# MATH4406 Homework 1 

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## Exercise 1

For linear interpolation, we obviously want $K(0)=1$, and we want $K$ to decay linearly until $n=1$, but we also want to it to build linearly from $n=-1$. Also, we want the data point at $t=0$ to effect only the previous and next points, so $K(t)=0$ for all $t \notin[-1,1]$. So we want

$$
K(t)=\left\{\begin{array}{cc}
1-t & -1 \leq t \leq 0 \\
t-1 & 0<t \leq 1 \\
0 & \text { otherwise }
\end{array}\right.
$$

For $-1 \leq t \leq 1$, this looks like the absolute value function. Outside of this region where the absolute value is positive, the function takes the value 0 . So this function can be represented by

$$
K(t)=\max \{1-|t|, 0\} .
$$

A simple example would be interpolating the points $u(2 n)=(2 n)^{2}+1$, i.e. the points $(0,1),(2,5),(4,17) \ldots$ Between the first two points we would expect the interpolation to be a line with gradient 2 and intercept 1, i.e. $\left.\tilde{u}\right|_{[0,2]}(t)=2 t+1$. This interpolation gives for $t \in[0,1]$

$$
\begin{aligned}
\tilde{u}(t) & =\sum_{n} u(2 n) K\left(\frac{t-2 n}{2}\right) \\
& =\sum_{n} u(2 n) \max \left\{1-\left|\frac{t-2 n}{2}\right|, 0\right\} \\
& =u(0)\left(1-\frac{t}{2}\right)+u(2)\left(1-\left|\frac{t-2}{2}\right|\right)+0+0+\ldots \\
& =\left(1-\frac{t}{2}\right)+5\left(\frac{t}{2}\right) \\
& =1+2 t
\end{aligned}
$$

which is what we expected.

## Exercise 2

I'm going to use the definition of $c_{j}$ as

$$
\sum_{k=\max \{0, j-n+1\}}^{\min \{j, n-1\}} a_{k} b_{k-j}
$$

as this avoids the convention of assuming non-defined co-efficients as 0 , a convention which can get messy in a proof by induction.

Consider the case where $n=1$, then $A(x)=a_{0}, B(x)=b_{0}$ and

$$
C(x)=a_{0} b_{0}=\sum_{k=0}^{0} a_{k} b_{k-j}
$$

so the result holds trivially. Now assume the result is true for $n=m$, and consider the case $n=m+1$. Then

$$
\begin{aligned}
C(x) & =A(x) B(x) \\
& =\sum_{j=0}^{m+1-1} a_{j} x^{j} \sum_{j=0}^{m+1-1} b_{j} x^{j} \\
& =\left(a_{m} x^{m}+\sum_{j=0}^{m-1} a_{j} x^{j}\right)\left(b_{m} x^{m}+\sum_{j=0}^{m-1} b_{j} x^{j}\right) \\
& =a_{m} x^{m} b_{m} x^{m}+b_{m} x^{m} \sum_{j=0}^{m-1} a_{j} x^{j}+a_{m} x^{m} \sum_{j=0}^{m-1} b_{j} x^{j}+\sum_{j=0}^{m-1} a_{j} x^{j} \sum_{j=0}^{m-1} b_{j} x^{j} \\
& =a_{m} b_{m} x^{2 m}+a_{j} b_{m} x^{2 m-1}+b_{j} a_{m} x^{2 m-1}+\sum_{j=0}^{m-2}\left(a_{j} b_{m} x^{j+m}+b_{j} a_{m} x^{j+m}\right)+\sum_{j=0}^{m-1} a_{j} x^{j} \sum_{j=0}^{m-1} b_{j} x^{j} .
\end{aligned}
$$

Define

$$
D(x)=\sum_{j=0}^{2 n-2} d_{j} x^{j}=\sum_{j=0}^{m-1} a_{j} x^{j} \sum_{j=0}^{m-1} b_{j} x^{j} .
$$

Thus, by the assumption

$$
D(x)=\sum_{j=0}^{2 m-2} \sum_{k=\max \{0, j-m+1\}}^{\min \{j, m-1\}} a_{k} b_{k-j} x^{j} .
$$

So we have

$$
\begin{aligned}
& C(x)= a_{m} b_{m} x^{2 m}+\left(a_{m} b_{m-1}+b_{m} a_{m-1}\right) x^{2 m-1} \\
&+\sum_{j=0}^{m-2}\left(a_{m-1} b_{m} x^{j+m}+b_{m-1} a_{m} x^{j+m}\right)+\sum_{j=0}^{2 m-2} \sum_{k=\max \{0, j-m+1\}}^{\min \{j, m-1\}} a_{k} b_{k-j} x^{j} \\
&= \sum_{j=2 m-1}^{2 m} \sum_{k=j-m}^{m} a_{k} b_{k-j} x^{j}+\sum_{j=m}^{2 m-2}\left(a_{j-m} b_{m} x^{j}+b_{j-m} a_{m} x^{j}\right) \\
&+\sum_{j=0}^{m-1} \sum_{k=\max \{0, j-m+1\}}^{\min \{j, m-1\}} a_{k} b_{k-j} x^{j}+\sum_{j=m}^{2 m-2} \sum_{k=\max \{0, j-m+1\}}^{\min \{j, m-1\}} a_{k} b_{k-j} x^{j} \\
&= \sum_{j=2 m-1}^{2 m} \sum_{k=j-m}^{m} a_{k} b_{k-j} x^{j}+\sum_{j=m}^{2 m-2}\left(a_{j-m} b_{m} x^{j}+b_{j-m} a_{m} x^{j}\right) \\
&+\sum_{j=0}^{m-1} \sum_{k=0}^{j} a_{k} b_{k-j} x^{j}+\sum_{j=m}^{2 m-2} \sum_{k=j-m+1}^{m-1} a_{k} b_{k-j} x^{j} \\
&= \sum_{j=2 m-1}^{2 m} \sum_{k=j-m}^{m} a_{k} b_{k-j} x^{j}+\sum_{j=0}^{m-1} \sum_{k=0}^{j} a_{k} b_{k-j} x^{j} \\
&+\sum_{j=m}^{2 m-2}\left(b_{m} a_{j-m} x^{j}+a_{m} b_{j-m} x^{j}+\sum_{k=j-m+1}^{m-1} a_{k} b_{k-j} x^{j}\right) \\
&= \sum_{j=2 m-1}^{2 m} \sum_{k=j-m}^{m} a_{k} b_{k-j} x^{j}+\sum_{j=0}^{m-1} \sum_{k=0}^{j} a_{k} b_{k-j} x^{j}+\sum_{j=m}^{2 m-2} \sum_{k=j-m}^{m} a_{k} b_{k-j} x^{j} \\
&= \sum_{j=0}^{m-1} \sum_{k=0}^{j} a_{k} b_{k-j} x^{j}+\sum_{j=m}^{2 m} \sum_{k=j-m}^{m} a_{k} b_{k-j} x^{j} \\
&= \sum_{j=0}^{2 m} \sum_{k=\max \{0, j-m\}} a_{k} b_{k-j} x^{j} \\
&= \sum_{j=0}^{2(m+1)-2} \min \{j,(m+1)-1\} \\
& k=\max \{0, j-(m+1)-1\} \\
& m_{k} b_{k-j} x^{j}
\end{aligned}
$$

so this result holds for $n=m+1$, thus, the statement is true by induction.

