Arrival Times to a Queue With Order Penalties

Liron Ravner

Department of Statistics,
Center for the Study of Rationality
The Hebrew University of Jerusalem

Joint work with Moshe Haviv

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Introduction - motivation

- The cafeteria opens for lunch at 12:30
- Customers wish to avoid standing in the queue
- The first customers catch the best seats
- The quality of food deteriorates with time
- Other examples: concert or flight with unmarked seats

Introduction - $?/M/1$ model

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- Single FCFS exponential server with rate μ
- Service starts at time *zero*

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  - **Number of previous arrivals**
- Customers choose when to arrive

Introduction - $?/M/1$ model - non-cooperative game

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- We focus on symmetric mixed strategies
- A mixed strategy is a distribution F on arrival times
- Nash equilibrium: no single customer can benefit by changing her own strategy

Introduction - literature

- Vickrey (1969) (Fluid congested bottleneck model)
- Glazer and Hassin (1983, 1987) ($N \sim Poisson$)
- Hassin and Kleiner (2011) (No early birds)
- Jain, Juneja and Shimkin (2011) (Customer types, fluid)
- Juneja and Shimkin (2012) (General N, tardiness)
- Haviv (2013) (Tardiness, fluid)

Two customer example

- Two customer: $N = \{1, 2\}$
- Server opens at time 0
- Service times are *iid* $\sim$ *exp*(μ)
- Server policy is FCFS

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- Server opens at time 0
- Service times are *iid* $\sim$ *exp*(μ)
- Server policy is FCFS
- Customers choose their arrival times
- Customers can queue before time zero

Two customer game - tardiness cost

Juneja and Shimkin (2012)

- The cost parameters:
  - α - cost per unit of waiting time
  - β - linear tardiness cost
    (cost of βt when entering service at time $t > 0$)

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- The cost parameters:
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    (cost of βt when entering service at time $t > 0$)
- The cost of arriving at time t, when the other plays F:

$$c_F(t) = -\alpha t \mathbb{1}\{t < 0\} + (\alpha + \beta)\frac{\mathbb{E}Q_F(t)}{\mu} + \beta t \mathbb{1}\{t > 0\} \qquad (1)$$

where $Q_F(t)$ is the number of customers in the system

Two customer game - tardiness cost - equilibrium

Theorem (Juneja and Shimkin)

In the two customer game, the unique equilibrium arrival distribution F is defined by the following density:

$$f(t) = \begin{cases} \mu\frac{\alpha}{\alpha+\beta} & , t \in [t_a, 0) \\ -\mu\frac{\beta}{\alpha+\beta} - \frac{\mu^2}{\alpha+\beta}(\beta t + \alpha t_a) & , t \in [0, t_b] \\ 0 & , o.w. \end{cases} \tag{2}$$

Where $t_b < 0$ and $0 < t_a < \infty$ define the support of F and satisfy:

$$-t_a = \frac{1}{\mu}\sqrt{\frac{\beta}{\alpha}\left(2 + \frac{\beta}{\alpha}\right)} \tag{3}$$

$$t_b = \frac{1}{\mu}\left(\sqrt{1 + \frac{2\alpha}{\beta}} - 1\right) \tag{4}$$

Two customer game - tardiness cost - equilibrium

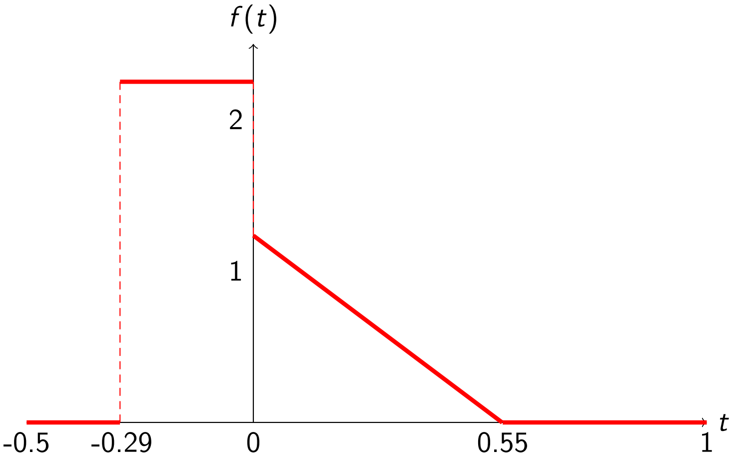

Figure : Equilibrium arrival density $(\mu = 3,\ \alpha = 6,\ \beta = 2)$

Two customer game - index cost

- The cost parameters:
  - α - cost per unit of waiting time
  - γ - cost of arrival index
    (the j'th arrival incurs a cost of $\gamma(j-1)$ *)*

