MATH4406 - Assignment 1

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Question 1

$$\tilde{u}(t) = \sum_n u(nT) K\left(\frac{t-nT}{T}\right), \quad K(0) = 1, \quad K(n) = 0 \ \forall n \in \mathbb{Z}\setminus\{0\}$$

We want linear interpolation i.e. let $p \in [0,1]$ be the linear proportion between inputs $u(n_t T)$ and $u(n_{t+1}T)$ with $n_t = \lfloor \frac{t}{T} \rfloor$

$$\tilde{u}(t) \quad = \quad pu(n_t T) + (1-p)\, u((n_t+1)T)$$

Consider

$$\begin{array}{ccc|ccc} n_t T \leq & t & \leq (n_t+1)T & n_t T \leq & t & \leq (n_t+1)T \\ 0 \leq & t - n_t T & \leq T & -T \leq & t - (n_t+1)T & \leq 0 \\ 0 \leq & \frac{t-n_t T}{T} & \leq 1 & -1 \leq & \frac{t-(n_t+1)T}{T} & \leq 0 \end{array}$$

Clearly

$$p = 1 - \frac{t - n_t T}{T}$$

Thus

$$\begin{aligned} K\left(x = \frac{t-nT}{T}\right) &= \begin{cases} 1-x & 0 \leq x \leq 1 \\ 1+x & -1 \leq x \leq 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Example

$$u(x) = \begin{cases} 2 & x = 0 \\ 1 & x = 0.5 \\ 3 & x = 1 \\ 0 & \text{otherwise} \end{cases}, \ T = 0.5$$

Let $t = 0.7$

$$\begin{aligned} \tilde{u}(0.7) &= K\left(\frac{0.7 - 0*0.5}{0.5}\right) u(0) + K\left(\frac{0.7-1*0.5}{0.5}\right) u(0.5) + K\left(\frac{0.7-2*0.5}{0.5}\right) u(1) + \ldots \\ &= K(1.4)\, u(0) + K(0.4)\, u(0.5) + K(-0.6)\, u(1) + 0 \\ &= 0 * u(0) + 0.6 * u(0.5) + 0.4 * u(1) \\ &= 0.6 + 0.4 * 3 \\ &= 1.8 \end{aligned}$$

Question 2

$$C\left(x\right)=A\left(x\right)B\left(x\right)=\sum_{j=0}^{2n-2}c_{j}x^{j}$$

$$\begin{aligned}A\left(x\right)B\left(x\right) &= \sum_{j=0}^{n-1}a_{j}x^{j}\sum_{j=0}^{n-1}b_{j}x^{j}\\ &= \left(a_{0}+a_{1}x^{1}+\cdots+a_{n-1}x^{n-1}\right)\left(b_{0}+b_{1}x^{1}+\cdots+b_{n-1}x^{n-1}\right)\end{aligned}$$

Let $a_{i},b_{i}=0\quad\forall i>n-1$ and collect terms up to largest term: $(n-1)+(n-1)=2n-2$

x^{0} $\qquad a_{0}b_{0}$

x^{1} $\qquad a_{0}b_{1}+a_{1}b_{0}$

x^{2} $\qquad a_{0}b_{2}+a_{1}b_{1}+a_{2}b_{0}$

$\vdots$

x^{n-1} $\qquad a_{0}b_{n-1}+a_{1}b_{n-2}+\cdots+a_{n-2}b_{1}+a_{n-1}b_{0}=\sum_{k=0}^{n-1}a_{k}b_{j-k}$

$\vdots$

x^{2n-2} $\qquad a_{0}b_{2n-2}+\cdots+a_{n-1}b_{n-1}+\cdots+a_{2n-2}b_{0}=\sum_{k=0}^{2n-2}a_{k}b_{j-k}$

Clearly

$$x^{j}\ :\ \sum_{k=0}^{j}a_{k}b_{j-k}$$

Question 3

$$D\left(g*h\right)=Dg*h=g*Dh$$

Discrete time (shift)

Recall $g*h=\sum_{k-\infty}^{\infty}g\left(k\right)h\left(n-k\right),\qquad D\left(g\right)\left(n\right)=g\left(n-1\right)$

$$\begin{aligned}D\left(g*h\right)\left(n\right) &= \left(g*h\right)\left(n-1\right)\\ &= \sum_{k=-\infty}^{\infty}g\left(k\right)h\left(\left(n-1\right)-k\right)\\ &= \sum_{k=-\infty}^{\infty}g\left(k\right)D\left(h\right)\left(n-k\right)\\ &= g*D\left(h\right)\end{aligned}$$

By relabeling variables we have

$$\begin{aligned}D\left(g*h\right)\left(n\right) &= \sum_{k=-\infty}^{\infty}g\left(k\right)h\left(n-1-k\right)\\ &= \sum_{j=-\infty}^{\infty}g\left(j-1\right)h\left(n-1-\left(j-1\right)\right)\\ &= \sum_{j=-\infty}^{\infty}D\left(g\right)\left(j\right)h\left(n-j\right)\\ &= D\left(g\right)*h\end{aligned}$$

Continuous time (differentiation)