## Exercise 3

Firstly, differentiation:

$$
\begin{gathered}
D(f * g)=D \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \mathrm{d} x=\int_{-\infty}^{\infty} D(f(\tau) g(t-\tau)) \mathrm{d} x=\int_{-\infty}^{\infty} f(\tau) D(g(t-\tau)) \mathrm{d} x \\
=\int_{-\infty}^{\infty} f(\tau)(D g)(t-\tau) D(t-\tau) \mathrm{d} x=\int_{-\infty}^{\infty} f(\tau)(D g)(t-\tau) 1 \mathrm{~d} x=f * D(g)
\end{gathered}
$$

by the chain rule. Similarly,

$$
D(g * f)=D(f) * g
$$

and since convolutions are commutative, we have

$$
f * D(g)=D(f * g)=D(g * f)=D(f) * g
$$

Now shift

$$
D(f * g)[n]=D \sum_{m=-\infty}^{\infty} f[m] g[n-m]=\sum_{m=-\infty}^{\infty} f[m] g[(n-1)-m]=f * D g
$$

and substituting $k=m-1$ gives
$D(f * g)[n]=\sum_{m=-\infty}^{\infty} f[m] g[(n-1)-m]=\sum_{k=-\infty}^{\infty} f[k-1] g[(n-1)-(k-1)]=\sum_{k=-\infty}^{\infty} D(f)[k] g[n-k]=D f * g$ thus

$$
D(f * g)=f * D(g)=D(f) * g .
$$

## Exercise 4

Firstly, notice that $f(t)=\mathbb{1}(t)-\mathbb{1}(t-1)$. Using the Laplace transform,

$$
\mathcal{L}(f * f)=\hat{f}(s)^{2}=\left(\frac{1}{s}-\frac{e^{-s}}{s}\right)^{2}=\frac{1}{s^{2}}-\frac{2 e^{-s}}{s^{2}}+\frac{e^{-2 s}}{s^{2}} .
$$

Transforming back (using the table of Laplace transforms) gives

$$
\left.\left.\begin{array}{rl}
f * f & =t \mathbb{1}(t)-2(t-1) \mathbb{1}(t-1)+(t-2) \mathbb{1}(t-2) \\
0 & t<0 \\
t & 0 \leq t<1 \\
t-2(t-1) & 1 \leq t<2 \\
t-2(t-1)+(t-2) & 2 \leq t
\end{array}\right\} \begin{array}{cc}
0 & t<0 \\
t & 0 \leq t<1 \\
2-t & 1 \leq t<2 \\
0 & 2 \leq t
\end{array}\right] .
$$

Repeating this process gives

$$
\mathcal{L}\left(f^{* 3}\right)=\hat{f}(s)^{3}=\frac{1}{s^{3}}-\frac{3 e^{-s}}{s^{3}}+\frac{3 e^{-2 s}}{s^{3}}-\frac{e^{-3 s}}{s^{3}}
$$

and transforming back gives

$$
\begin{array}{rl}
f^{* 3} & =\frac{1}{2} t^{2} \mathbb{1}(t)-\frac{3}{2}(t-1)^{2} \mathbb{1}(t-1)+\frac{3}{2}(t-2)^{2} \mathbb{1}(t-2)-\frac{1}{2}(t-3)^{2} \mathbb{1}(t-3) \\
0 & t<0 \\
0 & 0 \leq t<1 \\
\frac{t^{2}}{2} & 1 \leq t<2 \\
\frac{1}{2}\left(t^{2}-3(t-1)^{2}\right) & 2 \leq t<3 \\
\frac{1}{2}\left(t^{2}-3(t-1)^{2}+3(t-2)^{2}\right) & 3 \leq t \\
\frac{1}{2}\left(t^{2}-3(t-1)^{2}+3(t-2)^{2}-(t-3)^{2}\right) & \begin{array}{cc}
0 & t<0 \\
0 & 0 \leq t<1 \\
\frac{t^{2}}{2} & 1 \leq t<2 \\
-t^{2}+3 t-\frac{3}{2} & 1 \leq \begin{array}{cc}
2 \\
\frac{1}{2}\left(t^{2}-6 t+9\right) & 2 \leq t<3 \\
0 & 3 \leq t
\end{array}
\end{array}
\end{array}
$$

Finally, repeating this again gives

$$
\mathcal{L}\left(f^{* 4}\right)=\hat{f}(s)^{4}=\frac{1}{s^{4}}-\frac{4 e^{-s}}{s^{4}}+\frac{6 e^{-2 s}}{s^{4}}-\frac{4 e^{-3 s}}{s^{4}}+\frac{e^{-4 s}}{s^{4}}
$$

and transforming back gives

$$
\begin{aligned}
& f^{* 4}=\frac{1}{6} t^{3} \mathbb{1}(t)-\frac{4}{6}(t-1)^{3} \mathbb{1}(t-1)+\frac{6}{6}(t-2)^{3} \mathbb{1}(t-2)-\frac{4}{6}(t-3)^{3} \mathbb{1}(t-3)+\frac{1}{6}(t-4)^{4} \mathbb{1}(t-4) \\
& =\left\{\begin{array}{cc}
0 & t<0 \\
\frac{t^{3}}{6} & 0 \leq t<1 \\
\frac{1}{6}\left(t^{3}-4(t-1)^{3}\right) & 1 \leq t<2 \\
\frac{1}{6}\left(t^{3}-4(t-1)^{3}+6(t-2)^{3}\right) & 2 \leq t<3 \\
\frac{1}{6}\left(t^{3}-4(t-1)^{3}+6(t-2)^{3}-4(t-3)^{3}\right) & 3 \leq t<4 \\
\frac{1}{6}\left(t^{3}-4(t-1)^{3}+6(t-2)^{3}-4(t-3)^{3}+(t-4)^{3}\right) & 4 \leq t
\end{array}\right. \\
& =\left\{\begin{array}{cc}
0 & t<0 \\
\frac{t^{3}}{6} & 0 \leq t<1 \\
\frac{1}{6}\left(-3 t^{3}+12 t^{2}-12 t+4\right) & 1 \leq t<2 \\
\frac{1}{6}\left(3 t^{3}-24 t^{2}+60 t-44\right) & 2 \leq t<3 \\
\frac{1}{6}\left(-t^{3}+12 t^{2}-48 t+64\right) & 3 \leq t<4 \\
0 & 4 \leq t
\end{array} .\right.
\end{aligned}
$$

For $f_{2}(t)=e^{-t} \mathbb{1}(t)$ we have

$$
\begin{gathered}
f_{2}^{* 2}(t)=\int_{-\infty}^{\infty} f_{2}(\tau) f_{2}(t-\tau) \mathrm{d} \tau=\int_{-\infty}^{\infty} e^{-\tau} \mathbb{1}(\tau) e^{-t+\tau} \mathbb{1}(t-\tau) \mathrm{d} \tau=\int_{0}^{\infty} e^{-t+\tau-\tau} \mathbb{1}(t-\tau) \mathrm{d} \tau \\
=\left\{\begin{array}{cc}
\int_{0}^{t} e^{-t} \mathrm{~d} \tau & t>0 \\
0 & \text { otherwise }
\end{array}=t e^{-t} \mathbb{1}(t) .\right.
\end{gathered}
$$

Similarly

$$
\begin{gathered}
f_{2}^{* 3}(t)=\int_{-\infty}^{\infty} f_{2}(\tau) f_{2}^{* 2}(t-\tau) \mathrm{d} \tau=\int_{-\infty}^{\infty} e^{-\tau} \mathbb{1}(\tau) e^{-t+\tau}(t-\tau) \mathbb{1}(t-\tau) \mathrm{d} \tau \\
=\mathbb{1}(t) e^{-t} \int_{0}^{t} t-\tau \mathrm{d} \tau=e^{-t} \frac{t^{2}}{2} \mathbb{1}(t)
\end{gathered}
$$

and

$$
f_{2}^{* 4}(t)=\int_{0}^{\infty} e^{-\tau} e^{-t+\tau} \frac{(t-\tau)^{2}}{2} \mathbb{1}(t-\tau) \mathrm{d} \tau=\frac{1}{2} \mathbb{1}(t) e^{-t} \int_{0}^{t} t^{2}-2 t \tau+\tau^{2} \mathrm{~d} \tau=\frac{t^{3}}{6} e^{-t} \mathbb{1}(t)
$$

Here are the plots of these functions.
It is clear from these graphs that effect of repeated self convolutions on a function is to smooth it out. It appears that if the process was repeated indefinitely that resulting function would be a smooth bell curve, or Gaussian distribution. This is not surprising, as if the function $f(t)$ was probability distribution function of a random variable (of finite mean and variance), $f^{* n}$ would be the distribution of the sum of $n$ independent and identically distributed random variables. By the Central Limit Theorem, the mean of the sum of such variables is approximately normally distributed.


