)$

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  But our results hold for any stationary ergodic sequence

Tanny's Theorem implies that there exists a constant $\omega$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log\{e^{\Omega\xi_1} \Delta_\delta \, e^{\Omega\xi_2} \Delta_\delta \cdots e^{\Omega\xi_n} \Delta_\delta\}_{ij} = \omega \quad \text{a.s.,}$$

independently of $i$ and $j$, and

$$\mathbf{q} = \mathbf{1} \quad \Leftrightarrow \quad \omega \leq 0$$

# Looking for bounds

$$\omega = \lim_{n\to\infty} \frac{1}{n} \log\{e^{\Omega\xi_1}\Delta_\delta\, e^{\Omega\xi_2}\Delta_\delta \cdots e^{\Omega\xi_n}\Delta_\delta\}_{ij}$$

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$$\omega = \lambda E[\xi] + \log\delta,$$

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- If killing is not uniform, $\Delta_\delta$ modifies the eigenvectors of $e^{\Omega\xi}$ in different ways for different values of $\xi$

# A duality approach

(1) $\Omega^* := \Omega - \lambda I$: has one eigenvalue $0$, and all others have strictly negative real part

$$\begin{aligned}\omega &= \lim_{n\to\infty} \frac{1}{n} \log\{e^{\Omega\xi_1}\Delta_\delta \cdots e^{\Omega\xi_n}\Delta_\delta\}_{ij} \\ &= \lambda E[\xi] + \lim_{n\to\infty} \frac{1}{n} \log\{e^{\Omega^*\xi_1}\Delta_\delta \cdots e^{\Omega^*\xi_n}\Delta_\delta\}_{ij}\end{aligned}$$

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(2) Let $\mathbf{v}$ be the left eigenvector of $\Omega^*$ corresponding to $0$.
Define $\Theta := \Delta_v^{-1}\,\Omega^*\,\Delta_v$, with $\Delta_v = \mathrm{diag}(\mathbf{v})$.
$\Theta$ is a generator!

$$\rightarrow \quad \omega = \lambda E[\xi] + \lim_{n\to\infty} \frac{1}{n} \log\{e^{\Theta\xi_1}\Delta_\delta \cdots e^{\Theta\xi_n}\Delta_\delta\}_{ij}$$

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- Random matrices $e^{\Omega\xi}$ $\Rightarrow$ random stochastic matrices $e^{\Theta\xi}$

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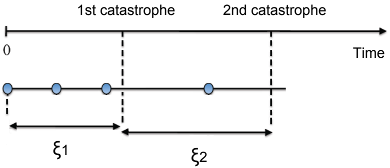
- Random matrices $e^{\Omega\xi}$ $\Rightarrow$ random stochastic matrices $e^{\Theta\xi}$
- The whole population of a branching process $\Rightarrow$ one single particle which evolves according to the Markov dual process $\{\varphi_t\}$ with generator $\Theta$



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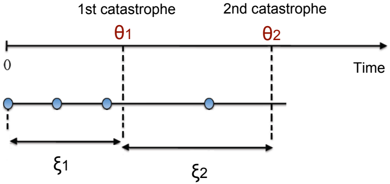
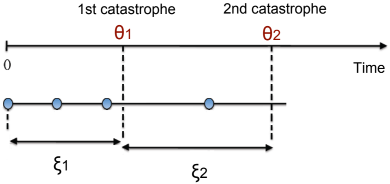



- $\{\theta_n, n \geq 1\}$: successive epochs of catastrophes
- $S$: first epoch when the single particle does not survive
- $\varphi_n$: the state of the single particle at catastrophe epoch $\theta_n$

# A duality approach

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- $S$: first epoch when the single particle does not survive
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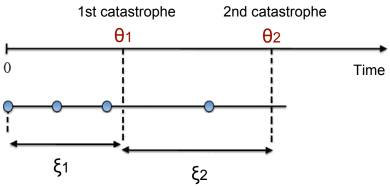
$$\omega = \lambda E[\xi] + \lim_{n\to\infty} \frac{1}{n} \log\{e^{\Theta \xi_1}\Delta_\delta \cdots e^{\Theta \xi_n}\Delta_\delta\}_{ij}$$

$$= \lambda E[\xi] + \lim_{n\to\infty} \frac{1}{n} \log P[S > \theta_n, \varphi_n = j | \varphi_0 = i, \theta_1, \ldots, \theta_n]$$

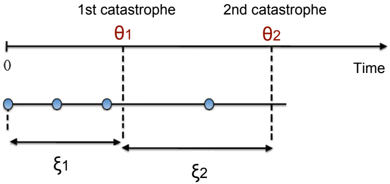
# An upper bound for $\omega$




$$\begin{aligned}
\omega &= \lambda E[\xi] + \lim_{n\to\infty} \frac{1}{n} \log P[S > \theta_n, \varphi_n = j | \varphi_0 = i, \theta_1, \ldots, \theta_n] \\
&\leq \lambda E[\xi] + \lim_{n\to\infty} \frac{1}{n} \log E[P[S > \theta_n, \varphi_n = j | \varphi_0 = i, \theta_1, \ldots, \theta_n]] \\
&= \lambda E[\xi] + \lim_{n\to\infty} \frac{1}{n} \log P[S > \theta_n, \varphi_n = j | \varphi_0 = i] \\
&= \lambda E[\xi] + \log sp\{\beta(\beta I - \Theta)^{-1}\Delta_\delta\}
\end{aligned}$$

$\beta(\beta I - \Theta)^{-1}\Delta_\delta$: transition matrix for $\{\varphi_t\}$ embedded immediately after catastrophe epochs

