Fundamentals of Control Theory

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Control theory is the discipline that concerns itself with the response of dynamical systems to input. Control theory is a field that spans both mathematics and engineering as it finds the vast majority of its applications while providing many complex problems as the topic of much mathematical research. Such problems often deal with questions such as stability, robustness (tolerance in variation of the parameters), tracking (small error for certain input) and others.

One often deals with the concept of a plant. The plant represents the dynamic system and has input $u(t)$,output $y(t)$ and state $x(t)$ signals which contain the information on the state of the system. This may be something simple like a scalar quantity like speed or it could be a vector that contains values for multiple parameters defining the state of the system; for example an airplane may need to take into account pitch, yaw and roll. For systems with scalar signals the system is referred to as SISO (single input single output) whereas when the signals are vectors the system is referred to as MIMO (multiple input multiple output). The goal in control theory is generally to find a control (often an input) to ensure that the state/output has the desired properties. There are two major types of control that need to be considered; open loop control and closed loop control. Open loop control is the simple situation where a choice of $u(t)$ is made to ensure the desired behaviour whereas closed loop control is where the input is dependent on the output: $u(t) = g(y(t), t)$ for some function $g(.)$ known as the control law. This control law is what must be chosen to ensure desired behaviour in the feedback scenario.

Often the dynamic system can be expressed as a differential equation (or system of differential equations). One approach to analyzing such a system is to transform from the time-domain to the frequency-domain by use of Laplace transforms. The system can then be expressed in the following way:

$$Y(s) = H(s)U(s)$$

Where $Y(s), U(s)$ are the transformed output and input respectively however here the $H(s)$ is what is known as the transfer function and is extremely important as it describes the properties of the dynamical system in its entirety. Often the analysis of the system reduces down to the analysis of the transfer function.

In many situations the stability of the system is of great importance. Stability can be defined as the requirement that a bounded input signal corresponds to a bounded output signal. This can in fact be assessed from the transfer function itself. This is known as BIBO (bounded input bounded output). The system is stable if the poles of $H(s)$ are in the left-hand plane, that is, the poles have negative real parts. In most cases the transfer function can be expressed in the following form:

$$H(s) = \frac{f(s)}{\prod_{i=1}^{n}(s + s_i)}$$

Here the s_i are the poles of the system and can be easily read off. If all $Re(s_i) < 0$ then the system is stable.

Here $G_1(s)$ and $G_2(s)$ are controllers that are generally linear and time independent. The task of controlling the system reduces down to the choice of $G_1(s)$ and $G_2(s)$. Here r is the reference state (with corresponding frequency-domain form $R(s)$) that enters the system. One can consider the entirety of this system as one plant with input $R(s)$ and transfer function $\widetilde{H}(s)$ which satisfies the following relation:

$$Y(s) = \widetilde{H}(s)R(s)$$

Which can be expressed in terms of the components as:

$$Y(s) = \frac{G_1(s)H(s)}{1 + G_2(s)G_1(s)H(s)}R(s)$$

The feedback system thus has a transfer function:

$$\widetilde{H}(s) = \frac{G_1(s)H(s)}{1 + G_2(s)G_1(s)H(s)}$$

The choice of $G_1(s)$ and $G_2(s)$ is often made with a certain goals in mind. Some of these include:

- Stability: The transfer function of the total system should have poles in the left-hand plane
- Regulation: Disturbances $W(s)$ and $V(s)$ that can occur in the system can be taken into account
- Tracking: For certain references the error $E(s)$ should be small
- Robustness: Tolerance of variation in the parameters of the plant e.g: $G'(s) = G(s)(1+\delta)$ for some δ small must also be controlled well
- Simplicity: Minimum number of components

A common example of such a system is the PID (Proportional-Integral-Derivative) controller. A PID controller consists of a proportional component, an integrative component and a differential component. Here $G_1(s)$ is replaced by each of these components in parallel with values $k_P, k_I, k_D$and $G_2(s) = 1$. So can express $G_1(s)$ as $G_1(s) = k_P + k_I\frac{1}{s} + k_D s$ which gives the plant transfer function $H_c(s)$ as:

$$H_c(s) = \frac{H(s)(k_P + k_I\frac{1}{s} + k_D s)}{1 + H(s)(k_P + k_I\frac{1}{s} + k_D s)}$$

Incorporating the different terms produces different effects. A constant term k_P will give rise to a constant error term whereas an integrator term k_I will be able to reject constant disturbances. A derivative term on the other hand sharply responds to changing signals.

A common method of assessing the behaviour of a control is to observe the response to a step function. This response is known as the step response of the system and demonstrates the dynamics of the system when it experiences a change in the input signal. Many properties of the response of a BIBO system such as 'rise time' (time taken to reach 0.9), 'settling time' (time taken for the response to decay to close to reference - usually within 0.01), 'overshoot' (the amount the response overshoots the reference) and 'peak time' (the time taken to reach the maximum) are important to understanding the behaviour of a system. For example a fast rise time is often accompanied by a high overshoot - i.e there is a tradeoff between the two properties. The step response assesses the systems response over time however one can also consider the response to different frequencies. This can be done by using a sine function as the input. The resulting output can then be assessed for resonances which are frequencies where there is a large gain (large magnitude of the response). A frequency response is often plotted on a Bode plot.