Figure 1: Self convolutions of $f(t)=\mathbb{1}(t) \mathbb{1}(1-t)$

## Exercise 5

Assume $f(t)$ is of exponential order as $t \rightarrow \infty$. Then there is a real $\sigma$ and an $M, T>0$ such that for all $t>T$,

$$
|f(t)|<M e^{\sigma t} .
$$

Then for some $\tilde{\sigma}>\sigma$, we have

$$
e^{-\tilde{\sigma} t}|f(t)|<M e^{(\sigma-\tilde{\sigma}) t}
$$

for all $t>T$. Noting that $-\tilde{\sigma} t \in \mathbb{R}, e^{-\tilde{\sigma} t}>0$, so

$$
M e^{(\sigma-\tilde{\sigma}) t}>e^{-\tilde{\sigma} t}|f(t)|=\left|e^{-\tilde{\sigma} t} f(t)\right| \geq 0
$$

Since $\sigma-\tilde{\sigma}<0$, the limit as $t \rightarrow \infty$ of the left hand side is 0 , as is the limit of the right hand side. So we have by the squeeze principle we have

$$
\lim _{t \rightarrow \infty}\left|e^{-\tilde{\sigma} t} f(t)\right|=0
$$



Figure 2: Self convolutions of $f_{2}(t)=e^{-t} \mathbb{1}(t)$

## Exercise 6

Let $p(t), q(t)$ be polynomials. Then there are a finite number of poles of the equation $\frac{p(t)}{q(t)}$. Thus there is some number $T$ such that for all $t>T, q(t) \neq 0$. Take $\tilde{\sigma}=1$. Now, if the order of $q$ is more than the order of $p$, then $\lim _{t \rightarrow \infty} \frac{p(t)}{q(t)}=0$, thus

$$
\lim _{t \rightarrow \infty}\left|e^{-t} \frac{p(t)}{q(t)}\right|=\lim _{t \rightarrow \infty} e^{-t} \lim _{t \rightarrow \infty} \frac{p(t)}{q(t)}=0
$$

If the order of $q$ is the same as the order of $p$, then $\lim _{t \rightarrow \infty} \frac{p(t)}{q(t)}=M$, for some $M \in \mathbb{R}$. Thus

$$
\lim _{t \rightarrow \infty}\left|e^{-t} \frac{p(t)}{q(t)}\right|=\lim _{t \rightarrow \infty} e^{-t}\left|\lim _{t \rightarrow \infty} \frac{p(t)}{q(t)}\right|=0 \times|M|=0
$$

Finally, if the order of $q$ is less than the order of $p$, then applying l'Hop ital's rule, we have

$$
\lim _{t \rightarrow \infty}\left|e^{-t} \frac{p(t)}{q(t)}\right|=\lim _{t \rightarrow \infty}\left|\frac{p^{\prime}(t)}{e^{t}(q(t)+q(t))}\right|=\lim _{t \rightarrow \infty}\left|e^{-t} \frac{p_{2}(t)}{q_{2}(t)}\right|
$$

where $q_{2}(t)$ is of the same order as $q(t)$ and $p_{2}(t)$ is of order one less. If the order of $p_{2}$ is the same as the order of $q_{2}$, then the previous case can be applied, otherwise the order of $p_{2}$ is
still greater than the order of $q_{2}$, in which case l'Hop ital's rule can be applied again. Since the order of a polynomial must be finite, eventually $p_{n}$ will have the same order as $q_{n}$, and then

$$
\lim _{t \rightarrow \infty}\left|e^{-t} \frac{p(t)}{q(t)}\right|=\lim _{t \rightarrow \infty}\left|e^{-t} \frac{p_{n}(t)}{q_{n}(t)}\right|=\lim _{t \rightarrow \infty} e^{-t}\left|\lim _{t \rightarrow \infty} \frac{p_{n}(t)}{q_{n}(t)}\right|=0 \times|M|=0 .
$$

## Exercise 7

$b(t)$ is a polynomial, so it has a finite number of roots. Thus there is a largest root. Let this root be $t_{1}$ and choose $T>t_{1}$. As the order of $b$ is more than the order of $a$, then $\lim _{t \rightarrow \infty} \frac{a(t)}{b(t)}=0$. This, combined with the fact that for $t>T \frac{a(t)}{b(t)}$ is a continuous function means that $\sup _{t>T} \frac{a(t)}{b(t)}<\infty$. Thus, there exists an $M>0$ such that for all $t>T,\left|\frac{a(t)}{b(t)}\right|<$ $M=M e^{0 t}$. So the infimum over possible $\sigma$ is less than or equal to 0 .

Now assume that the infimum is not 0 . Then there exists $\sigma>0$ such that for some $M>0$ and for all $t>T$ for some $T \in \mathbb{R}$ we have

$$
\left|\frac{a(t)}{b(t)}\right|<M e^{-\sigma t}
$$

i.e. the abscissa of convergence is $-\sigma<0$. But $e^{-\sigma t} \rightarrow 0$ faster than a polynomial fraction goes to 0 as $t \rightarrow \infty$, for any $\sigma>0$. So we have a contradiction, which means the infimum is greater than or equal to 0 . Combined with the above reasoning, the infimum over possible $\sigma$ must be 0 , so the abscissa of convergence is 0 .

## Exercise 8

Consider the curve in the complex plain defined by $C=\{\sigma+b i:-M \leq b \leq M\} \cup\{z$ : $|z-\sigma|=M\}=C_{1} \cup C_{2}$. So

$$
f(t)=\int_{C} \frac{e^{s t}}{(a+s)^{2}} \mathrm{~d} s=\int_{C} \frac{e^{s t}}{(a+s)^{2}} \mathrm{~d} s-\int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \frac{e^{M e^{i x} t}}{\left(a+M e^{i x}\right)^{2}} \mathrm{~d} x .
$$

Since $e^{s t}$ is an analytic function, we can use Cauchy's integral tehorem here. This theorem states that the complex integral of a closed curve is equal to the sum over the number of poles of the residues at each pole. For suffeciently large $M$, there is only one pole of the integrand (at $s=-a$ ) in $C$, and the residue at this pole is $\left.\frac{\mathrm{d}}{\mathrm{d} s}\left(e^{s t}\right)\right|_{-a}=t e^{-a t}$. So we have

$$
f(t)=t e^{-a t}-\int_{\frac{\pi}{2}}^{\frac{3 \pi}{2}} \frac{e^{M t \cos (x)} e^{i M t \sin (x)}}{\left(a+M e^{i x}\right)^{2}} \mathrm{~d} x
$$

