An optimal sequential procedure for a multiple selling problem

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A multiple selling problem

Let $y_1, y_2, \ldots, y_N$ be a sequence of independent random variables. We observe these random variables sequentially and have to decide when we must stop.

Our decision to stop depends on the observations already made, but does not depend on the future which is not yet known.

After k ($k \geqslant 2$) stoppings at times $m_1, m_2, \ldots, m_k, 1 \leqslant m_1 < m_2 < \cdots < m_k \leqslant N$ we get a gain

$$Z_{m_1,m_2,\ldots,m_k} = y_{m_1} + y_{m_2} + \cdots + y_{m_k}.$$

The problem consists of finding a procedure for maximizing the expected gain.

Interpretation

The random variable y_n can be interpreted as a value of asset (for example, a house) at time n.

So we consider the problem of selling k identical objects with the finite horizon N, with one offer per time period and no recall of past offers.

This model can also be used to analyse some behavioural ecology problems such as the sequential mate choice or the optimal choice of a place of foraging.

Optimal multiple stopping rules

Let $y_1, y_2, \ldots$ be a sequence of random variables with known joint distribution. We are allowed to observe the y_n sequentially, stopping anywhere we please. If we stop at time m_1 after observations $(y_1, \ldots, y_{m_1})$, then we begin to observe another sequence y_{m_1,m_1+1}, $y_{m_1,m_1+2}, \ldots$ (depending on $(y_1, \ldots, y_{m_1})$) and must solve the problem of an optimal stopping of the new sequence. If we made i stoppings at times $m_1, m_2, \ldots, m_i$ $(1 \leqslant i \leqslant k-1)$, then we observe a sequence of random variables $y_{m_1,\ldots,m_i,m_i+1}, y_{m_1,\ldots,m_i,m_i+2}, \ldots$ whose distribution depends on $(y_1, \ldots, y_{m_1}, y_{m_1,m_1+1}, \ldots, y_{m_1,m_2}, \ldots, y_{m_1,\ldots,m_i})$.

A gain

Our decision to stop at times m_i $(i = 1, 2, \ldots, k)$ depends solely on the values of the basic random sequence already observed and not on any future values. After k $(k \geqslant 2)$ stoppings we receive a gain

$$Z_{m_1,\ldots,m_k} = g_{m_1,\ldots,m_k}(y_1, \ldots, y_{m_1,m_1+1}, \ldots, y_{m_1,\ldots,m_k}),$$

where $g_{m_1,\ldots,m_k}$ is the known function.

We are interested in finding stopping rules which maximize our expected gain.

Assumptions

(a) a probability space $(\Omega, \mathcal{F}, \mathbf{P})$;

(b) a non-decreasing sequence of σ-subalgebras $\{\mathcal{F}_{m_1,\ldots,m_{i-1},m_i}, m_i > m_{i-1}\}$ of σ-algebra $\mathcal{F}$ such that

$$\mathcal{F}_{m_1,\ldots,m_{i-1}} \subseteq \mathcal{F}_{m_1,\ldots,m_i} \subseteq \mathcal{F}_{m_1,\ldots,m_{i-1},m_i+1}$$

for all $i = 1, 2, \ldots, k$, $0 \equiv m_0 < m_1 < \cdots < m_{i-1}$;

(c) a random process

$$\{Z_{m_1,\ldots,m_{k-1},m_k}, \mathcal{F}_{m_1,\ldots,m_{k-1},m_k}, m_k > m_{k-1}\}$$

for any fixed integer $m_1, \ldots, m_{k-1}$, $1 \leqslant m_1 < m_2 < \cdots < m_{k-1}$.

A multiple stopping rule

A collection of integer-valued random variables $(\tau_1, \ldots, \tau_i)$ is called an i-*multiple stopping rule* $(1 \leqslant i \leqslant k)$ if the following conditions hold:

a) $1 \leqslant \tau_1 < \tau_2 \cdots < \tau_i < \infty$ (**P**-a.s.),

b_j) $\{\omega : \tau_1 = m_1, \ldots, \tau_j = m_j\} \in \mathcal{F}_{m_1,\ldots,m_j}$ for all $m_j > m_{j-1} > \ldots > m_1 \geqslant 1; j = 1, 2, \ldots, i$.

A k-multiple stopping rule with $k > 1$ is called a *multiple stopping rule*.

The value of the game

Let S_m be a class of multiple stopping rules $\tau = (\tau_1, \ldots \tau_k)$ such that $\tau_1 \geqslant m$ (**P**-a.s.).

The function

$$v_m = \sup_{\tau \in S_m} \mathbf{E} Z_\tau$$

is called the m-*value of the game*. In particular, if $m = 1$ then $v = v_1$ is called the *value of the game*.