In general systems can be expressed in differential form (or difference form for the discrete case) as below:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad x(n+1) = Ax(n) + Bu(n) \\ y(t) &= Cx(t) + Du(t) \qquad y(n) = Cx(n) + Du(n) \end{aligned}$$

where $x(t), u(t), y(t)$ need not be scalars. In fact in general the components are vectors and matrices of the following dimensions $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$where the input, state and output are vectors $u(t) \in \mathbb{R}^m, x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^p$. In such a case the transfer function $H(s)$ has the following form:

$$H(s) = C(sI - A)^{-1}B + D$$

So then the relation: $y(s) = H(s)u(s)$ still holds. Often it is convenient to work with certain systems (ie certain forms of (A, B, C, D)) and not others. In these situations it is possible to make a similarity transform to a new system (A', B', C', D') by transforming the state vector: $P\widetilde{x} = x$ for some invertible transformation matrix P. The new system is then $(P^{-1}AP, PB, CP, D)$. This new system is known as an equivalent system and has the same transfer function and thus the same response to signals such as the step.

There are some important properties of MIMO systems known as controllability and observability. A state x is controllable if there is an input $u(.)$ that takes the system from x to the zero-state in a finite amount of time. Thus a system is said to be controllable if any state is controllable. If this is indeed the case one says that (A, B) is controllable. How is controllability analyzed? One important way is by considering what is referred to as the controllability matrix $con_k(A, B) = [B, AB, A^2B, ..., A^{k-1}B]$. If this matrix has a rank of n (the dimension of the state vector) then the system is considered to be controllable.

Observability is defined in analogous ways. A system is said to be observable if one can determine the initial state from the input and output states over a finite time period. Similarly to controllability one defines the observability matrix:

$$obs_k(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{k-1} \end{bmatrix}$$

Then a system is said to be observable if the rank of the observability matrix is n as in the controllability case. There exists a relationship between observability and controllability when one considers the dual system (A^T, B^T, C^T, D^T) which is just the transpose of the system. Then $con(A, B) = obs(A^T, B^T)^T, obs(A, C)^T = con(A^T, B^T)$. The concept of controllability and observability is important when considering Linear State Feedback (LSF) and Observers.

LSF is the situation where the input is defined in terms of the state and the reference $r(t) \in \mathbb{R}^m$. That is, $u(t) = Fx(t) + r(t)$ for some matrix $F \in \mathbb{R}^{m\times n}$. The system is then:

$$\begin{aligned} \dot{x}(t) &= (A + BF)x(t) + Br(t) \\ y(t) &= (C + DF)x(t) + Dr(t) \end{aligned}$$

Often it is required to find an F such that the system is stable. This is equivalent to choosing an F such that $A + BF$ has eigenvalues in the left hand plane. Such an F can always be found if and only if (A, B) is controllable.

One can also design a system such that a state $\hat{x}(t)$ is an estimate of the state $x(t)$ which may not be observable. This is known as a Luenberger observer. There are two systems, the observer system and the original system:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + K(y - \hat{y}) \\ \hat{y}(t) &= C\hat{x}(t) + Du(t) \end{aligned}$$

Combined the total system is:

$$\begin{aligned}\dot{\hat{x}}(t) &= (A-KC)\hat{x}(t) + [B-KD, K]u(t)\begin{bmatrix} u \\ y \end{bmatrix} \\ \hat{y}(t) &= C\hat{x}(t) + [D, 0]\begin{bmatrix} u \\ y \end{bmatrix}\end{aligned}$$

In such a system it is important to know how much error there is in the estimate, that is the error $e(t) = x(t) - \hat{x}(t)$. In fact the error behaves as a system itself evolving according to a differential equation:

$$\dot{e}(t) = (A-KC)e(t)$$

If $(A-KC)$ has eigenvalues in the left hand plane then $e(t) \to 0$ as $t \to \infty$. In a case such as this the observer is a powerful tool as it provides very accurate estimates of the state of the system. So to ensure such a situation one needs to choose a K such that $(A-KC)$ has eigenvalues in the left hand plane. A K can always be chosen to ensure this provided (A, C) is an observable pair.

As is evident by the content so far stability is very important in control theory and as such there is a wealth of theory that deals with stability. One of the most important is the Lyapunov stability theory. There are two forms of stability - stability and asymptotic stability. If a system remains within a bounded region then it is said to be stable, i.e an equilibrium point $x = 0$ is stable if $|\phi(t)| < \epsilon$ for all t provided x_0 starts within some region $x_0 < \delta$. Asymptotic stability on the other hand requires that the system converges to equilibrium, that is, $|\phi(t)| \to 0$ as $t \to \infty$ for the initial point $x_0 < \eta$. The Lyapunov theorem is used very often to prove whether a system is stable or asymptotically stable. The basic idea of the approach is to define a function that represents a kind of potential (positive everywhere in the domain and zero at the critical point) centered on the critical point and then if the gradient is less than or equal to zero at every point (except the critical point) then the system is stable if it is negative (strictly less than zero) then the system is asymptotically stable.

