The Hausdorff spectra of a class of multifractal processes

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Multifractal measures

Given a measure μ the local dimension at x is

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

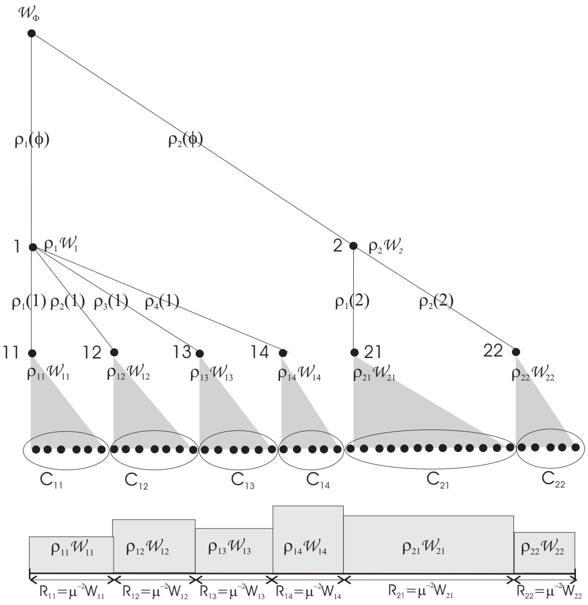
(when the limit exists)

The multifractal or Hausdorff spectrum of μ at α is

$$D_\mu(\alpha) = \text{dim}_H(\{x : d_\mu(x) = \alpha\})$$

where dim_H indicates the Hausdorff dimension.

Cascade measures on trees



We index the nodes of the tree Υ using the sibling number at each generation.

The length of a node $\mathbf{i} = i_1 \dots i_n$ is $|\mathbf{i}| = n$.
If $|\mathbf{i}| > n$, $\mathbf{i}|_n$ is the curtailment of $\mathbf{i}$ after n terms.

Denote by Υ_n the n-th generation of the tree.
We will identify $\mathbf{i} \in \Upsilon_n$ with the subset of $\partial\Upsilon$ descended from it.

Measure on the tree boundary

We have weight $\rho_{\mathbf{i}}(j)$ on the edge from $\mathbf{i}$ to $\mathbf{i}j$,
all i.i.d. with mean $1/m$ where m is the mean family size
The weight at node $\mathbf{i}$ is the product of the edge weights down its line of descent, and is denoted $\rho_{\mathbf{i}}$.

The weights define a measure on the tree boundary.
For $|\mathbf{i}| < \infty$

$$\mu(\mathbf{i}) = \rho_{\mathbf{i}} \mathcal{W}_{\mathbf{i}}, \text{ with mean } (1/m)^{|\mathbf{i}|}$$

(The $\mathcal{W}_{\mathbf{i}}$ are needed to ensure conservation of mass, but we can ignore them for now.)

$d(\mathbf{i}, \mathbf{j}) = m^{-n}$ where m is the mean family size, and n is the generation of the last common ancestor. Thus for $\mathbf{i} \in \partial\Upsilon$

$$d_{\mu}(\mathbf{i}) = \lim_{n \to \infty} \frac{\log \rho_{\mathbf{i}|_n}}{n \log 1/m}.$$

SLLN gives

$$d_\mu(\mathbf{i}) = \frac{\mathbb{E}\log\rho}{\log 1/m} \text{ almost surely}$$

For the other points, noting that $|\Upsilon_n| \approx m^n$, we define

$$g(\alpha) = \lim_{n\to\infty} \frac{\log|\{\mathbf{i}\in\Upsilon_n : \mu(\mathbf{i}) \approx m^{-n\alpha}\}|}{\log m^n}$$

That is, the number of level n sets of size $m^{-n\alpha}$ grows like $m^{ng(\alpha)}$.

Thus in the limit, the set points with local dimension α should have dimension $g(\alpha)$.

(The intuition here is actually that of the Minkowski dimension, which gives us an upper bound on the Hausdorff dimension.)

Multifractal formalism

For $\mathbf{i}$ chosen at random from $\partial\Upsilon$ we define

$$\begin{aligned} 1+\gamma(q) &= \lim_{n\to\infty} \frac{\log \mathbb{E}\mu(\mathbf{i}|_n)^q}{n\log 1/m} = \frac{\log \mathbb{E}\rho^q}{\log 1/m} \\ &= \lim_{n\to\infty} \frac{\log m^{-n}\sum_{\mathbf{i}\in\Upsilon_n}\mu(\mathbf{i})^q}{n\log 1/m} \\ &= 1+\lim_{n\to\infty} \frac{\log \sum_{\mathbf{i}\in\Upsilon_n}\mu(\mathbf{i})^q}{n\log 1/m} \end{aligned}$$

That is, for $\mathbf{i}\in\partial\Upsilon$

$$\mathbb{E}\mu(\mathbf{i}|_n)^q \approx m^{-n(1+\gamma(q))}$$

$1+\gamma(\cdot)$ is called a partition function.