A multiple stopping rule $\tau^* \in S_m$ is called an *optimal multiple stopping rule* in S_m if $\mathbf{E} Z_{\tau^*}$ exists and $\mathbf{E} Z_{\tau^*} = v_m$.

The selling problem: the value of the game

Let $y_1, y_2, \ldots, y_N$ be a sequence of independent random variables with known distribution functions $F_1, F_2, \ldots, F_N$,

$$Z_{(m)_k} = y_{m_1} + y_{m_2} + \cdots + y_{m_k}.$$

Let $v^{L,l}$ be the value of a game with l, $l \leqslant k$, stoppings and L, $L \leqslant N$, steps. If there exist $\mathbf{E}y_1, \mathbf{E}y_2, \ldots, \mathbf{E}y_N$, then the value $v = v^{N,k}$, where

$$\begin{aligned} v^{n,1} &= \mathbf{E}\big(\max\{y_{N-n+1}, v^{n-1,1}\}\big),\ 1 \leqslant n \leqslant N,\ v^{0,1} = -\infty, \\ v^{n,k-i+1} &= \mathbf{E}\big(\max\{v^{n-1,k-i} + y_{N-n+1}, v^{n-1,k-i+1}\}\big), \\ & \quad k-i+1 \leqslant n \leqslant N, \\ v^{k-i,k-i+1} &= -\infty,\ i = k-1, \ldots, 1. \end{aligned}$$

The selling problem: the optimal rule

We put

$$\begin{aligned}
\tau_1^* &= \min\{m_1 : 1 \leqslant m_1 \leqslant N-k+1, \\
&\qquad y_{m_1} \geqslant v^{N-m_1,k} - v^{N-m_1,k-1}\}, \\
\tau_i^* &= \min\{m_i : \tau_{i-1}^* < m_i \leqslant N-k+i, \\
&\qquad y_{m_i} \geqslant v^{N-m_i,k-i+1} - v^{N-m_i,k-i}\},\ i = 2,\dots,k-1, \\
\tau_k^* &= \min\{m_k : \tau_{k-1}^* < m_k \leqslant N, y_{m_k} \geqslant v^{N-m_k,1}\},
\end{aligned}$$

then $\tau^* = (\tau_1^*, \dots, \tau_k^*)$ is the optimal multiple stopping rule.

A generalisation of the selling problem

After k, $1 \leqslant k \leqslant K$, stoppings at times $m_1, m_2, \ldots, m_k, 1 \leqslant m_1 < m_2 < \cdots < m_k \leqslant N$ we get a gain

$$Z_{m_1,m_2,\ldots,m_k} = c_1 y_{m_1} + c_2 y_{m_2} + \cdots + c_k y_{m_k},$$

where $c_1 + c_2 + \cdots + c_k = K$, $1 \leqslant c_n \leqslant C$.

The problem consists of finding a procedure for maximizing the expected gain.

This problem is a generalisation of the problem with one offer per time period (that is, $C = 1$) considered before.

Interpretation

The random variable y_n can be interpreted as a value of asset (for example, a house) at time n, c_n is a number of the objects sold at time n.

We consider the problem of selling K identical objects with finite horizon N, with a fixed rate of offers C, that is, a number of offers per time period, and no recall of past offers.

If a decision-maker stops at time n, he or she can sell $1, 2, \ldots, C$ objects. Clearly, this decision may affect the decision-maker's further strategy to sell the remaining objects.

We need to find a decision rule for identifying the number of stoppings k and a corresponding optimal procedure for this selling problem.

The number of stoppings

The optimal multiple stopping rule with a smaller number of stoppings is more efficient (see Sofronov, 2013), that is,

$$v^{N,1} \geqslant \frac{v^{N,2}}{2} \geqslant \cdots \geqslant \frac{v^{N,k}}{k}.$$

For example, if $C \geqslant K$ and we use the optimal stopping rule with 1 stopping, then we obtain the expected gain $Kv^{N,1}$.

In other words, we get a higher expected gain if we stop once and sell all of the K objects than we stop more than once and sell the objects in parts.

Example 1: Uniform distribution

Let $y_1, y_2, \ldots, y_N$ be a sequence of independent random variable having uniform distribution $U(a, b)$, a, b are fixed numbers. We have

$$
\begin{aligned}
v^{n,1} &= (v^{n-1,1} - a)^2/(2(b-a)) + (a+b)/2, \\
v^{n,k} &= (v^{n-1,k} - v^{n-1,k-1} - a)^2/(2(b-a)) + (a+b)/2 + v^{n-1,k-1},
\end{aligned}
$$

where $v^{0,1} = a$, $v^{k,k+1} = a + k(a+b)/2$, $1 \leqslant n \leqslant N$.