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$$c_F(t) = -\alpha t \mathbb{1}\{t < 0\} + \frac{\alpha}{\mu}\mathbb{E}Q(t) + \gamma \mathbb{E}A(t) \tag{5}$$

$Q(t)$ - number of customers in the system at time t
$A(t)$ - number of customer arrivals up until time t

Two customer game - index cost - equilibrium analysis

- Customers are indifferent between arrival times

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- F is an equilibrium if for every customer:

$$\begin{aligned} c_F(t) &= c^e, \forall t \in \tau_F \\ c_F(t) &\geq c^e, \forall t \end{aligned}$$

  where c^e is some constant

- τ_F is the corresponding support - $(\{t : f(t) > 0\})$

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- We next present an equilibrium strategy

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Two customer game - index cost - equilibrium Analysis

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- The equilibrium dynamics are given by $\frac{d}{dt}c_F(t) = 0$ where:

$$c_F(t) = -\alpha t \mathbb{1}\{t < 0\} + \frac{\alpha}{\mu}\mathbb{P}(Q(t) = 1) + \gamma F(t) \tag{6}$$

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- Which can be solved using:

$$\mathbb{P}(Q(t) = 1) = \mathbb{E}Q(t) = F(t) - \mu \int_0^t \mathbb{P}(Q(s) = 1)ds$$

Two customer game - index cost - equilibrium

Theorem

The following density defines a symmetric equilibrium arrival distribution:

$$f(t) = \begin{cases} \frac{\alpha}{\gamma + \frac{\alpha}{\mu}} & , t \in \left[-\frac{\gamma}{\alpha}, 0\right) \\ \frac{\mu}{\left(1+\frac{\alpha}{\gamma\mu}\right)\left(1+\frac{\gamma\mu}{\alpha}\right)} e^{-\frac{\mu}{1+\frac{\alpha}{\gamma\mu}} t} & , t \in [0, \infty) \\ 0 & , o.w. \end{cases} \tag{9}$$

The expected cost for each customer in this equilibrium is γ.

Two customer game - index cost - equilibrium

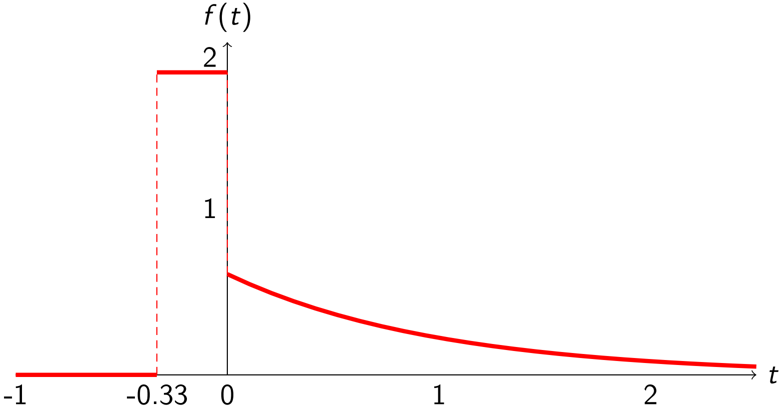

Figure : Equilibrium arrival density $(\mu = 3,\ \alpha = 6,\ \gamma = 1)$

Two customer game - index cost - no early birds

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- Otherwise, p_0 satisfies:

$$\frac{p_0}{2}\left(\frac{\alpha}{\mu}+\gamma\right) = \gamma \Leftrightarrow p_0 = \frac{2\gamma}{\gamma+\frac{\alpha}{\mu}} \tag{11}$$

Two customer game - index cost - no early birds

- The cost of arriving shortly after time zero is:

$$c(t) = p_0 \left(\gamma + e^{-\mu t} \frac{\alpha}{\mu} \right) \quad (12)$$

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- There is an interval $(0, t^e)$ with no arrivals
- $c(t^e) = \gamma$
- After t^e the dynamics are as before

Two customer game - index cost - no early birds

Theorem

For the two customer game with no early birds the symmetric equilibrium arrival distribution is given by:

1. *If* $\frac{\alpha}{\mu} \leq \gamma$ *then* $p_0 = 1$*, i.e. both customers arrive at time zero and are admitted into service in random order. The expected cost for each customer is* $\frac{1}{2}\left(\frac{\alpha}{\mu} + \gamma\right)$.
2. *If* $\frac{\alpha}{\mu} > \gamma$ *then* $p_0 = \frac{\gamma}{\gamma + \frac{\alpha}{\mu}}$ *and:* $f(t) = (1 - p_0)\frac{\mu}{2} p_0 e^{-\frac{\mu}{2} p_0 t} \mathbb{1}\{t \geq t^e\}$, $t^e = -\frac{1}{\mu} \log\left(\frac{1}{2}\left(1 - \frac{\gamma\mu}{\alpha}\right)\right)$*. The expected cost is* γ *for each customer.*