Consider

$$\begin{aligned} m^{-n\gamma(q)} &\approx \sum_{\mathbf{i}\in\Upsilon_n} \mu(\mathbf{i})^q \\ &= \sum_{\alpha} \sum_{\mu(\mathbf{i})\approx m^{-n\alpha}} \mu(\mathbf{i})^q \\ &= \sum_{\alpha} m^{ng(\alpha)} m^{-n\alpha q} \\ &= \sum_{\alpha} m^{-n(\alpha q - g(\alpha))} \\ &\approx m^{-n \inf_\alpha (\alpha q - g(\alpha))} \end{aligned}$$

That is $\gamma = g^*$, the Legendre transform of g.
If γ is nice (convex, twice differentiable), then applying the transform again

$$g = \gamma^*$$

Multifractal processes

For a stochastic process X the local Hölder exponent is

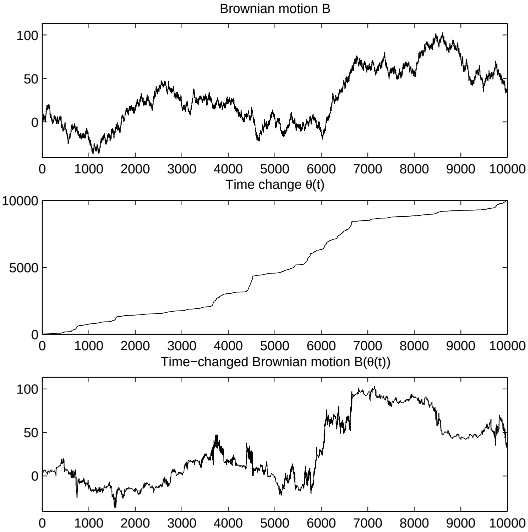
$$h_X(t) = \liminf_{\epsilon \to 0} \frac{\log \sup_{|s-t|<\epsilon} |X(s) - X(t)|}{\log \epsilon}$$

and the multifractal spectrum is

$$D_X(\alpha) = \dim_H(\{t : h_X(t) = \alpha\})$$

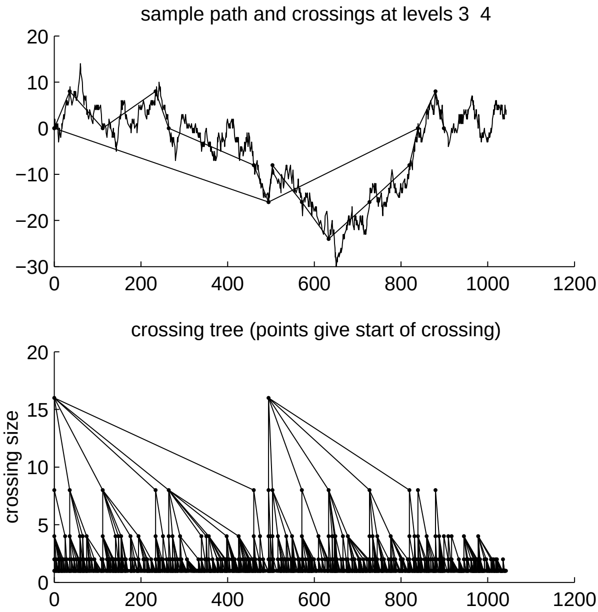
The most popular way to construct a multifractal process is to take a monofractal process and apply a multifractal time-change.

Example: time-changed Brownian motion



The crossing tree of a continuous process

Sample path decomposition using a nested spatial grid



EBP processes

A continuous process X is called an **Embedded Branching Process** (EBP) process if the crossing tree is a Bienaymé-Galton-Watson process.

We construct a multifractal process by taking an EBP, constructing a multiplicative cascade on the crossing tree, then using this to give a multifractal time change

We have that the measure of node $\mathbf{i}$ on the crossing tree is the *duration* of the crossing.

Canonical EBP

We can construct a class of EBP with constant modulus of continuity

$$\sup_{|t-s|<\delta} |X(t) - X(s)| \approx |t-s|^H |\log|t-s||^{1-H}$$

where $H = \log 2 / \log m$ and m is the average family size of the crossing tree.

Such an X is monofractal:

$$h_X(t) = H \text{ for all } t$$

Given a canonical EBP X, we construct a cascade measure on its crossing tree, map this to $\mathbb{R}^+$ and then integrate to get a non-decreasing process $\mathcal{M}$.

Our time-change process is $Y = X \circ \mathcal{M}^{-1}$.

Let T_k^n be the k-th time X makes a size 2^n crossing, and $\mathcal{T}_k^n$ the k-th time Y makes a size 2^n crossing, then by construction

$$\begin{aligned} Y(\mathcal{T}_k^n) &= X(T_k^n) \\ \mathcal{T}_k^n &= \mathcal{M}(T_k^n) \end{aligned}$$

The multifractal spectrum of $\mathcal{M}$ is

$$D_{\mathcal{M}}(\alpha) = \gamma^*(\alpha)$$

where

$$\gamma(q) = \frac{\log \mathbb{E}\rho^q}{\log 1/m} - 1$$

The multifractal spectrum of $\mathcal{M}^{-1}$ is

$$D_{\mathcal{M}^{-1}}(\alpha) = \alpha\gamma^*(1/\alpha)$$

The multifractal spectrum of Y is

$$D_Y(\alpha) \;=\; D_{\mathcal{M}^{-1}}(\alpha/H) \;=\; \frac{\alpha}{H}\gamma^*\left(\frac{H}{\alpha}\right)$$

A process X is self-similar if

$$X(at) \stackrel{fdd}{=} a^H X(t)$$

The partition function for such a function (analogous to the partition function for a measure) is given by $\mathbb{E}X(t)^q = ct^{1+\gamma(q)}$.

Here we have

$$\mathbb{E}X(t)^q = \mathbb{E}(t^H X(1))^q = ct^{Hq}$$

so the partition function is

$$\gamma(q) = Hq - 1$$

It follows that X is monofractal with dimension H.

For a multifractal Y the analogous scaling relation is

$$Y(at) \stackrel{fdd}{=} M(a)Y(t) \text{ where } M(ab) \stackrel{d}{=} M_1(a)M_2(a)$$

By construction, for a canonical EBP we have discrete self-similarity

$$X(m^n t) \stackrel{fdd}{=} 2^n X(t)$$

and similarly for a multitype EBP

$$Y(\rho_{1\cdots 1} t) \stackrel{fdd}{=} 2^{-n} Y(t)$$