# A lower bound for $\omega$




$$\omega = \lambda E[\xi] + \lim_{n\to\infty} \frac{1}{n} \log P[S > \theta_n, \varphi_n = j | \varphi_0 = i, \theta_1, \ldots, \theta_n]$$
$$\geq \lambda E[\xi] + \lim_{n\to\infty} \frac{1}{n} \log(P[S > \theta_n, \varphi_n = j | \varphi_0 = i, \varphi_1, \ldots, \varphi_{n-1}, \theta_1, \ldots, \theta_n])$$
$$= \lambda E[\xi] + \lim_{n\to\infty} \log \delta_1^{n_1/n} \delta_2^{n_2/n} \cdots \delta_m^{n_m/n} [(e^{\Theta \xi_n} \Delta_\delta)_{\varphi_{n-1},j}]^{1/n}$$
$$= \lambda E[\xi] + \sum_{1\leq i\leq m} \pi_i \log \delta_i,$$

$\boldsymbol{\pi}$: stationary distribution of $\{\varphi_t\}$, $\boldsymbol{\pi} = \mathbf{u}^T \Delta_v$

$\mathbf{u}$ and $\mathbf{v}$: left and right eigenvectors of $\Omega^*$ corresponding to $0$

# Bounds for Poisson catastrophes

In summary,

**Theorem**

$$\frac{\lambda}{\beta} + \sum_{1 \leq i \leq n} u_i v_i \log \delta_i \;\leq\; \omega \;\leq\; \frac{\lambda}{\beta} + \log sp\left[\beta\left(\beta I - \Theta\right)^{-1} \Delta_\delta\right]$$

where

$$\begin{aligned} \lambda &= \text{dominant eigenvalue of } \Omega, \\ \mathbf{u}, \mathbf{v} &= \text{ left \& right eigenvectors of } \Omega \text{ corresp. to } \lambda, \text{ with } \mathbf{u1} = 1,\ \mathbf{uv} = 1, \\ \Theta &= \Delta_v^{-1}(\Omega - \lambda I)\Delta_v. \end{aligned}$$

# The bounds are tight

Recall that when killing is uniform:

$$\omega = \lambda E[\xi] + \log \delta$$

In this case,

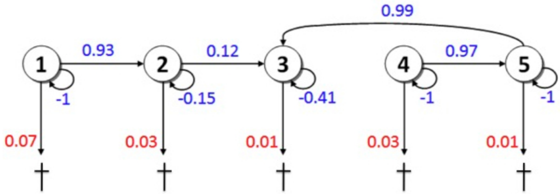
$$\lambda E[\xi] + \sum_{1 \leq i \leq m} \pi_i \log \delta_i = \omega = \lambda E[\xi] + \log sp\{\beta(\beta I - \Theta)^{-1} \Delta_\delta\}$$

# North Atlantic right whales

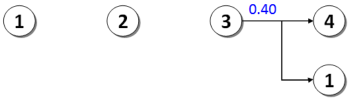
# North Atlantic right whales: The model

1=calf, 2=immature, 3=mature, 4=reproducing, 5=post-breeding

Transition rates

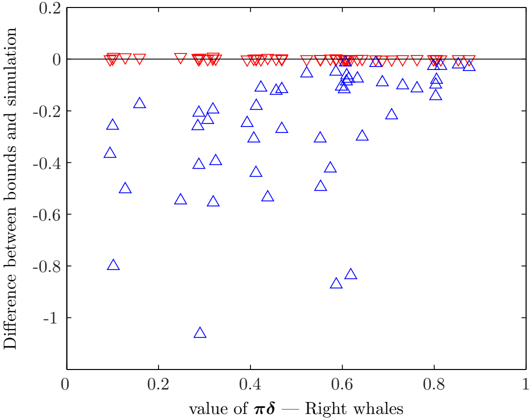



Birth rate



# North Atlantic right whales: The effect of survival probabilities

Catastrophes follow a Poisson process with $E(\xi) = 25$ years



# Back to random environments

We have high hopes that the same type of duality approach may be used to find bounds for

$$\omega = \lim_{n\to\infty} \frac{1}{n} \log \left\{ e^{\Omega^{(\hat{X}_0)}\xi_0} \, e^{\Omega^{(\hat{X}_1)}\xi_1} \, \ldots \, e^{\Omega^{(\hat{X}_{n-1})}\xi_{n-1}} \right\}_{ij}$$

for more general random environments.

# Work in progress

Markovian random environment with two states:

**Theorem**

$\omega_\ell \leq \omega \leq \omega_u$ *with*

$$\begin{aligned} \omega_\ell &= \frac{1}{2}[(\lambda_1/c_1 + \lambda_2/c_2) + (\boldsymbol{\pi}_1 - \boldsymbol{\pi}_2)\log(\Delta_{\mathbf{v}_1}^{-1}\,\mathbf{v}_2)] \\ \omega_u &= \frac{1}{2}[(\lambda_1/c_1 + \lambda_2/c_2) + \log sp(\tilde{M})] \end{aligned}$$

where

$$\begin{aligned} \tilde{M} &= c_1\, c_2\, [(c_1+\lambda_1)I - \Omega^{(1)}]^{-1}\, [(c_2+\lambda_2)I - \Omega^{(2)}]^{-1} \\ c_i &= \text{parameter of the exponential sojour time in environment } i \\ \lambda_i &= \text{max. eigenvalue of } \Omega^{(i)} \\ \mathbf{u}_i, \mathbf{v}_i &= \text{left \& right eigenvectors corresp. to } \lambda_i \text{, s.t. } \mathbf{u}_i^T \mathbf{1} = 1,\ \mathbf{u}_i^T \mathbf{v}_i = 1 \\ \boldsymbol{\pi}_i &= \mathbf{u}_i^T\, \Delta_{\mathbf{v}_i} \end{aligned}$$

Thank you for your attention.