Using fancy arguments from complex analysis we can show that teh second integral $\rightarrow 0$ as $M \rightarrow \infty$. Thus we have

$$
f(t)=t e^{-a t}
$$

## Exercise 9

$$
\begin{gathered}
\mathcal{L}\left(e^{\alpha t}\right)=\lim _{T \rightarrow \infty} \int_{0}^{T} e^{-s t} e^{\alpha t} \mathrm{~d} t=\lim _{T \rightarrow \infty}\left[\frac{1}{\alpha-s} e^{\alpha t-s t}\right]_{0}^{T}=\lim _{T \rightarrow \infty} \frac{1}{\alpha-s}\left[e^{\alpha T-s T}-1\right] \\
=\frac{1}{s-\alpha}\left[1-\lim _{T \rightarrow \infty} e^{(a-\sigma) T}(\cos ((b+\omega) T)+i \sin ((b+\omega) T))\right]
\end{gathered}
$$

where $\alpha=a+b i$. This converges if $a-\sigma<0$, i.e. $\Re(\alpha)<\Re(s)$. If this is true, then the transform converges to

$$
\mathcal{L}\left(e^{\alpha t}\right)=\frac{1}{s-\alpha} .
$$

## Exercise 10

Note that $\cos (b t)=\frac{1}{2}\left(e^{i b t}+e^{-i b t}\right)$, so $e^{-a t} \cos (b t)=\frac{1}{2} e^{(-a+i b) t}+\frac{1}{2} e^{(-a-i b) t}$. Since Laplace transforms are linear, we can just apply the result of Exercise 9 twice with $\alpha=-a+b i$ and $\alpha=-a-b i$ to yield

$$
\mathcal{L}\left(e^{-a t} \cos (b t)\right)=\frac{1}{2} \mathcal{L}\left(e^{(-a+b i) t}\right)+\frac{1}{2} \mathcal{L}\left(e^{(-a-b i) t}\right)=\frac{1}{2 s+2 a-2 b i}+\frac{1}{2 s+2 a+b i}
$$

with a region of convergence that satisfies both $\Re(-a+b i)<\Re(s)$ and $\Re(-a-b i)<\Re(s)$, i.e $\Re(s)>\max \{\Re(-a-b i), \Re(-a+b i)\}$. The transform can be simplified to give

$$
\mathcal{L}\left(e^{-a t} \cos (b t)\right)=\frac{s+a+b i}{2 s^{2}+4 s a-2 a^{2}+2 b^{2}}+\frac{s+a-b i}{2 s^{2}+4 s a+2 a^{2}+2 b^{2}}=\frac{s+a}{(s+a)^{2}+b^{2}} .
$$

## Exercise 11

Let $X(s)=\mathcal{L}(x(t))$ and note that $\mathcal{L}\left(\cos \left(\frac{t}{2}\right)\right)=\frac{4 s}{4 s^{2}+1}$. Transforming the equation gives

$$
-\dot{x}(0)-s x(0)+s^{2} X(s)+6 X(s)=\frac{4 s}{4 s^{2}+1}
$$

Subbing in the initial conditions gives

$$
0-0+\left(s^{2}+6\right) X(s)=\frac{4 s}{4 s^{2}+1}
$$

and rearranging for $X(s)$,
$X(s)=\frac{4 s}{\left(4 s^{2}+1\right)\left(s^{2}+6\right)}=\frac{A s+B}{4 s^{2}+1}+\frac{C s+D}{s^{2}+6}=\frac{A s^{3}+6 A s+B s^{2}+6 B+4 C s^{3}+C s+4 D s^{2}+D}{\left(4 s^{2}+1\right)\left(s^{2}+6\right)}$
where partial fractions are being used. Equating coefficients of $s^{0}$ and $s^{2}$ gives $D=-6 B$ and $B=-4 D$, so $B=D=0$. Equating coefficients of $s^{3}$ gives $A=-4 C$ so coefficients of $s^{1}$ gives the equation $4=-24 C+C=-23 C$. So we have

$$
X(s)=\frac{16 s}{23\left(4 s^{2}+1\right)}-\frac{4 s}{23\left(s^{2}+6\right)}
$$

Transforming back (using the table of transforms) gives

$$
x(t)=\frac{4}{23} \cos \left(\frac{t}{2}\right)-\frac{4}{23} \cos (\sqrt{6} t)
$$

## Exercise 12

Assume $f, g$ are of exponential order, and that they are both only defined for $t \geq 0$, and 0 otherwise. Then

$$
\begin{gathered}
\mathcal{L}(f(t) * g(t))=\int_{0}^{\infty} e^{-s t} f(t) * g(t) \mathrm{d} t=\int_{0}^{\infty} e^{-s t} \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \mathrm{d} \tau \mathrm{~d} t \\
=\int_{0}^{\infty} e^{-s t} \int_{0}^{t} f(\tau) g(t-\tau) \mathrm{d} \tau \mathrm{~d} t
\end{gathered}
$$

since the integrand is zero when $\tau<0$ and when $\tau>t$. The integration is over the set $\{(t, \tau): 0<t<\infty, 0<\tau<t\}$, which is the same set as $\{(t, \tau): 0<\tau<\infty, \tau<t<\infty\}$. Assuming these functions are nice enough, we can change the order of integration by Fubini's theorem. This gives

$$
\int_{0}^{\infty} \int_{\tau}^{\infty} e^{-s t} f(\tau) g(t-\tau) \mathrm{d} t \mathrm{~d} \tau
$$

Changing variables by $r=t-\tau$ with $\mathrm{d} r=\mathrm{d} t$ and $t=-\tau \Rightarrow r=0$ gives

$$
\begin{gathered}
\int_{0}^{\infty} \int_{0}^{\infty} e^{-s(r+\tau)} f(\tau) g(r) \mathrm{d} \tau \mathrm{~d} t=\int_{0}^{\infty} \int_{0}^{\infty} e^{-s \tau} f(\tau) e^{-s r} g(r) \mathrm{d} r \mathrm{~d} \tau \\
=\int_{0}^{\infty} e^{-s r} g(r) \mathrm{d} r \int_{0}^{\infty} e^{-s \tau} f(\tau) \mathrm{d} \tau=\mathcal{L}(f(t)) \mathcal{L}(g(t))
\end{gathered}
$$

Now, if we can't assume that $f, g$ are 0 for $t<0$, we have to take the Bilateral Laplace transform:

$$
\begin{gathered}
\mathcal{L}_{B}(f(t) * g(t))=\int_{-\infty}^{\infty} e^{-s t} f(t) * g(t) \mathrm{d} t=\int_{-\infty}^{\infty} e^{-s t} \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \mathrm{d} \tau \mathrm{~d} t \\
=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-s t} f(\tau) g(t-\tau) \mathrm{d} t \mathrm{~d} \tau
\end{gathered}
$$

by Fubini's theorem. Changing variables by $r=t-\tau$ gives $\mathrm{d} r=\mathrm{d} t$ and

$$
\begin{gathered}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-s(r+\tau)} f(\tau) g(r) \mathrm{d} r \mathrm{~d} \tau=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-s \tau} f(\tau) e^{-s r} g(r) \mathrm{d} r \mathrm{~d} \tau \\
=\int_{-\infty}^{\infty} e^{-s r} g(r) \mathrm{d} r \int_{-\infty}^{\infty} e^{-s \tau} f(\tau) \mathrm{d} \tau=\mathcal{L}_{B}(f(t)) \mathcal{L}_{B}(g(t))
\end{gathered}
$$

## Exercise 13

$$
\begin{aligned}
& \begin{array}{lll}
-s^{2} & & -1 \\
\hline & -s & +3
\end{array}
\end{aligned}
$$