An important part of control is the area concerned with optimal control. That is, where a control is needed to optimize the system. This is often phrased in terms of a cost function J which needs to be minimized by the control. A common system is a Linear Quadratic Regulator and is of the form $\dot{x}(t) = Ax(t) + Bu(t)$ with the cost function $J(u) = \int_0^T x(t)^T Q x(t) + u(t)^T R u(t) dt + x(T)^T Q_f x(T)$. Here T is referred to as the time horizon and R, Q, Q_f contain the coefficients of the x_i^2 (quadratic) terms. It is known that the optimal control for such a system is easily computed according to the following relations: $u(t) = (-R^{-1}B^T P(t))x(t)$. Here $P(t)$ is a solution to the Ricatti equation: $-\dot{P}(t) = A^T P(t) + P(t)A - P(t)BR^{-1}B^T P(t) + Q$ where $P(T) = Q_f$. This equation is generally not easily solvable as it is a matrix differential equation.

Another example is the sub-optimal control known as Model Predictive Control (MPC). This system is a discrete system that resembles a discrete form of the continuous LQR system presented before. However MPC has a further constraint $F\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \leq b$ for some b. One method of solving such a system is to formulate it in terms of a quadratic program. The general from of a quadratic program is $min_z z^T Q z + Pz, Fz \leq b$. It is possible rearrange the MPC to fit such a form and if $Q > 0$ then there is a solution that can be found efficiently.

Often optimal control problems require the minimization of a cost in the form of a functional $\int f(x, u, t)dt$ where one would like to find a control $u(t)$ to minimize it. One such approach is to use calculus of variations to extremize such a problem by using the Euler-Lagrange equation:

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0$$

Solutions to this equation extremise the functional. So in this situation conditions on $u(t)$ can be found. When considering conditions such as the conditions imposed by the system (a differential equation) one must introduce Lagrange multipliers $p(t)$. A convenient way to deal with such situations is to use the Hamiltonian formalism and introduce the Hamiltonian:

$$\mathcal{H} = f(x,u,t) + p(t)[Ax(t) + Bu(t)]$$

Then the following equations give conditions on the $p(t)$ and $u(t)$.

$$\frac{\partial \mathcal{H}}{\partial u} = 0, \frac{\partial \mathcal{H}}{\partial p} = Ax(t) + Bu(t) = \dot{x}(t), -\frac{\partial \mathcal{H}}{\partial x} = \dot{p}$$

With these equations it is possible to find controls $u(t)$ that extremise (optimize) the cost of the system.

Up until now the systems that have been considered assume perfect measurements of a perfect dynamic system however in reality this is not necessarily the case. It is possible to take into account noise in both the measurements and disturbance noise in the system. One can then write the system with extra terms representing these factors:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \xi_x(t) \\ y(t) &= Cx(t) + Du(t) + \xi_y(t) \end{aligned}$$

Where ξ_x and ξ_y are the disturbance and measurement noise factors and are random processes of normal distribution with zero mean. Often a practical way of treating such a problem is to consider the discrete form of the system which enables the use of computers and simplifies the mathematics somewhat. There are two well known treatments of this model; State Estimation (Kalman filtering) and Optimal control (Linear Quadratic Gaussian). Kalman filtering deals with introducing an estimator state $\hat{x}(n)$ which is solved by using what is called the LMMSE estimator (Linear Minimum Mean Square Error estimator) which is a condition that minimizes the measure of dependence between two random vectors X_a and X_b. This can be used to create what is known as the Kalman filtering algorithm which estimates the state of the system $x(n)$.

Linear quadratic gaussian (LQG) is a generalization of the LQR problem discussed earlier but to a stochastic system. Where dynamic programming can be applied to LQR stochastic dynamic programming is used for LQG.

This outline of some of the fundamentals of control theory are all results found before the 1970s. What is the state of the field today? Modern control theory can be divided into the following sections:

- Non-linear control: Control of systems that are non-linear. Such systems may have squares of derivatives or other expressions that make analysis much more difficult
- Adaptive control: Occurs when a system of unknown parameters (eg (A, B, C, D)) needs to be controlled. This can be done by estimating the parameters or by simply adapting the control continuously to ensure desired output. Such a controlled system is often non-linear thus making it a difficult subject.
- Robust control: Deals with situations where the nature of the plant is not necessarily known. The robustness of the control allows a tolerance to variation of the parameters.
- Supervisory control: Controls a set of discrete events with complicated discrete state spaces
- Control of inherently stochastic systems: Deals with systems akin to the one seen in the section on LQG where there is a stochastic term. A lot of research revolves around approximate dynamic programming for such systems and stability analysis.

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