Recall $g * h = \int_{-\infty}^{\infty} g(\tau)\, h(t-\tau)\, d\tau$, $\qquad D(g)(t) = \frac{d}{dt}(g(t))$

$$\begin{aligned}
\frac{d}{dt}(g*h) &= \frac{d}{dt}\left(\int_{-\infty}^{\infty} g(\tau)\, h(t-\tau)\, d\tau\right) \\
&= \int_{-\infty}^{\infty} \frac{d}{dt}\left(g(\tau)\, h(t-\tau)\right) d\tau \\
&= \int_{-\infty}^{\infty} g(\tau)\, \frac{d}{dt}(h)(t-\tau)\, d\tau \\
&= g * D(h)
\end{aligned}$$

Similar to the discrete case, changing variables, or via communitive property

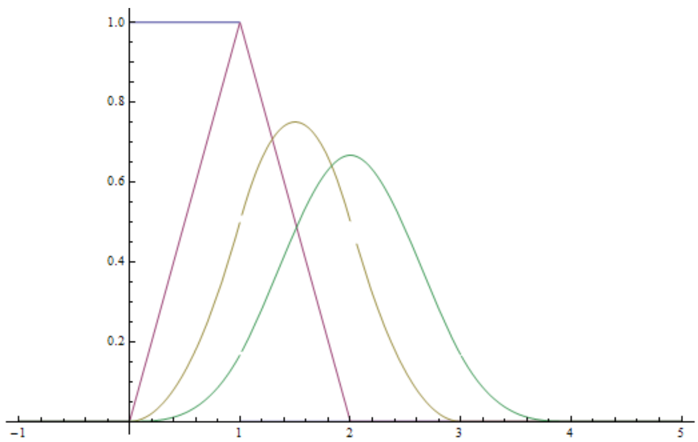
$$\begin{aligned}
\frac{d}{dt}(g*h) &= \frac{d}{dt}(h*g) \\
&= h * D(g) \\
&= D(g) * h
\end{aligned}$$

Question 4

Let $f(t) = \mathbf{1}(t)\,\mathbf{1}(1-t)$

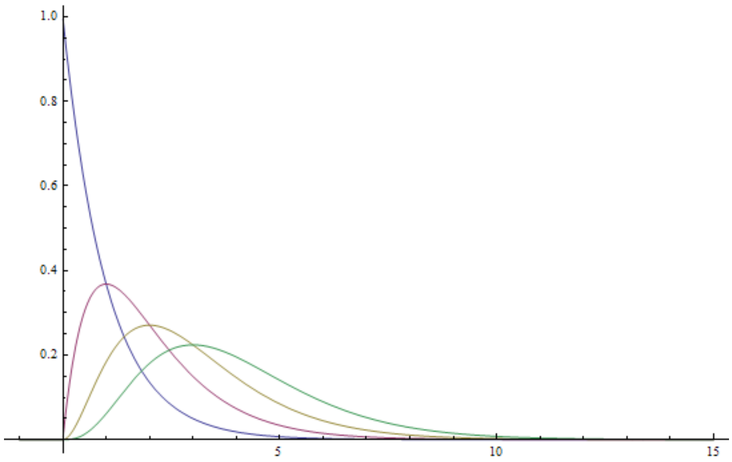
$$f^{*2}(t) = \begin{cases} 1 & t=1 \\ 2-t & 1<t<2 \\ 4-2t & t=2 \\ t & 0\le t<1 \end{cases}, \quad f^{*3}(t) = \begin{cases} \frac{1}{2} & t=2 \\ \frac{t^2}{2} & 0<t<1 \\ -t^2+3t-\frac{3}{2} & 1<t<2 \\ -\frac{1}{2}(t-2)t & t=1 \\ \frac{1}{2}(t-3)^2 & 2<t<3 \end{cases}$$

$$f^{*4}(t) = \begin{cases} \frac{1}{6} & t=1 \vee t=3 \\ \frac{2}{3} & t=2 \\ \frac{t^3}{6} & 0<t<1 \\ -\frac{t^3}{2}+2t^2-2t+\frac{2}{3} & 1<t<2 \\ -\frac{1}{6}(t-4)^3 & 3<t<4 \\ \frac{t^3}{2}-4t^2+10t-\frac{22}{3} & 2<t<3 \end{cases}$$



Let $f_2(t) = e^{-t}\mathbf{1}(t)$

$$f_2^{*2}(t) = e^{-t}t\mathbf{1}(t), \quad f_2^{*3}(t) = \frac{1}{2}e^{-t}t^2\mathbf{1}(t), \quad f_3^{*4}(t) = \frac{1}{6}e^{-t}t^3\mathbf{1}(t)$$



Since $f(t)$ and $f_2(t)$ are probability distributions (Uniform and Exponential respectfully), and convolution is addition of distributions it is clear that the CLT follows from the limit of convolutions, which can be seen visibily in the above plots.

Question 5

Recall from defnition of limit

$$\lim_{t\to\infty} |f(t)| = L \qquad \Leftrightarrow \forall\epsilon > 0\, \exists T\, s.t.\, t > T \Rightarrow ||f(t)| - L| < \epsilon$$

Assuming σ, $M > 0$ exists such that the following holds as $t \to \infty$, i.e. $\forall t > T$

$$\begin{aligned} |f(t)| &< Me^{\sigma t} \\ \left|f(t)\, e^{-\sigma t}\right| &< M \\ \left|f(t)\, e^{-(\sigma-\alpha)t}\right| &< Me^{-\alpha t} \end{aligned}$$

