Point processes characterized by their one dimensional distributions

Aihua Xia

Department of Mathematics and Statistics

The University of Melbourne, VIC 3010

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Independence and uncorrelation

For a bivariate random vector (X, Y) with finite second moments, we can define $\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$ and its joint joint cdf $F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y)$.

- X and Y are uncorrelated if $\mathrm{Cov}(X, Y) = 0$
- X and Y are indept if $F_{X,Y}(x, y) = F_X(x)F_Y(y)$ for all x and y
- If X and Y are independent, then they are uncorrelated.
- When does uncorrelation imply independence?

Independence and uncorrelation (2)

- If X and Y takes two values?

$Y \backslash X$	0	1
0	p_{00}	p_{01}
1	p_{10}	p_{11}

$\mathbb{E}(XY) = p_{11}$, $\mathbb{E}X = p_{01} + p_{11} =: p_{\cdot 1}$ and $\mathbb{E}Y = p_{10} + p_{11} =: p_{1\cdot}$, so $\mathrm{Cov}(X, Y) = 0$ iff $p_{11} = p_{\cdot 1} p_{1 \cdot}$ iff X and Y are indept.

In general,

$Y \backslash X$	a_0	a_1
b_0	p_{00}	p_{01}
b_1	p_{10}	p_{11}

X and Y are indept iff they are uncorrelated.

- One takes two values and the other takes more than two?

Reformulation

- Given that F is an n-dimensional df and G an m dimensional df, a coupling of F and G is a random vector $(X_1, \ldots, X_n; Y_1, \ldots, Y_m)$ such that $(X_1, \ldots, X_n) \sim F$ and $(Y_1, \ldots, Y_m) \sim G$.
- Assume that both F and G have finite second moments, what are the conditions such that any uncorrelated coupling must be an independent coupling?

Rank

We say that F has *rank* k if its support is k-dimension.

He and X. (1987): if F has rank k and G has rank l, then any uncorrelated coupling is an independent coupling iff F has at most $k+1$ points and G has at most $l+1$ values.

In the context of processes

Viewing $(X_1, \ldots, X_n; Y_1, \ldots, Y_m)$ as a process on $\{1, 2, \ldots, n+m\}$, the problem becomes

How to specify the distribution of a process from its marginal distributions plus something else?

What's something else?

Example. $X = (I_1, I_2) =: I_1\delta_1 + I_2\delta_2$ with I_1, I_2 two indicator rv's and assume we know $\mathbb{P}(I_1 = 0)$, $\mathbb{P}(I_2 = 0)$ and $\mathbb{P}(I_1 + I_2 = 0)$ (abstraction: *avoidance function*), then

$$\begin{aligned}
\mathbb{P}(I_1 = 0, I_2 = 0) &= \mathbb{P}(I_1 + I_2 = 0),\\
\mathbb{P}(I_1 = 0, I_2 = 1) &= \mathbb{P}(I_1 = 0) - \mathbb{P}(I_1 = 0, I_2 = 0),\\
\mathbb{P}(I_1 = 1, I_2 = 0) &= \mathbb{P}(I_2 = 0) - \mathbb{P}(I_1 = 0, I_2 = 0),\\
\mathbb{P}(I_1 = 1, I_2 = 1) &= \text{easy.}
\end{aligned}$$

Remark

- $\text{Cov}(I_1, I_2) = 0$ specifies $\mathbb{P}(I_1 = 1, I_2 = 1)$
- avoidance function specifies $\mathbb{P}(I_1 = 0, I_2 = 0)$

Generally

If $I_1, \ldots, I_k$ are indicator rv's, then the distribution of $(I_1, I_2, \ldots, I_k)$ is uniquely determined by the probabilities of

$$\mathbb{P}(I_{i_1} + \cdots + I_{i_l} = 0)$$

for all $1 \leq l \leq k$ and $1 \leq i_1 < i_2 < \cdots < i_l \leq k$.