Remark: The cost is lower (weakly) when early arrivals are not allowed

Two customer game - index cost - no early birds - equilibrium

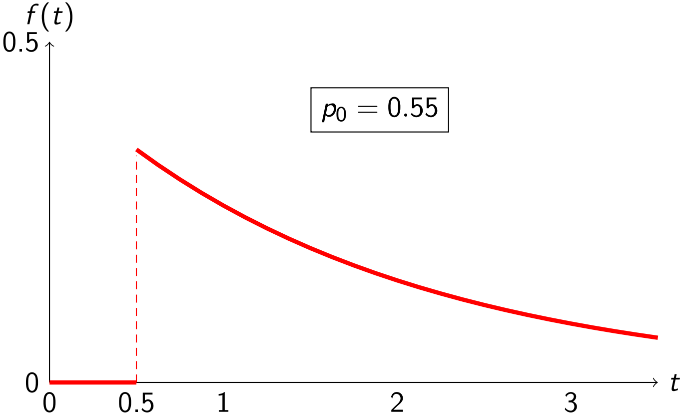

Figure : Equilibrium arrival density $(\mu = 2,\ \alpha = 6,\ \gamma = 1)$

Two customer game - index cost - finite closing time

- Arrivals are not allowed after time T

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Two customer game - index cost - finite closing time

- Arrivals are not allowed after time T
- The dynamics are as before with $F(T) = 1$
- The cost is higher than γ:

$$c(T) = \gamma + \frac{\alpha}{\mu}\mathbb{P}(Q(T) = 1) \tag{13}$$

Two customer game - index Cost - finite closing time

Theorem

The equilibrium arrival distribution for the two customer game with closing time T is defined by the following density:

$$f(t) = \begin{cases} \frac{\alpha}{\gamma+\frac{\alpha}{\mu}},\ t \in \left[-\frac{\gamma}{\alpha}\left(1 - \frac{e^{-\frac{\mu T}{1+\frac{\alpha}{\gamma\mu}}}}{1+\frac{\gamma\mu}{\alpha}}\right)^{-1}, 0\right) & \\ \frac{\mu}{\left(1+\frac{\alpha}{\gamma\mu}\right)\left(1+\frac{\gamma\mu}{\alpha}-e^{-\frac{\mu T}{1+\frac{\alpha}{\gamma\mu}}}\right)} e^{-\frac{\mu t}{1+\frac{\alpha}{\gamma\mu}}} & ,\ t \in [0, T] \\ 0 & ,\ o.w. \end{cases} \tag{14}$$

And the equilibrium cost of every customer is $\gamma\left(1 - \frac{e^{-\frac{\mu T}{1+\frac{\alpha}{\gamma\mu}}}}{1+\frac{\gamma\mu}{\alpha}}\right)^{-1}$.

Two customer game - index cost - finite closing time - Equilibrium

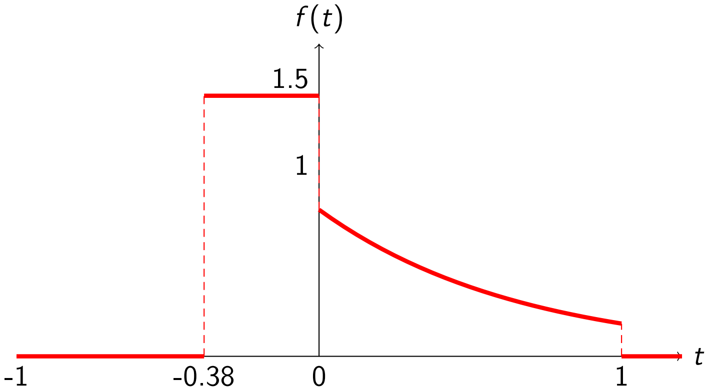


Figure : Equilibrium arrival density $(\mu = 3,\ \alpha = 6,\ \gamma = 2,\ T = 1)$

Two customer game - index cost - finite closing time and no early birds

Theorem

The equilibrium arrival distribution for the two customer game with arrivals allowed only in the interval $[0, T]$, *is given by:*