Choosing α to make $\epsilon = Me^{-\alpha t} > 0$ arbitrary, then there exists a $\tilde{\sigma}$ such that

$$\left|f(t)\, e^{-\tilde{\sigma} t}\right| - 0 < \epsilon$$

Which by the above definition is

$$\lim_{t\to\infty} \left|f(t)\, e^{-\tilde{\sigma} t}\right| = 0$$

Question 6

Consider P, Q polynomial functions and repetitively applying l'hopitals

$$\lim_{t\to\infty} \frac{P(t)}{Q(t)} e^{-\sigma t} = \lim_{t\to\infty} \frac{1}{Me^{-\sigma t}} = 0, \qquad M \in \mathbb{R}$$

Thus any rational function is of exponential order

Question 7

Consider the rational polynomial function $f(t) = \frac{a(t)}{b(t)}$ with $a(t)$ having lower degree than $b(t)$. Clearly for large t we have $\frac{a(t)}{b(t)} < 1$ and

$$\begin{aligned} \lim_{t\to\infty} |f(t)| &= 0 \\ \lim_{t\to\infty} \left|f(t)\,e^{-0t}\right| &= 0 \end{aligned}$$

Thus $\sigma_c = 0$.

Question 8

Optionally skipped

Question 9

$$\begin{aligned} \hat{f}(s) &= \mathcal{L}\left(e^{\alpha t}\right) \\ &= \int_0^\infty e^{-st} e^{\alpha t}\, dt \\ &= \left[\frac{1}{\alpha - s} e^{(\alpha - s)t}\right]_0^\infty \\ &= \frac{1}{s-\alpha} \end{aligned}$$

With $Re(s) > Re(\alpha)$

Question 10

Find laplace transform of $f(t) = e^{-at}\cos(bt)$

$$\hat{f}(s) = \int_0^\infty e^{-st} e^{-at} \cos(bt)\; dt$$

$$\begin{aligned} \int e^{-(s+a)t} \cos(bt)\; dt &= \frac{e^{-(s+a)t}\cos(bt)}{-(s+a)} + \frac{b}{s+a}\int e^{-(s+a)t}\sin(bt)\; dt + C \\ &= \frac{e^{-(s+a)t}\cos(bt)}{-(s+a)} + \frac{b}{s+a}\,\frac{e^{-(s+a)t}\sin(bt)}{-(s+a)} \\ &\quad - \frac{b^2}{(s+a)^2}\int e^{-(s+a)t}\cos(bt)\; dt + C \\ \left(1 + \frac{b^2}{(s+a)^2}\right)\int e^{-(s+a)t}\cos(bt)\; dt &= \frac{e^{-(s+a)t}\left((s+a)\cos(bt) + b\sin(bt)\right)}{-(s+a)^2} + C \\ \int e^{-(s+a)t}\cos(bt)\; dt &= -\frac{e^{-(s+a)t}\left((s+a)\cos(bt) + b\sin(bt)\right)}{(s+a)^2 + b^2} + C \\ \int_0^\infty e^{-st}e^{-at}\cos(bt)\; dt &= \left[-\frac{e^{-(s+a)t}\left((s+a)\cos(bt) + b\sin(bt)\right)}{(s+a)^2 + b^2}\right]_0^\infty \\ &= \frac{s+a}{(s+a)^2 + b^2} \end{aligned}$$

Where $b \in \mathbb{R}$

Question 11

$$\ddot{x}+6x=\cos\left(\frac{t}{2}\right), \quad x(0)=0,\ \dot{x}(0)=0$$

Taking laplace transform gives

$$\begin{aligned}
s^2Y-sx(0)-\dot{x}(0)+6Y &= \frac{s}{s^2+\frac{1}{4}}\\
Y &= \frac{s}{\left(s^2+\frac{1}{4}\right)(s^2+6)}\\
&= \frac{4}{23}\frac{s}{s^2+\left(\frac{1}{2}\right)^2}-\frac{4}{23}\frac{s}{s^2+\left(\sqrt{6}\right)^2}
\end{aligned}$$

Inverse laplace transform gives

$$x(t)=\frac{4}{23}\left(\cos\frac{t}{2}-\cos\sqrt{6}t\right)$$

Question 12

Prove $\mathcal{L}(f_1 * f_2)=\hat{f}_1\cdot\hat{f}_2$

$$\begin{aligned}
\mathcal{L}(f_1 * f_2) &= \int_0^\infty e^{-st}\int_{-\infty}^t f_1(\tau)\, f_2(t-\tau)\ d\tau\, dt\\
&= \int_0^\infty \int_{-\infty}^t f_1(\tau)\, e^{-st} f_2(t-\tau)\ d\tau\, dt
\end{aligned}$$

Let $u=t-\tau$ and change order of integration

$$\begin{aligned}
&= \int_0^\infty \int_0^\infty f_1(\tau)\, e^{-s(u+\tau)} f_2(u)\ du\, d\tau\\
&= \int_0^\infty e^{-s\tau} f_1(\tau)\ d\tau \int_0^\infty e^{-su} f_2(u)\ du\\
&= \hat{f}_1\cdot\hat{f}_2
\end{aligned}$$

Question 13

Long division for $\frac{s^4+2s^3+s+2}{s^2+1}$

$$\begin{array}{rr}
 & s^2+2s-1\\
s^2+1 & \overline{)\,s^4+2s^3+s+2}\\
 & s^4+s^2\\
\hline
 & 2s^3-s^2+s+2\\
 & 2s^3+2s\\
\hline
 & -s^2-s+2\\
 & -s^2-1\\
\hline
 & -s+3
\end{array}$$