## Exercise 14

From example:

$$
\frac{s-1}{(s+1)^{2}(s-2)}=\frac{A}{s+1}+\frac{B}{(s+1)^{2}}+\frac{C}{s-2}=\frac{A\left(s^{2}-s-2\right)+B(s-2)+C\left(s^{2}+2 s+1\right)}{(s+1)^{2}(s-2)} .
$$

Equating numerators gives

$$
s-1=(A+C) s^{2}+(B-A+2 C) s+(-2 A-2 B+C)
$$

Equating co-efficients of $s^{2}$ gives $A=-C$, so we get the pair of equations $1=B-A+2 C=$ $B+3 C$ and $-1=-2 A-2 B+C=-2 B+3 C$. Rearranging for $3 C$ gives $1-B=2 B-1$ which implies $B=\frac{2}{3}, C=\frac{1}{9}$ and $A=\frac{-1}{9}$. Thus,

$$
\frac{s-1}{(s+1)^{2}(s-2)}=\frac{-1}{9(s+1)}+\frac{2}{3(s+1)^{2}}+\frac{1}{9(s-2)} .
$$

## Exercise 15

First note that $s^{2}+2 s+5$ has no real roots $\left(2^{2}-4 \cdot 1 \cdot 5=-16<0\right)$.

$$
\frac{s+3}{\left(s^{2}+2 s+5\right)(s+1)}=\frac{A s+B}{s^{2}+2 s+5}+\frac{C}{s+1}=\frac{A s^{2}+A s+B s+B+C s^{2}+2 C s+5 C}{\left(s^{2}+2 s+5\right)(s+1)}
$$

Equating co-efficients of $s^{2}$ gives $A=-C$ so equating the other co-efficients gives $1=B-A$ and $3=B-5 A$. Thus $A=\frac{-1}{2}, B=\frac{1}{2}$ and $C=\frac{1}{2}$. So we have

$$
\frac{s+3}{\left(s^{2}+2 s+5\right)(s+1)}=\frac{-s+1}{2\left(s^{2}+2 s+5\right)}+\frac{1}{2(s+1)} .
$$

## Exercise 16

Note that $\sin (t)=\frac{e^{i t}-e^{-i t}}{2 i}$. So we have

$$
\hat{f}(\omega)=\int_{-\infty}^{\infty} \frac{e^{i t}-e^{-i t}}{2 i t} e^{i \omega t} \mathrm{~d} t=\int_{-\infty}^{\infty} \frac{e^{(\omega+1) i t}-e^{(\omega-1) i t}}{2 i t} \mathrm{~d} t
$$

The integral of $\frac{e^{-t}}{t}$ is not an analytic function. This question could be continued using the special function $E i(t)$, which is the exponential integral function, however, a different method will be used to do this question. Sine, when broken into exponentials, looks a bit like the result of the fundamental theorem of calculus. Define $F(x)=e^{x t}$. Then we have

$$
\begin{gathered}
\hat{f}(\omega)=\int_{-\infty}^{\infty} \frac{e^{i t}-e^{-i t}}{2 i t} e^{i \omega t} \mathrm{~d} t=\int_{-\infty}^{\infty} \frac{e^{i \omega t}}{2 i t}(F(i)-F(-i)) \mathrm{d} t \\
=\int_{-\infty}^{\infty} \frac{e^{i \omega t}}{2 i t} \int_{-i}^{i} F^{\prime}(x) \mathrm{d} x \mathrm{~d} t=\int_{-\infty}^{\infty} \frac{e^{i \omega t}}{2 i t} \int_{-i}^{i} t e^{x t} \mathrm{~d} x \mathrm{~d} t \\
=\int_{-\infty}^{\infty} \frac{e^{i \omega t} t}{2 i t} \int_{-\infty i}^{\infty i} \chi_{[-1,1]}(x) e^{x t} \mathrm{~d} x \mathrm{~d} t
\end{gathered}
$$

using $x$ as a dummy variable. Switching variables to $x=s i, \mathrm{~d} x=i \mathrm{~d} x$ gives

$$
\begin{aligned}
\hat{f}(\omega) & =\int_{-\infty}^{\infty} \frac{e^{i \omega t}}{2 i} \int_{-\infty}^{\infty} \chi_{[-1,1]}(s i) e^{i s t} \mathrm{~d} x \mathrm{~d} t=\int_{-\infty}^{\infty} e^{i \omega t} \frac{1}{2 \pi} \int_{-\infty}^{\infty} \pi \chi_{[-1,1]}(i s) e^{i s t} \mathrm{~d} x \mathrm{~d} t \\
& =\int_{-\infty}^{\infty} e^{i \omega t} \mathcal{F}^{-1}\left(\pi \chi_{[-1,1]}(t)\right) \mathrm{d} t=\mathcal{F}\left(\mathcal{F}^{-1}\left(\pi \chi_{[-1,1]}(t)\right)\right)=\pi \chi_{[-1,1]}(\omega) .
\end{aligned}
$$

So this is the Laplace transform.

## Exercise 17

Using Matlab, the commands bode(tf(1,[1, 1,2$]))$ and nyquist $(t f(1,[1,1,2]))$ create the Bode (figure 3) and Nyquist (figure 4) plots for the system

$$
H(s)=\frac{1}{s^{2}+s+2} .
$$



Figure 3: Bode Plot for the transfer function $H(s)=\frac{1}{s^{2}+s+2}$.

## Exercise 18

Two generalised functions $\eta_{1}, \eta_{2}$ are are equal if

$$
\int_{-\infty}^{\infty} \eta_{1}(t) \phi(t) \mathrm{d} t=\int_{-\infty}^{\infty} \eta_{2}(t) \phi(t) \mathrm{d} t
$$

for $\phi \in C_{0}^{\infty}(\mathbb{R})$. Consider

$$
\begin{gathered}
\int_{-\infty}^{\infty}\left(\alpha_{1} \delta(t)+\alpha_{2} \delta(t)\right) \phi(t) \mathrm{d} t=\int_{-\infty}^{\infty} \alpha_{1} \delta(t) \phi(t)+\alpha_{2} \delta(t) \phi(t) \mathrm{d} t=\int_{-\infty}^{\infty} \alpha_{1} \delta(t) \phi(t) \mathrm{d} t+\int_{-\infty}^{\infty} \alpha_{2} \delta(t) \phi(t) \mathrm{d} t \\
=\alpha_{1} \phi(0)+\alpha_{2} \phi(0)=\left(\alpha_{1}+\alpha_{2}\right) \phi(0)=\int_{-\infty}^{\infty}\left(\alpha_{1}+\alpha_{2}\right) \delta(t) \phi(t) \mathrm{d} t .
\end{gathered}
$$

Thus

$$
\alpha_{1} \delta(t)+\alpha_{2} \delta(t)=\left(\alpha_{1}+\alpha_{2}\right) \delta(t)
$$



Figure 4: Nyquist Plot for the transfer function $H(s)=\frac{1}{s^{2}+s+2}$.