Proof. By math induction on k. ∎

Why not point processes?

- Γ is a metric space, typically $\mathbb{R}_+$, $\mathbb{R}$ or $\mathbb{R}^d$
- We define $\mathcal{H}$ as the class of all integer-valued locally finite measures on $\mathcal{H}$ equipped with a σ-field
- Ξ is a measurable mapping from a probability space to $\mathcal{H}$ and is called a *point process*
- A point process Ξ is called *simple* if, almost surely, $\Xi(\omega)$ takes either 1 point or no points at each location.
- The previous example is a simple point process

The complete distribution of a PP

[Kallenberg (1983) or Daley and Vere-Jones (1988)] To specify the complete distribution of a point process Ξ, it is necessary and sufficient to specify all finite distributions $(\Xi(B_1), ..., \Xi(B_k))$ for all $k \geq 1$ and all disjoint Borel sets B_1, ..., B_k.

Simple point processes

Renyi (1967) and Mönch (1971): the distribution of a simple point process is determined by the probability of there being 0 points (avoidance function) in each of the Borel sets.

Example

A simple point process Ξ is a Poisson process on Γ iff for any Borel $B \subseteq \Gamma$, $\Xi(B) \sim \mathrm{Pn}$.

- $\Xi(B) \sim \mathrm{Pn}$ can be replaced by $\mathbb{P}(\Xi(B) = 0) = e^{-\mathbb{E}\Xi(B)}$.

Remark Lee (1968) and Moran (1967): it's not sufficient to specify the Poisson property on *intervals*.

An application in extreme value theory

Let $\eta_1, \eta_2, \ldots, \eta_n$ be iid (or weakly dependent with α mixing or β mixing conditions) and define

$$\Xi_n = \sum_{i=1}^{n} \mathbf{1}_{\eta_i \geq u_n} \delta_{i/n}.$$

If $n\mathbb{P}(\eta_1 \geq u_n) \to c$, then Ξ_n converges in distribution to $\mathrm{Pn}(\lambda)$ with $\lambda(ds) = cds$.

• Using this theorem, with $\eta_{(i)}$ being the ith smallest order statistics, we get