Thus we have

$$\hat{f}(s)=s^2+2s-1+\frac{3-s}{s^2+1}$$

Question 14

$$\begin{aligned}
\frac{A_{11}}{s+1}+\frac{A_{12}}{(s+1)^2}+\frac{A_{21}}{s-2} &= \frac{s-1}{(s+1)^2(s-2)} \\
A_{11}(s+1)(s-2)+A_{12}(s-2)+A_{21}(s+1)^2 &= s-1 \\
A_{11}\left(s^2-s-2\right)+A_{12}(s-2)+A_{21}\left(s^2+2s+1\right) &= s-1
\end{aligned}$$

s^2 $\qquad A_{11}+A_{21}=0 \quad \Rightarrow \quad A_{11}=-A_{21}$

s $\qquad -A_{11}+A_{12}+2A_{21}=1$

1 $\qquad -2A_{11}-2A_{12}+A_{21}=-1$

Subtracting and substituting above equations gives

$$A_{11}=-\frac{1}{9},\ A_{12}=\frac{2}{3},\ A_{21}=\frac{1}{9}$$

$$\frac{-\frac{1}{9}}{s+1}+\frac{\frac{2}{3}}{(s+1)^2}+\frac{\frac{1}{9}}{s-2}=\frac{s-1}{(s+1)^2(s-2)}$$

Question 15

Partial fractions for

$$\begin{aligned}
\frac{As+B}{s^2+2s+5}+\frac{C}{s+1} &= \frac{s+3}{(s^2+2s+5)(s+1)} \\
(As+B)(s+1)+C\left(s^2+2s+5\right) &= s+3
\end{aligned}$$

s^2 $\qquad A+C=0 \quad \Rightarrow \quad A=-C$

s $\qquad \begin{aligned} A+B+2C&=1 \\ B+C&=1 \end{aligned}$

1 $\qquad B+5C=3$

$$C=\frac{1}{2},\ B=\frac{1}{2},\ A=-\frac{1}{2}$$

$$\frac{\frac{1}{2}-\frac{1}{2}s}{s^2+2s+5}+\frac{\frac{1}{2}}{s+1}=\frac{s+3}{(s^2+2s+5)(s+1)}$$

Question 16

Find Fourier of $f(t)=\frac{\sin t}{t}$.

As a guess we first consider the inverse of the function $\hat{f}(\omega)=\begin{cases}1 & -L\le\omega\le L \\ 0 & \text{otherwise}\end{cases}$

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\,e^{i\omega t}\,d\omega \\
&= \frac{1}{2\pi}\int_{-L}^{L} e^{i\omega t}\,d\omega \\
&= \frac{1}{2\pi}\left[\frac{-ie^{i\omega t}}{t}\right]_{-L}^{L} \\
&= \frac{i}{2\pi}\left(\frac{-e^{itL}}{t}+\frac{e^{-itL}}{t}\right) \\
&= \frac{i}{2\pi}\left(\frac{-\cos(Lt)-i\sin(Lt)+\cos(-Lt)+i\sin(-Lt)}{t}\right) \\
&= \frac{i}{2\pi}\left(\frac{-2i\sin(Lt)}{t}\right) \\
&= \frac{\sin(Lt)}{\pi t}
\end{aligned}
$$

Thus our Fourier transform must be scaled and have $L = 1$ such that

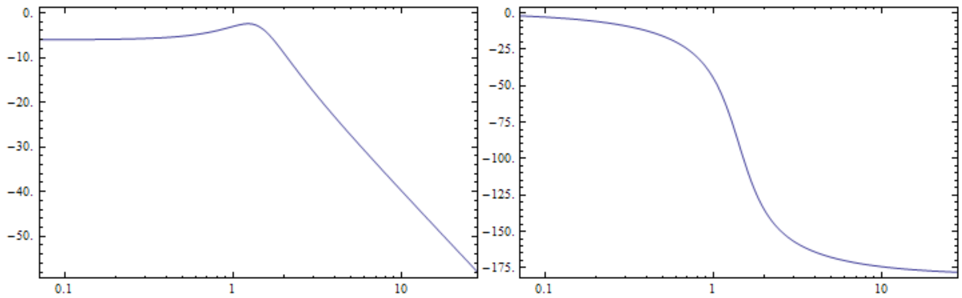
$$
\hat{f}(\omega) = \begin{cases} \pi & -1 \le \omega \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

Question 17

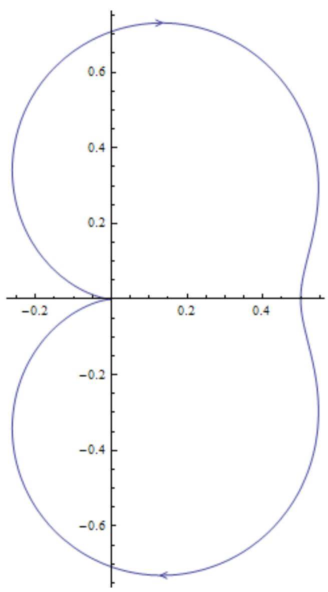
Bode and Nyquist plot for

$$
H(s) = \frac{1}{s^2 + s + 2}
$$

Bode (Magnitude / Frequency)