## Exercise 19

Again, two generalised functions $\eta_{1}, \eta_{2}$ are are equal if

$$
\int_{-\infty}^{\infty} \eta_{1}(t) \phi(t) \mathrm{d} t=\int_{-\infty}^{\infty} \eta_{2}(t) \phi(t) \mathrm{d} t
$$

for $\phi \in C_{0}^{\infty}(\mathbb{R})$. Consider

$$
\int_{-\infty}^{\infty} f(t) \delta(t-\tau) \phi(t) \mathrm{d} t=\int_{-\infty}^{\infty} f(s+\tau) \delta(s) \phi(s+\tau) \mathrm{d} s=f(\tau) \phi(\tau)
$$

by a change of variables $s=t-\tau$ and the definition of $\delta(t)$. Now consider
$\int_{-\infty}^{\infty} f(\tau) \delta(t-\tau) \phi(t) \mathrm{d} t=\int_{-\infty}^{\infty} f(\tau) \delta(s) \phi(s+\tau) \mathrm{d} s=f(\tau) \phi(\tau)=\int_{-\infty}^{\infty} f(t) \delta(t-\tau) \phi(t) \mathrm{d} t$.
Thus

$$
f(t) \delta(t-\tau)=f(\tau) \delta(t-\tau)
$$

## Exercise 20

We may assume

$$
\int_{0}^{\infty} \phi(t) \mathrm{d} t=\int_{-\infty}^{\infty} \mathbb{1}(t) \phi(t) \mathrm{d} t
$$

for all integrable $\phi(t)$. So consider integrable $\phi(t)$ and define

$$
\psi(t)=\left\{\begin{array}{cc}
\phi(t) & t<0 \\
0 & t \geq 0
\end{array}\right.
$$

Since $\phi$ and 0 are integrable, $\psi$ is integrable also. Then we have

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \mathbb{1}(t) \psi(t) \mathrm{d} t=\int_{0}^{\infty} \psi(t) \mathrm{d} t \\
& \int_{-\infty}^{0} \mathbb{1}(t) \phi(t) \mathrm{d} t=\int_{0}^{\infty} 0 \mathrm{~d} t=0
\end{aligned}
$$

But $\phi$ is arbitrary. So $\mathbb{1}(t)=0$ for all $t<0$.
Now consider integrable $\phi(t)$ and define

$$
\psi(t)=\left\{\begin{array}{cc}
0 & t<0 \\
\phi(t) & t \geq 0
\end{array}\right.
$$

Again, $\phi$ and 0 are integrable, so $\psi$ is integrable also. Then we have

$$
\begin{gathered}
\int_{-\infty}^{\infty} \mathbb{1}(t) \psi(t) \mathrm{d} t=\int_{0}^{\infty} \psi(t) \mathrm{d} t \\
\int_{0}^{\infty} \mathbb{1}(t) \phi(t) \mathrm{d} t=\int_{0}^{\infty} \phi(t) \mathrm{d} t \\
\int_{0}^{\infty}(\mathbb{1}(t)-1) \phi(t) \mathrm{d} t=0 .
\end{gathered}
$$

Again, $\phi$ is arbitrary. So $\mathbb{1}(t)-1=0$ for all $t \geq 0$. Thus, we must have

$$
\mathbb{1}(t)= \begin{cases}0 & t<0 \\ 1 & t \geq 0 .\end{cases}
$$

## Exercise 21

Fix $\theta \in \mathbb{R}$. Changing variables and integration by parts gives

$$
\int_{-\infty}^{\infty} \mathbb{1}^{\prime}(t-\theta) \phi(t) \mathrm{d} t=\int_{-\infty}^{\infty} \mathbb{1}(t)^{\prime} \phi(t-\theta) \mathrm{d} t=-\int_{-\infty}^{\infty} \mathbb{1}(t) \phi^{\prime}(t+\theta) \mathrm{d} t
$$

applying the definition of the step function and remembering that $\phi(t)$ has compact support (so $\phi(\infty)=0$ ), we get
$=-\int_{0}^{\infty} \phi^{\prime}(t-\theta) \mathrm{d} t=-(\phi(\theta)-\phi(\theta+\infty))=\phi(\theta+0)=\int_{-\infty}^{\infty} \delta(t) \phi(\theta+t) \mathrm{d} t=\int_{-\infty}^{\infty} \delta(t-\theta) \phi(t) \mathrm{d} t$ after applying the definition of $\phi(t)$ again and performing another change of variables. Since

$$
\int_{-\infty}^{\infty} \mathbb{1}^{\prime}(t-\theta) \phi(t) \mathrm{d} t=\int_{-\infty}^{\infty} \delta(t-\theta) \phi(t) \mathrm{d} t
$$

for arbitrary integrable $\phi(t)$ with compact support, we have

$$
\mathbb{1}^{\prime}(t-\theta)=\delta(t-\theta) .
$$

## Exercise 22

The linearity property holds for

$$
\sum_{i=1}^{2} \alpha_{i} u_{i}(t)
$$

so assume it holds for $N$, i.e.

$$
\mathcal{O}\left(\sum_{i=1}^{N} \alpha_{i} u_{i}(t)\right)=\sum_{i=1}^{N} \alpha_{i} \mathcal{O}\left(u_{i}(t)\right) .
$$

Consider

$$
\sum_{i=1}^{N+1} \alpha_{i} u_{i}(t)
$$

and define

$$
\sum_{i=1}^{N} \alpha_{i} u_{i}(t)=v(t)
$$

Then

$$
\sum_{i=1}^{N+1} \alpha_{i} u_{i}(t)=v(t)+\alpha_{n+1} u_{n+1}(t) .
$$

So we have

$$
\begin{gathered}
\mathcal{O}\left(\sum_{i=1}^{N+1} \alpha_{i} u_{i}(t)\right)=\mathcal{O}\left(v(t)+\alpha_{n+1} u_{n+1}(t)\right)=\mathcal{O}(v(t))+\mathcal{O}\left(\alpha_{n+1} u_{n+1}(t)\right) \\
=\left(\sum_{i=1}^{N} \alpha_{i} u_{i}(t)\right)+\mathcal{O}\left(\alpha_{n+1} u_{n+1}(t)\right)=\sum_{i=1}^{N} \alpha_{i} \mathcal{O}\left(u_{i}(t)\right)+\mathcal{O}\left(\alpha_{n+1} u_{n+1}(t)\right)=\sum_{i=1}^{N} \alpha_{i} \mathcal{O}\left(u_{i}(t)\right) .
\end{gathered}
$$

Thus the proposition is true by induction.

## Exercise 23

Memory-less: In this case $y(n)$ can only depend on the current value of $u$ i.e $u(n)$, so $m$ can only be 0 , so for this system to be memoryless, $M=N=0$. The other parameters are free $\alpha, \beta \in \mathbb{R}$.

Causal: In this case $y(n)$ can only depend on the current and previous values of $u$ i.e $u(m), \forall m \leq n$. So, $m$ cannot be greater than 0 for this system, which means $N \leq 0$. The other parameters are free $\alpha, \beta \in \mathbb{R}$ and $M \in \mathbb{N}$. To prove this is causal, fix $k \in \mathbb{Z}$ and let $N \leq 0$ and let $u_{1}$ and $u_{2}$ be two inputs such that $u_{1}(n)=u_{2}(n)$ for all $n \leq k$. Then
$y_{1}(n)=\frac{1}{M+N+1} \sum_{m=-M}^{N}\left(u_{1}(n+m)\right)^{\alpha+\beta \cos (n)}=\frac{1}{M+N+1} \sum_{m=-M}^{N}\left(u_{2}(n+m)\right)^{\alpha+\beta \cos (n)}=y_{2}(n)$.
So the system is casual.
Linear: Let $u(n)=\alpha_{1} u_{1}(n)+\alpha_{2} u_{2}(n)$. Then,

$$
y(n)=\frac{1}{M+N+1} \sum_{m=-M}^{N}(u(n+m))^{\alpha+\beta \cos (n)}=\frac{1}{M+N+1} \sum_{m=-M}^{N}\left(\alpha_{1} u_{1}(n)+\alpha_{2} u_{2}(n)\right)^{\alpha+\beta \cos (n)} .
$$

This will be linear iff $\alpha=1, \beta=0$. If this is the case, we have

$$
\begin{gathered}
y(n)=\frac{1}{M+N+1} \sum_{m=-M}^{N} \alpha_{1} u_{1}(n)+\alpha_{2} u_{2}(n)=\frac{\alpha_{1}}{M+N+1} \sum_{m=-M}^{N} u_{1}(n)+\frac{\alpha_{2}}{M+N+1} \sum_{m=-M}^{N} u_{1}(n) \\
=\alpha_{1} y_{1}(n)+\alpha_{2} y_{2}(n) .
\end{gathered}
$$

Any other value of $\alpha$ or $\beta$ will result in a $u_{1}(n) u_{2}(n)$ term in the sum, which won't be 0 for general $u_{1}, u_{2}$, so the result wouldn't be linear. $M$ and $N$ can take any values in $\mathbb{Z}$.