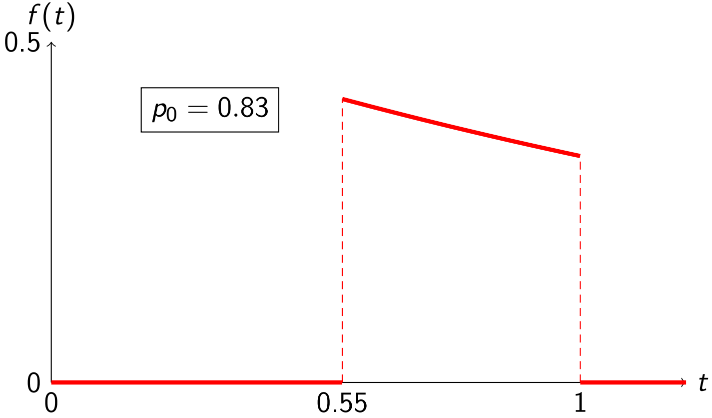
(1) *If* $\frac{\alpha}{\mu}(1 - 2e^{-\mu T}) \le \gamma$ *then* $p_0 = 1$, *i.e. both customers arrive at time zero and are admitted into service in random order. The expected cost for each customer is* $\frac{1}{2}\left(\frac{\alpha}{\mu} + \gamma\right)$.

(2) *If* $\frac{\alpha}{\mu}(1 - 2e^{-\mu T}) > \gamma$ *then* $p_0 = \frac{2}{1+\frac{\alpha}{\gamma\mu}-\left(\frac{\alpha}{\gamma\mu}-1\right)e^{-\frac{\mu}{1+\frac{\alpha}{\gamma\mu}}(T-t^e)}}$ *and:*

$$f(t) = \frac{p_0\mu}{2}\frac{\frac{\alpha}{\gamma\mu}-1}{\frac{\alpha}{\gamma\mu}+1}e^{-\frac{\mu}{1+\frac{\alpha}{\gamma\mu}}(t-t^e)}\mathbb{1}\{t \in [t^e, T]\} \qquad (15)$$

The expected cost is $\frac{p_0}{2}\left(\gamma + \frac{\alpha}{\mu}\right)$ *for each customer.*

Two customer game - index cost - finite closing time and no early birds - equilibrium



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General model

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- Both index and tardiness costs:

$$c_F(t)^* = -\alpha t \mathbb{1}\{t < 0\} + \frac{\alpha + \beta}{\mu} \mathbb{E}Q(t) + \beta t \mathbb{1}\{t \geq 0\} + \gamma \mathbb{E}A(t) \quad (16)$$

- *Cost of arriving at t when all other customers play F

General model - equilibrium

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- Before time zero, $t \in [t_a, 0)$:

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- $F(t_b) = 1$ and $f(t_b) = 0$

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- $F(t_b) = 1$ and $f(t_b) = 0$
- Numerical analysis is required

General model - equilibrium cost - numerical analysis

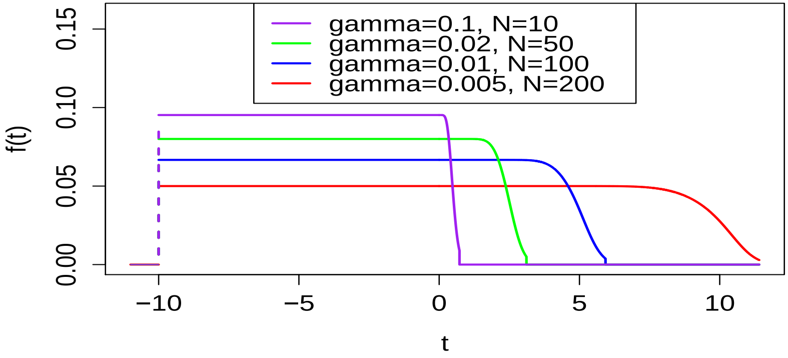
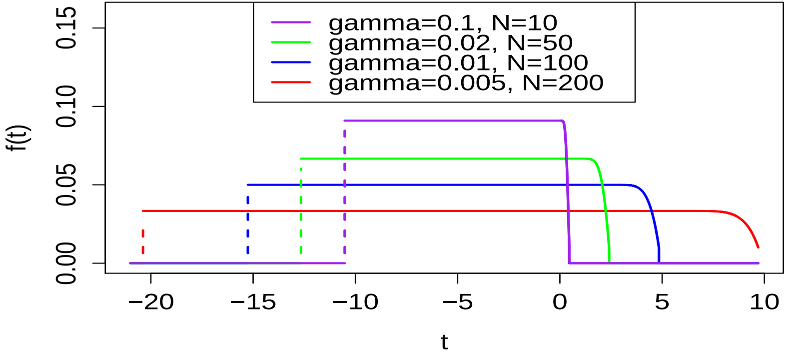


Figure : Equilibrium arrival density $(\mu = 20,\ \alpha = 0.1,\ \beta = 0)$

General model - equilibrium cost - numerical analysis



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Concluding remarks

We have shown the equilibrium properties for linear index costs and various arrival constraints.
What next?

- Non-linear cost functions

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- Non-homogeneous customers

Questions?

Thank You!