Nyquist



Question 18

Prove that

$$\alpha_1 \delta + \alpha_2 \delta = (\alpha_1 + \alpha_2)\,\delta$$

That is:

$$\begin{aligned}\int_{-\infty}^{\infty} (\alpha_1 \delta + \alpha_2 \delta)\,\phi(t)\; dt &= \alpha_1 \phi(0) + \alpha_2 \phi(0) \\ &= (\alpha_1 + \alpha_2)\,\phi(0) \\ &= \int_{-\infty}^{\infty} (\alpha_1 + \alpha_2)\,\delta \phi(t)\; dt\end{aligned}$$

Question 19

Fixing τ and let $t \in \mathbb{R}$, show that

$$f(t)\,\delta(t-\tau) = f(\tau)\,\delta(t-\tau)$$

We have:

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\,\delta(t-\tau)\,\phi(t)\; dt &= \int_{-\infty}^{\infty} f(u+\tau)\,\delta(u)\,\phi(u+\tau)\; du \\ &= f(\tau)\,\phi(\tau) \\ &= \int_{-\infty}^{\infty} f(\tau)\,\delta(u)\,\phi(u+\tau)\; du \\ &= \int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,\phi(t)\; dt\end{aligned}$$

Thus

$$f(t)\,\delta(t-\tau) = f(\tau)\,\delta(t-\tau)$$

Question 20

From $\int_{-\infty}^{\infty} \mathbf{1}(t)\,\phi(t)\; dt = \int_0^{\infty} \phi(t)\; dt$ show definition of $\mathbf{1}(t)$. Consider $\mathbf{1}(t)$ to be an arbitrary function

$$\int_{-\infty}^{\infty} \mathbf{1}(t)\,\phi(t)\; dt \;\; = \textstyle\int_{-\infty}^{0} \mathbf{1}(t)\,\phi(t)\; dt + \int_0^{\infty} \mathbf{1}(t)\,\phi(t)\; dt = \displaystyle\int_0^{\infty} \phi(t)\; dt$$

since $\int_{-\infty}^{0} \mathbf{1}(t)\,\phi(t)\; dt = 0$ for arbitrary $\phi(t)$ we must have $\mathbf{1}(t) = 0\,\forall t < 0$

since $\int_0^{\infty} \mathbf{1}(t)\,\phi(t)\; dt = \int_0^{\infty} \phi(t)\; dt$ for arbitrary $\phi(t)$ we must have $\mathbf{1}(t) = 1\,\forall t \geq 0$

thus

$$\mathbf{1}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Question 21

Show $\mathbf{1}'(t-\theta)=\delta(t-\theta)$, i.e. show $\int_{-\infty}^{\infty}\mathbf{1}'(t-\theta)\,\phi(t)\,dt=\int_{-\infty}^{\infty}\delta(t-\theta)\,\phi(t)\,dt$

$$\begin{aligned}
\int_{-\infty}^{\infty}\mathbf{1}'(t-\theta)\,\phi(t)\,dt &= \int_{-\infty}^{\infty}\mathbf{1}'(u)\,\phi(u+\theta)\,du \\
&= -\int_{-\infty}^{\infty}\mathbf{1}(u)\,\phi'(u+\theta)\,du \\
&= -\int_{0}^{\infty}\phi'(u+\theta)\,du \\
&= -\left(\phi(\infty)-\phi(\theta)\right) \\
&= \phi(\theta) \\
&= \int_{0}^{\infty}\delta(u)\,\phi(u+\theta)\,du \\
&= \int_{-\infty}^{\infty}\delta(t-\theta)\,\phi(t)\,dt
\end{aligned}$$

Thus $\mathbf{1}'(t-\theta)=\delta(t-\theta)$

Question 22

Show linear property for N inputs, that is: $\mathcal{O}\left(\sum_{i=1}^{N}\alpha_i u_i(t)\right)=\sum_{i=1}^{N}\alpha_i y_i(t)$
When $N=2$ this holds due to standard linear property $\mathcal{O}(\alpha_1 u_1(t)+\alpha_2 u_2(t))=\alpha_1 y_1(t)+\alpha_2 y_2(t)$, assume holds for k and let $y(t)=\mathcal{O}(u(t))=\mathcal{O}\left(\sum_{i=1}^{k}\alpha_i u_i(t)\right)=\sum_{i=1}^{k}\alpha_i y_i(t)$, we show that this holds for $k+1$

$$\begin{aligned}
\mathcal{O}\left(\sum_{i=1}^{k+1}\alpha_i u_i(t)\right) &= \mathcal{O}\left(\sum_{i=1}^{k}\alpha_i u_i(t)+\alpha_{k+1}u_{k+1}(t)\right) \\
&= \mathcal{O}\left(u(t)+\alpha_{k+1}u_{k+1}(t)\right) \\
&= y(t)+\alpha_{k+1}y_{k+1}(t) \\
&= \sum_{i=1}^{k+1}\alpha_i y_i(t)
\end{aligned}$$

Thus by induction this holds for arbitrarty N.