Time-invariant: Fix some $k \in \mathbb{Z}$.

$$
\mathcal{O}\left(u(n-k)=\frac{1}{M+N+1} \sum_{m=-M}^{N}(u((n-k)+m))^{\alpha+\beta \cos (n)}\right.
$$

but

$$
y(n-k)=\frac{1}{M+N+1} \sum_{m=-M}^{N}(u(n-k+m))^{\alpha+\beta \cos (n-k)} .
$$

So the system is time invariant (i.e. $\mathcal{O}(u(n-k)=y(n-k))$ iff $\beta \cos (n-k)=\beta \cos (n) . k$ was arbitrary, so this must be true for all $k \in \mathbb{Z}$. Thus $\beta=0$. The other parameters are free, $M, N \in \mathbb{N}$ and $\alpha \in \mathbb{R}$.

## Exercise 24

Assume the LTI system $y(t)=\mathcal{O}(u(t))$ is memoryless. Then $y(t)=g(u(t))$ for some scalar function $g$. But the system is linear, so for input $u_{1}$ and $u_{2} g\left(u_{1}(t)+u_{2}(t)\right)=g\left(u_{1}(t)\right)+$ $g\left(u_{2}(t)\right)$. This implies the system has the form $y(t)=K u(t)$ for some $A \in \mathbb{R}$. Then $h(t)=\mathcal{O}(\delta(t))=K \delta(t)$.

Now assume $h(t)=K \delta(t)$ for some LTI system. So $y(t)=(u * h)(t)=(u * K \delta)(t)=$ $K(u * \delta)(t)=K u(t)=g(u(t))$ for a scalar function $g(\cdot)=K \times \cdot$ Thus the system is memory-less. Thus an LTI system is memory-less iff $h(t)=K \delta(t)$.

## Exercise 25

Assume the LTI system $y(t)=\mathcal{O}(u(t))$ is causal. Then $\forall t_{0} \in \mathbb{R}$ and $u_{1}$ and $u_{2}$ such that $u_{1}(t)=u_{2}(t)$ for all $t \leq t_{0}, y_{1}(t)=y_{2}(t)$ for all $t \leq t_{0}$. The system is linear, so for input $u_{1}(t) \equiv 0$ and arbitrary $u_{2}(t)$, we have $y_{1}(t)+y_{2}(t)=\mathcal{O}\left(u_{1}(t)+u_{2}(t)\right)=\mathcal{O}\left(0+u_{2}(t)\right)=y_{2}(t)$, thus $y_{1}(t) \equiv 0$. Now, as the system is causal, choose $t_{0}=0$. For all $t<0 u_{1}(t)=\delta(t)$, so $h(t)=h(t)+y_{1}(t)=\mathcal{O}(\delta(t))+\mathcal{O}\left(u_{1}(t)\right)=0$.

Now assume for all $t<0, h(t)=0$ for an LTI system. Let $t_{0} \in \mathbb{R}$ be arbitrary and consider $u_{1}$ and $u_{2}$ such that $u_{1}(t)=u_{2}(t)$ for all $t \leq t_{0}$. So

$$
y_{i}(t)=\left(h * u_{i}\right)(t)=\int_{-\infty}^{\infty} u_{i}(\tau) h(t-\tau) \mathrm{d} \tau .
$$

But $h(t)=0$ for all $t<0$, so for all $\tau>t$ the integrand becomes 0 . So we have

$$
y_{i}(t)=\int_{-\infty}^{t} u_{i}(\tau) h(t-\tau) \mathrm{d} \tau .
$$

If we fix $T \leq t_{0}$, then

$$
y_{i}(T)=\int_{-\infty}^{T} u_{i}(\tau) h(T-\tau) \mathrm{d} \tau
$$

but $u_{1}(t)=u_{2}(t)$ for all $t \leq T$. Thus $y_{1}(T)=y_{2}(T)$ for all $t \leq T$, and since $T$ and $t_{0}$ were arbitrary, this system is causal.

## Exercise 26

Assume this system is BIBO. Then $\|h(t)\|_{1}<\infty$. If $u(t)=0$, then $\|u\|_{\infty}=0$ and $y(t)=$ $h(t) * 0=0$, so $\|y\|_{\infty}=0$. Thus $\|y\|_{\infty}=0=0\|h(t)\|_{1}=\|h(t)\|_{1}\|u\|_{\infty}$ and equality is achieved.

## Exercise 27

Assume this system is BIBO. Then $\|h(t)\|_{1}<\infty$. Then

$$
\begin{aligned}
|y(t)|=\left|\int_{-\infty}^{\infty} h(t-\tau) u(\tau) \mathrm{d} \tau\right| & \leq \int_{-\infty}^{\infty}|h(t-\tau) u(\tau)| \mathrm{d} \tau \leq \int_{-\infty}^{\infty}|h(t-\tau)|\|u\|_{\infty} \mathrm{d} \tau \\
& =\|u\|_{\infty}\|h(t-\tau)\|_{1} .
\end{aligned}
$$

Again, to prove this is a necessary condition, use $u(t)=\operatorname{sgn}(h(-t))$. Then

$$
y(0)=\int_{-\infty}^{\infty} h(0-\tau) \operatorname{sgn}(h(-\tau)) \mathrm{d} \tau=\int_{-\infty}^{\infty}|h(-\tau)| \mathrm{d} \tau=\|h(t-\tau)\|_{1},
$$

thus if $\|h(t)\|_{1} \nless \infty$, then $y$ is not necessarily bounded. So $\|h(t-\tau)\|_{1}<\infty$ is a necessary condition.

## Exercise 28

The proof in Exercise 27 still holds for complex valued functions, other than the proof $\|h(t)\|_{1}<\infty$ is a necessary condition.

To prove this is a necessary condition, use $u(t)=\frac{h(-t)}{\mid h(-t)}$. Then

$$
y(0)=\int_{-\infty}^{\infty} h(0-\tau) \frac{h(-t)}{\mid h(-t)} \mathrm{d} \tau=\int_{-\infty}^{\infty} \frac{|h(-t)|^{2}}{\mid h(-t)} \mathrm{d} \tau=\int_{-\infty}^{\infty}|h(-\tau)| \mathrm{d} \tau=\|h(t)\|_{1},
$$

thus if $\|h(t)\|_{1} \nless \infty$, then $y$ is not necessarily bounded. So $\|h(t-\tau)\|_{1}<\infty$ is a necessary condition.