For further details, see Nikolaev and Sofronov (2007), Sofronov *et al.* (2006).

Example 1: the values for U(0,1)

L	$v^{L,1}$	$v^{L,2}$	$v^{L,3}$	$v^{L,4}$	$v^{L,5}$	$v^{L,6}$	$v^{L,7}$
0	0.0000						
1	0.5000	0.5000					
2	0.6250	1.0000	1.0000				
3	0.6953	1.1953	1.5000	1.5000			
4	0.7417	1.3203	1.7417	2.0000	2.0000		
5	0.7751	1.4091	1.9091	2.2751	2.5000	2.5000	
6	0.8004	1.4761	2.0341	2.4761	2.8004	3.0000	3.0000
7	0.8203	1.5287	2.1318	2.6318	3.0287	3.3203	3.5000
8	0.8364	1.5712	2.2105	2.7568	3.2105	3.5712	3.8364
9	0.8498	1.6064	2.2756	2.8597	3.3597	3.7756	4.1064
10	0.8611	1.6360	2.3303	2.9462	3.4847	3.9462	4.3303

Example 1: the optimal rule

If $K = 2$, $C = 3$, $N = 10$, then $k = 1$ and the value of the game of the reduced problem

$$v = v^{10,1} = 0.8611.$$

This yields the expected gain for the initial problem $2 \cdot 0.8611 = 1.7222$, which is higher than $v^{10,2} = 1.6360$ if we had used the double optimal stopping rule.

We have the optimal stopping rule $\tau^* = (\tau_1^*)$, where

$$\tau_1^* = \min\{m_1 : 1 \leqslant m_1 \leqslant 10, y_{m_1} \geqslant v^{10-m_1,1}\}.$$

Example 2: Normal distribution

Let $y_1, y_2, \ldots, y_N$ be a sequence of independent random variable having normal distribution $N(\mu, \sigma^2)$, here $\mu \in (-\infty, \infty)$, $\sigma > 0$ are fixed numbers. We obtain

$$\begin{aligned} v^{n,1} &= \sigma\psi\left(\frac{v^{n-1,1}-\mu}{\sigma}\right)+\mu, \\ v^{n,k} &= \sigma\psi\left(\frac{v^{n-1,k}-v^{n-1,k-1}-\mu}{\sigma}\right)+\mu+v^{n-1,k-1}, \end{aligned}$$

where $\psi(x) = \varphi(x) + x\Phi(x)$, $\varphi(x)$ is the density function of the standard normal distribution, $\Phi(x)$ is the distribution function of the standard normal distribution, $v^{0,1} = -\infty$, $v^{k,k+1} = -\infty$, $1 \leqslant n \leqslant N$.

Example 2: the values for N(0,1)

L	$v^{L,1}$	$v^{L,2}$	$v^{L,3}$	$v^{L,4}$	$v^{L,5}$	$v^{L,6}$	$v^{L,7}$
0	$-\infty$						
1	0.0000	$-\infty$					
2	0.3989	0.0000	$-\infty$				
3	0.6297	0.6297	0.0000	$-\infty$			
4	0.7904	1.0287	0.7904	0.0000	$-\infty$		
5	0.9127	1.3198	1.3198	0.9127	0.0000	$-\infty$	
6	1.0108	1.5478	1.7187	1.5478	1.0108	0.0000	$-\infty$
7	1.0924	1.7344	2.0380	2.0380	1.7344	1.0924	0.0000
8	1.6121	1.8918	2.3034	2.4369	2.3034	1.8918	1.1621

Example 2: the optimal rule

If $K = 6$, $C = 2$, $N = 8$, then $k = 3$ and the value of the game of the reduced problem

$$v = v^{8,3} = 2.3034.$$

This yields the expected gain for the initial problem $2 \cdot 2.3034 = 4.6068$, which is higher than $v^{8,6} = 1.8918$ if we had used the optimal stopping rule with 6 stoppings. We have the optimal stopping rule $\tau^* = (\tau_1^*, \tau_2^*, \tau_3^*)$:

$$\tau_1^* = \min\{m_1 \,:\, 1 \leqslant m_1 \leqslant 6, y_{m_1} \geqslant v^{8-m_1,3} - v^{8-m_1,2}\},$$
$$\tau_2^* = \min\{m_2 \,:\, \tau_1^* < m_2 \leqslant 7, y_{m_2} \geqslant v^{8-m_2,2} - v^{8-m_2,1}\},$$
$$\tau_3^* = \min\{m_3 \,:\, \tau_2^* < m_3 \leqslant 8, y_{m_3} \geqslant v^{8-m_3,1}\}.$$

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