Question 23

Determine properties for

$$y(n)=\frac{1}{N+M+1}\sum_{m=-M}^{N}\left(u(n+m)\right)^{\alpha+\beta\cos(n)}$$

memoryless $M=0$, i.e. uses no historical $<n$ values

causal $N=0$, that is uses no future $>n$ values

linear $\alpha=1$, $\beta=0$, since $y(n)=\frac{1}{N+M+1}\sum_{m=-M}^{N}u(n+m)$ is linear from previous exercise

time-invariant $\beta=0$, removing the $\beta\cos(n)$ periodic effect - which dramatically changes the result based on time

Question 24

Show LTI memoryless $\Leftrightarrow$ impulse has the form $h(t) = K\delta(t)$

$\Leftarrow$assuming impulse has the form $h(t) = K\delta(t)$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} u(\tau) K\delta(t-\tau)\, d\tau \\ &= Ku(t) \end{aligned}$$

thus output does not rely on historical time values, aka memoryless

$\Rightarrow$ assuming LTI memoryless
By looking at the convolution

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} u(\tau) h(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) u(t-\tau)\, d\tau \end{aligned}$$

we see that due to memoryless, we must have $h(\tau) = 0$ for $\tau > t$ (else $u(\tau)$ uses history) and we must have $h(\tau) = 0$ for $\tau < t$ (else $u(t-\tau)$ uses history)
Thus the convolution requires that the impluse is non-zero only at t. Thus our generalized linear impulse must be defined as a scaled δ function, that is:

$$h(t) = K\delta(t)$$

Question 25

Show LTI system is causal $\Leftrightarrow$ $h(t) = 0$ for all $t < 0$

$\Rightarrow$ assuming LTI causal, i.e. independent of future values
Considering the convolution of a causal system, the impulse response must only be non-zero over region $(-\infty, t]$ i.e.

$$y(t) = \int_{-\infty}^{t} u(\tau) h(t-\tau)\, d\tau$$

that is $h(t-\tau) = 0$ for $t < \tau \Rightarrow t - \tau < 0$. Relabelling gives us

$$h(t) = 0, \qquad \forall t < 0$$

$\Leftarrow$ assuming $h(t) = 0$ for $t < 0$
Similar to above in reverse. By looking at the resulting convolutions

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} u(\tau) h(t-\tau)\, d\tau \\ &= \int_{0}^{\infty} h(\tau) u(t-\tau)\, d\tau \end{aligned}$$

only depends on inputs during times up to t, thus system is causal.

Question 26

$\|y\|_\infty = \|h\|_1 \|u\|_\infty$ when $u(t) = 0$

Question 27

A SISO LTI system (impulse response $h(\cdot)$) is BIBO stable $\Leftrightarrow$ $\|h\|_1 < \infty$ (i.e. $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau$ exists) and further $\|y\|_\infty \leq \|h\|_1 \|u\|_\infty$ for any bounded input

$\Leftarrow$ assume $\|h\|_1 < \infty$

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau \right| \\ &\leq \int_{-\infty}^{\infty} |u(\tau)|\, |h(t-\tau)|\, d\tau \\ &\leq \|u\|_\infty \int_{-\infty}^{\infty} |h(t-\tau)|\, d\tau \\ &= \|u\|_\infty \int_{-\infty}^{\infty} |h(\tau)|\, d\tau \\ &= \|h\|_1 \|u\|_\infty \end{aligned}$$

Since $\|h\|_1$ is finite, then for every bounded input, we have a bounded output (in particular $\|y\|_\infty \leq \|h\|_1 \|u\|_\infty$) thus our system is BIBO stable.

$\Rightarrow$ assume BIBO stable and assuming input is real (complex is covered in next question). We choose the input

$$u(t) = \operatorname{sign}(h(-t))$$

then

$$\begin{aligned} y(0) &= \int_{-\infty}^{\infty} u(\tau)\, h(0-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} \operatorname{sign}(h(-\tau))\, h(-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} |h(-\tau)|\, d\tau \\ &= \|h\|_1 \end{aligned}$$

Thus if $\|h\|_1$ is unbounded, then our output is also unbounded (for a bounded input) - but since BIBO stable output must be bounded, thus $\|h\|_1 < \infty$.

Question 28

We consider the complex case in the above question. Here we use the complex conjugate as our input function

$$u(t) = \bar{h}(-t)$$

and the rest follows as above.

Question 29

Choosing transfer function (with 3 distinct poles with negative real part):

$$H(s) = \frac{1}{(s+1)(s+2)(s+3)} = \frac{\frac{1}{2}}{s+1} + \frac{-1}{s+2} + \frac{\frac{1}{2}}{s+3}$$

We have partial fraction result (working removed for brevity)

$$\begin{aligned} Y(s) &= H(s)\frac{\omega_0}{s^2+\omega_0^2} \\ &= \frac{\frac{\omega_0}{2(1+\omega_0^2)}}{s+1} + \frac{-\frac{\omega_0}{(4+\omega_0^2)}}{s+2} + \frac{\frac{\omega_0}{2(9+\omega_0^2)}}{s+3} + \frac{\frac{\omega_0(\omega_0^2-11)+i(6-6\omega_0^2)}{2(1+\omega_0^2)(4+\omega_0^2)(9+\omega_0^2)}}{(s+i\omega_0)} + \frac{\frac{\omega_0(\omega_0^2-11)-i(6-6\omega_0^2)}{2(1+\omega_0^2)(4+\omega_0^2)(9+\omega_0^2)}}{(s-i\omega_0)} \end{aligned}$$

Thus in the form $Y(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s-p_i} + \frac{\alpha_0}{s+i\omega_0} + \frac{\bar{\alpha_0}}{s-i\omega_0}$ we have

$$\alpha_1 = \frac{\omega_0}{2(1+\omega_0^2)}, \quad \alpha_2 = -\frac{\omega_0}{(4+\omega_0^2)}, \quad \alpha_3 = \frac{\omega_0}{2(9+\omega_0^2)}$$

$$\alpha_0 = \frac{\omega_0(\omega_0^2-11)+i(6-6\omega_0^2)}{2(1+\omega_0^2)(4+\omega_0^2)(9+\omega_0^2)}$$