$$\mathbb{P}(\eta_{(n)} \geq u_n) \approx \mathrm{Pn}(c)\{1, 2, ...\},$$

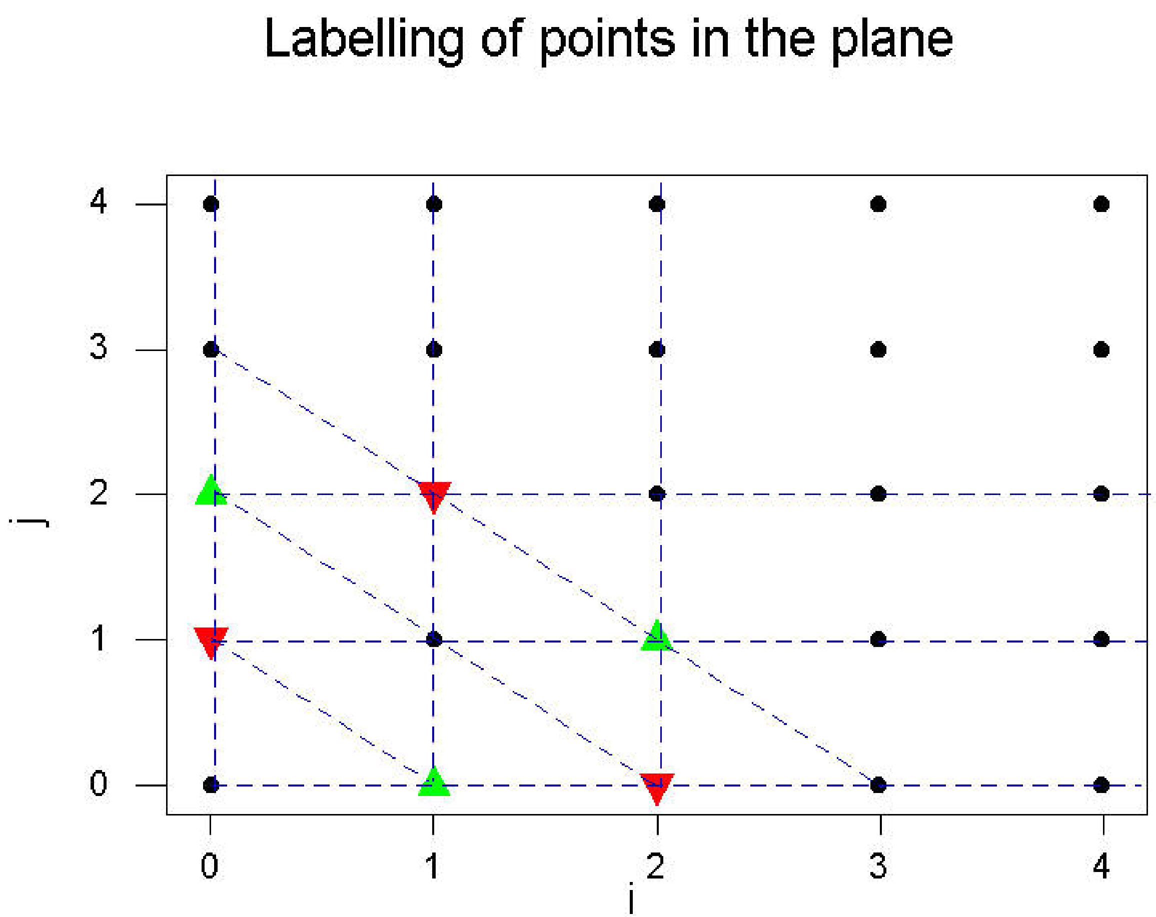
$$\mathbb{P}(\eta_{(n-1)} \geq u_n) \approx \mathrm{Pn}(c)\{2, 3, ...\},$$

etc.

Why simple point processes?

Example Let X be a nonnegative integer valued rv (e.g., Poisson), Y be an indicator rv. If we know the distributions of X, Y and $X + Y$, then we know the distribution of (X, Y).

Example (Brown and X. (2002)) If $\{p_{ij}\}$ is a joint probability mass function (that is an array of non-negative numbers whose sum is one) on $\{0, 1, 2, ...\}^2$ with strictly positive probabilities, then there are infinitely many joint probability mass functions for random variables (X, Y) for which the distributions of X, Y and $X + Y$ coincided with the corresponding distributions for $\{p_{ij}\}$.



Labelling of points in the plane
4
3
2
1
0
j
0
1
2
3
4
i

Theorem. (Brown and X. (2002)) For any measure λ on Γ, there is one distribution or infinitely many Poisson processes with mean measure λ according to whether the number of atoms of λ is less than or equal to 1 or greater than or equal to 2.

General PP

Example [cf Brown and X. (2002), Moran (1967) and Lee (1968)] Let (X_ϵ, Y_ϵ), $\epsilon < 1/9$, be a random vector with the following joint distribution:

	0	1	2	X_ϵ
0	$1/9$	$1/9+\epsilon$	$1/9-\epsilon$	$1/3$
1	$1/9-\epsilon$	$1/9$	$1/9+\epsilon$	$1/3$
2	$1/9+\epsilon$	$1/9-\epsilon$	$1/9$	$1/3$
Y_ϵ	$1/3$	$1/3$	$1/3$	

so that the distributions of X_ϵ, Y_ϵ and $X_\epsilon + Y_\epsilon$ do not depend on ϵ:

Values of $X_\epsilon + Y_\epsilon$	0	1	2	3	4
Probabilities	1/9	2/9	1/3	2/9	1/9

Let U and V be independent random variables uniformly distributed on $[0, 0.5]$ and $(0.5, 1]$ respectively and (U, V) be independent of (X_ϵ, Y_ϵ). Define $\Xi_\epsilon = X_\epsilon \delta_U + Y_\epsilon \delta_V$, where δ_z is the Dirac measure at z. Then, the mean measure of Ξ_ϵ is $2\mathbf{L}(B)$ with no atoms, where $\mathbf{L}$ is the Lebesgue measure. For every Borel set $B \subset [0, 1]$, $i \geq 1$, let $A_1 = \{U \in B\}$, $A_2 = \{V \in B\}$, A_j^c be the complement of A_j, by the total probability formula,

$$\begin{aligned}\mathbb{P}(\Xi_\epsilon(B) = i) &= \mathbb{P}(X_\epsilon + Y_\epsilon = i)\mathbb{P}(A_1 A_2) + \mathbb{P}(Y_\epsilon = i)\mathbb{P}(A_1^c A_2) \\ &\quad + \mathbb{P}(X_\epsilon = i)\mathbb{P}(A_1 A_2^c),\end{aligned}$$

hence one dimensional distributions are completely determined by the distributions of X_ϵ, Y_ϵ and $X_\epsilon + Y_\epsilon$, which don't depend on ϵ. However, choose $B_1 = [0, 0.5]$, $B_2 = (0.5, 1]$, $i, j \geq 1$, we have

$$\mathbb{P}(\Xi(B_1) = i, \Xi(B_2) = j) = \mathbb{P}(X_\epsilon = i, Y_\epsilon = j),$$

which depends on the joint distribution of (X_ϵ, Y_ϵ), therefore, on ϵ. ■

From simple to weakly orderly

A point process Ξ on Γ is said to be *weakly ordinary* if $\Xi(\omega)$ takes at most two values at each location.

X. (2004): if there is at most one point x_0 on Γ such that $\Xi|_{\Gamma\setminus\{x_0\}}$ is weakly orderly, then $\mathcal{L}(\Xi)$ is uniquely specified by its one dimensional distributions of $\Xi(B)$ for all Borel $B \subset \Gamma$. The condition is essentially necessary.

Sequence with strong dependence

It has been shown decades ago that the limit of Ξ_n defined above for strongly dependent sequence $\eta_1, \eta_2, \ldots, \eta_n$ will converge to compound Poisson process if converges.

• Compound Poisson process: Let ξ be a nonnegative integer-valued random variable, for each point of the Poisson process X, we replace it with an independent copy of ξ, the resulting process Ξ is called a *compound Poisson process*.

• **Question:** to determine the distribution of Ξ, how many dimensional distributions are sufficient?

• (G. Last, personal communication) We can introduce marks and use avoidance function.

- Back to "all finite distributions"

Example: Let

$$X = \xi_1 \delta_{x_1} + \xi_2 \delta_{x_2} + \xi_3 \delta_{x_3}$$

with ξ_1, ξ_2 and ξ_3 being $\{0, 1, 2\}$ valued rv's. Then the distribution of X is uniquely specified by two dimensional distributions of X:

$$\{\mathcal{L}(X(A), X(B)) : \ A, B \subset \{x_1, x_2, x_3\}, \ A \cap B = \emptyset\}.$$

Proof. Use generating functions. ■

A formula

For a compound Poisson process with mean measure λ and ξ takes k values, then the number of dimensions needed to determine the distribution of Ξ is

$$\text{number of atoms in } \lambda \vee (k-1)$$

Sketch of the proof. Assume the number of atoms in λ is l, we need at least l dimensions.

Next, we need at least $k-1$ dimensions by math induction and generating functions. ■

A generalization

Let Ξ be a point process with mean measure λ (not necessary compound Poisson). Assume λ has l atoms, and at the remaining locations, Ξ takes at most k values. Suppose that of the l atoms, Ξ takes more than k values at $\tilde{l}$ locations, then the distribution of Ξ is specified by

$$\tilde{l} \vee (k-1)$$

dimensional distributions.

Thank you for your time!