Giving

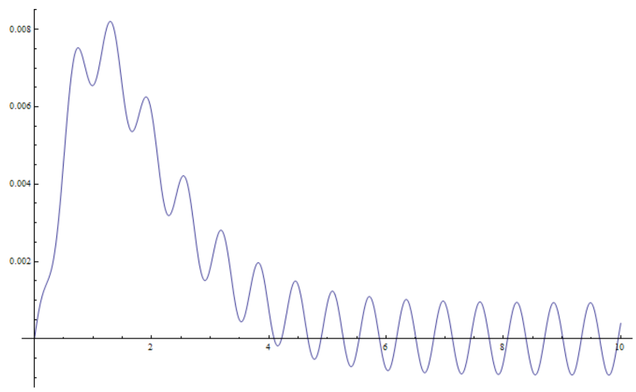
$$y(t) = \frac{\omega_0 e^{-t}}{2(1+\omega_0^2)} - \frac{\omega_0 e^{-2t}}{(4+\omega_0^2)} + \frac{\omega_0 e^{-3t}}{2(9+\omega_0^2)} + 2|\alpha_0|\cos(\omega_0 t+\phi), \qquad t \geq 0$$

Question 30

In above question

$$\begin{aligned} \phi &= \tan^{-1}\left(\frac{\Im(\alpha_0)}{\Re(\alpha_0)}\right) \\ &= \tan^{-1}\left(\frac{\frac{6-6\omega_0^2}{2(1+\omega_0^2)(4+\omega_0^2)(9+\omega_0^2)}}{\frac{\omega_0(\omega_0^2-11)}{2(1+\omega_0^2)(4+\omega_0^2)(9+\omega_0^2)}}\right) \\ &= \tan^{-1}\left(\frac{6-6\omega_0^2}{\omega_0(\omega_0^2-11)}\right) \end{aligned}$$

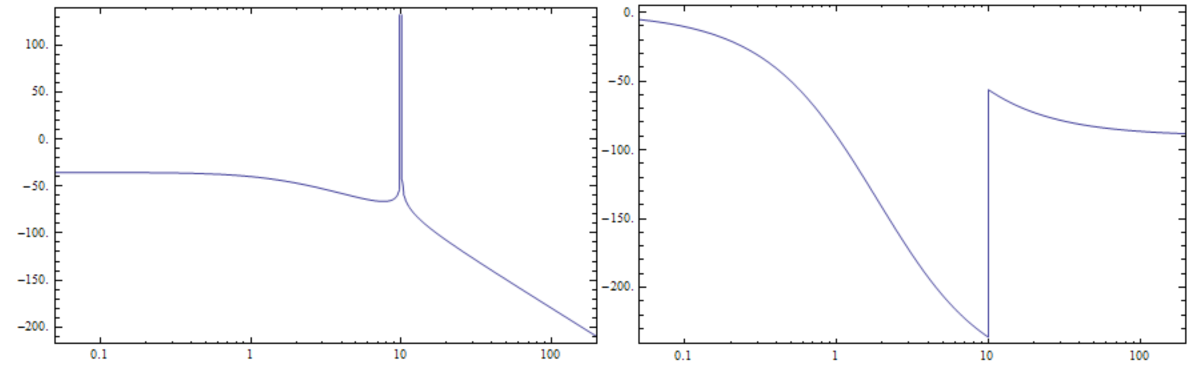
The figure below is a plot of our system, and it shows the convergence to a pure sinusoidal form (here with $\omega_0 = 10$)



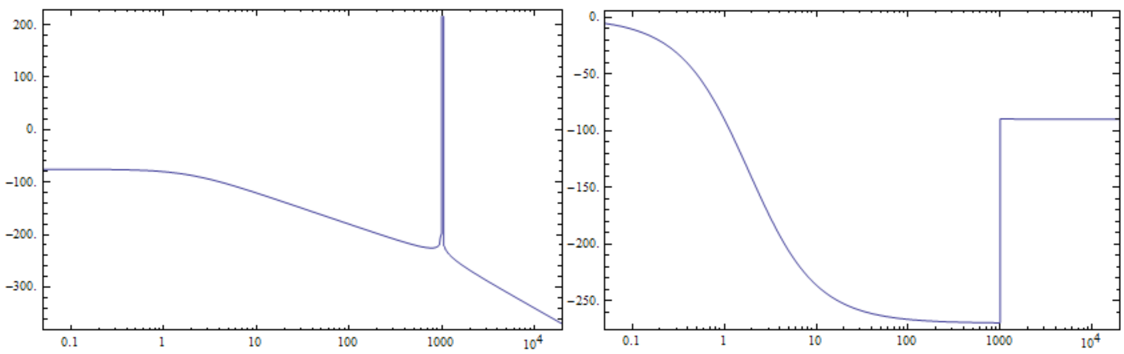
Question 31

Bode plots for system (Magnitude / Frequency). You can see that as we increase ω_0 the we have a point of convergence such that the frequency becomes a constant value (i.e. gives a pure sinusoidal wave).

$\omega_0 = 10$



$\omega_0 = 1000$



Question 32

Not assessable - skipped

Question 33

$$\frac{1}{\lambda}\dot{h}(t) + h(t) = \delta(t), \quad h(0^-) = 0$$

Solving via laplace

$$\begin{aligned}
\frac{1}{\lambda}sH(s) - h(0^-) + H(s) &= 1 \\
H(s) &= \frac{\lambda}{s+\lambda} \\
h(t) &= \lambda e^{-\lambda t}, \qquad t > 0 \\
&= \lambda e^{-\lambda t}\mathbf{1}(t)
\end{aligned}$$

Question 34

Show transfer function as Laplace transform for

$$\begin{aligned}
\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) &= \omega_n^2 u(t) \\
\ddot{h}(t) + 2\zeta\omega_n\dot{h}(t) + \omega_n^2 h(t) &= \omega_n^2 \delta(t), \quad \dot{h}(0) = 0,\ h(0) = 0 \\
s^2 H(s) + 2\zeta\omega_n s H(s) + \omega_n^2 H(s) &= \omega_n^2 \\
H(s)\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right) &= \omega_n^2 \\
H(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\end{aligned}$$

Question 35

Assume $\zeta \neq 1$, find partial fractions

$$\begin{aligned}
\frac{A}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} + \frac{B}{s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}} &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
(A+B)s + (A+B)\zeta\omega_n + (A-B)\,\omega_n\sqrt{\zeta^2 - 1} &= \omega_n^2
\end{aligned}$$

$s \qquad A + B = 0 \quad \Rightarrow \quad A = -B$

$$\begin{aligned}
(A+B)\zeta\omega_n + (A-B)\,\omega_n\sqrt{\zeta^2 - 1} &= \omega_n^2 \\
(2A)\,\omega_n\sqrt{\zeta^2 - 1} &= \omega_n^2 \\
A &= \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}
\end{aligned}$$

Thus system is

$$H(s) = \frac{M}{s - c_1} - \frac{M}{s - c_2}$$

where $M = \frac{\omega_n}{2\sqrt{\zeta^2-1}}$, $c_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Question 36

Apply inverse transform to above. Recall that $\mathcal{L}^{-1}\left(\frac{M}{s-\alpha}\right) = Me^{-\alpha t}\mathbf{1}(t)$

$$\begin{aligned}
h(t) &= Me^{c_1 t}\mathbf{1}(t) - Me^{c_2 t}\mathbf{1}(t) \\
&= M\left(e^{c_1 t} - e^{c_2 t}\right)\mathbf{1}(t)
\end{aligned}$$

Question 37

Assume $\zeta = 1$. Find $H(s)$ and $h(t)$

$$\begin{aligned}
H(s) &= \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} \\
&= \frac{\omega_n^2}{(s + \omega_n)^2}
\end{aligned}$$

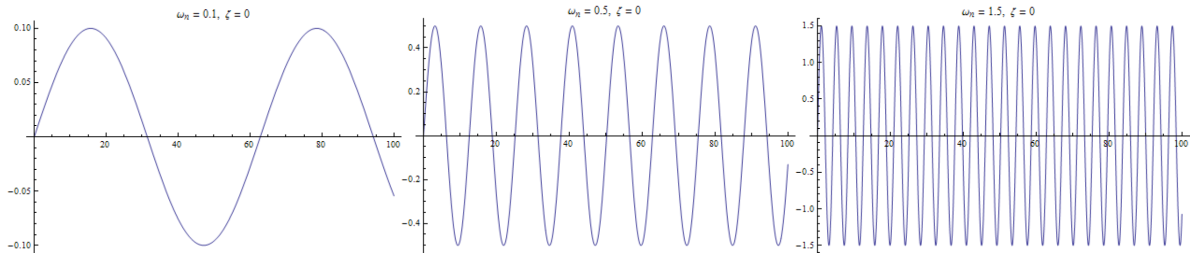
From lookup table $\mathcal{L}^{-1}\left(\frac{\omega_n^2}{(s+\omega_n)^2}\right) = te^{-\omega_n t}\mathbf{1}(t)$ thus

$$h(t) = \omega_n^2 te^{-\omega_n t}\mathbf{1}(t)$$

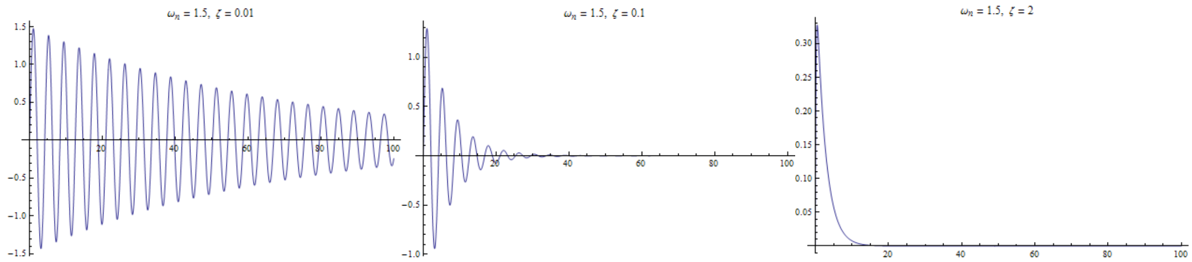
Question 38

It is clear from the below plots that ω_n controls the undampened frequency while ζ controls the dampening effect, this is obviously why they are called *undamped natural frequency* and *damping ratio* (repectfully).

Changing ω_n



Changing ζ



Question 39

This system often has exponential growth (unstable)

$$\begin{aligned} Y &= (R + G_2 Y)\, G_1 H \\ Y\,(1 - G_1 G_2 H) &= R G_1 H \\ Y &= \frac{R G_1 H}{1 - G_1 G_2 H} \end{aligned}$$