## Exercise 29

Choose $H(s)=\frac{1}{(s+1)\left(s^{2}+2 s+2\right)}$, then

$$
Y(s)=\frac{\omega_{0}}{\left(s^{2}+\omega_{0}^{2}\right)(s+1)\left(s^{2}+2 s+2\right)}=\frac{A}{(s+1)}+\frac{B s+C}{\left(s^{2}+2 s+2\right)}+\frac{D s+E}{\left(s^{2}+\omega_{0}^{2}\right)}
$$

$$
\begin{gathered}
=\frac{D\left(s^{4}+3 s^{3}+4 s^{2}+2 s\right)+E\left(s^{3}+3 s^{2}+4 s+2\right)+A\left(s^{4}+2 s^{3}+2 s^{2}+\omega_{0}^{2} s^{2}+2 \omega_{0}^{2} s+2 \omega_{0}^{2}\right)}{\left(s^{2}+\omega_{0}^{2}\right)(s+1)\left(s^{2}+2 s+2\right)} \\
+\frac{B\left(s^{4}+s^{3}+\omega_{0}^{2} s^{2}+\omega_{0}^{2} s\right)+C\left(s^{3}+s^{2}+\omega_{0}^{2} s+\omega_{0}^{2}\right)}{\left(s^{2}+\omega_{0}^{2}\right)(s+1)\left(s^{2}+2 s+2\right)}
\end{gathered}
$$

Equating co-efficients gives the equations

$$
\begin{gathered}
D=-A-B \\
0=3 D+E+2 A+B+C \\
0=4 D+3 E+\left(2+\omega_{0}^{2}\right) A+\omega_{0}^{2} B+C \\
0=2 D+4 E+2 \omega_{0}^{2} A+B \omega_{0}^{2}+C \omega_{0}^{2} \\
\omega_{0}=2 E+2 A \omega_{0}^{2}+\omega_{0}^{2} C
\end{gathered}
$$

Subbing in the first equation gives

$$
\begin{gathered}
A=-2 B+E+C \\
0=-4 B+3 E+\left(-2+\omega_{0}^{2}\right) A+\omega_{0}^{2} B+C \\
0=-2 A-2 B+4 E+2 \omega_{0}^{2} A+B \omega_{0}^{2}+C \omega_{0}^{2} \\
\omega_{0}=2 E+2 A \omega_{0}^{2}+\omega_{0}^{2} C
\end{gathered}
$$

Subbing in the first of these equations gives

$$
\begin{gathered}
0=E+\omega_{0}^{2}(E+C-B)-C \\
0=2 B-2 C+2 E-3 B \omega_{0}^{2}+2 E \omega_{0}^{2}+3 C \omega_{0}^{2} \\
\omega_{0}=2 E-4 B \omega_{0}^{2}+2 E \omega_{0}^{2}+3 C \omega_{0}^{2}
\end{gathered}
$$

Subbing in

$$
B=\frac{1}{\omega_{0}^{2}} E+E+C-\frac{1}{\omega_{0}^{2}} C
$$

gives

$$
\begin{gathered}
0=2 \frac{1}{\omega_{0}^{2}} E+E+3 C-2 \frac{1}{\omega_{0}^{2}} C-E \omega_{0}^{2} \\
\omega_{0}=-2 E-2 E \omega_{0}^{2}-C \omega_{0}^{2}+4 C
\end{gathered}
$$

Finally, subbing

$$
C=\frac{2 E+E \omega_{0}^{2}-E \omega_{0}^{4}}{-3 \omega_{0}^{2}+2}
$$

and re-arranging for $E$

$$
\begin{aligned}
\omega_{0} & =-2 E-2 E \omega_{0}^{2}-\frac{2 E \omega_{0}^{2}+E \omega_{0}^{4}-E \omega_{0}^{6}}{-3 \omega_{0}^{2}+2}+\frac{8 E+4 E \omega_{0}^{2}-4 E \omega_{0}^{4}}{-3 \omega_{0}^{2}+2} \\
& =E \frac{1}{\omega_{0}^{2}}\left(\frac{\left(-2-2 \omega_{0}^{2}\right)\left(-3 \omega_{0}^{2}+2\right)}{-3 \omega_{0}^{2}+2}-\frac{4 \omega_{0}^{2}-5 \omega_{0}^{4}+\omega_{0}^{6}+8}{-3 \omega_{0}^{2}+2}\right) \\
& =E \frac{1}{\omega_{0}^{2}}\left(\frac{4 \omega_{0}^{2}+\omega_{0}^{4}+\omega_{0}^{6}+4}{-3 \omega_{0}^{2}+2}\right) \\
E & =\frac{-3 \omega_{0}^{3}+2 \omega_{0}}{4 \omega_{0}^{2}+\omega_{0}^{4}+\omega_{0}^{6}+4}
\end{aligned}
$$

Substituting this back into the other variables (using Mathematica) gives

$$
\begin{gathered}
C=\frac{-\omega_{0}^{3}+2 \omega_{0}}{\omega_{0}^{4}+4} \\
B=\frac{-\omega_{0}^{3}}{\omega_{0}^{4}+4} \\
A=\frac{\omega_{0}}{\omega_{0}^{2}+1}
\end{gathered}
$$

and

$$
D=\frac{\omega_{0}^{3}-4 \omega_{0}}{4 \omega_{0}^{2}+\omega_{0}^{4}+\omega_{0}^{6}+4} .
$$

So we have

$$
Y(s)=\frac{\omega_{0}}{\left(\omega_{0}^{2}+1\right)(s+1)}+\frac{-\omega_{0}^{3} s-\omega_{0}^{3}+2 \omega_{0}}{\left(\omega_{0}^{4}+4\right)\left(s^{2}+2 s+2\right)}+\frac{\left(\omega_{0}^{3}-4 \omega_{0}\right) s-3 \omega_{0}^{3}+2 \omega_{0}}{\left(4 \omega_{0}^{2}+\omega_{0}^{4}+\omega_{0}^{6}+4\right)\left(s^{2}+\omega_{0}^{2}\right)}
$$

Finally, noting that

$$
\frac{a+b i}{s+i \omega_{0}}+\frac{a-b i}{s-i \omega_{0}}=\frac{a s-a \omega_{0} i+b i s+b \omega_{0}+a s+a \omega_{0} i-b s i+b \omega_{0}}{s+\omega_{0}^{2}}=\frac{2 a s+2 b \omega_{0}}{s^{2}+\omega_{0}^{2}},
$$

so

$$
\frac{D s+E}{s^{2}+\omega_{0}^{2}}=\frac{\omega_{0} D+i E}{2 \omega_{0}\left(s+i \omega_{0}\right)}+\frac{\omega_{0} D-i E}{2 \omega_{0}\left(s-i \omega_{0}\right)}
$$

and the expression becomes

$$
\begin{gathered}
Y(s)=\frac{\omega_{0}}{\left(\omega_{0}^{2}+1\right)(s+1)}+\frac{-\omega_{0}^{3} s-\omega_{0}^{3}+2 \omega_{0}}{\left(\omega_{0}^{4}+4\right)\left(s^{2}+2 s+2\right)}+\frac{\left(\omega_{0}^{4}-4 \omega_{0}^{2}-3 i \omega_{0}^{3}+2 i \omega_{0}\right.}{\left(4 \omega_{0}^{2}+\omega_{0}^{4}+\omega_{0}^{6}+4\right)\left(s+i \omega_{0}\right)} \\
+\frac{\omega_{0}^{4}-4 \omega_{0}^{2}+3 i \omega_{0}^{3}-2 i \omega_{0}}{2 \omega_{0}\left(4 \omega_{0}^{2}+\omega_{0}^{4}+\omega_{0}^{6}+4\right)\left(s-i \omega_{0}\right)}
\end{gathered}
$$

So from the question,

$$
\alpha_{0}=\omega_{0}^{4}-4 \omega_{0}^{2}-3 i \omega_{0}^{3}+2 i \omega_{0} .
$$

## Exercise 30

Transforming back (using the formula in the notes) gives
$y(t)=\frac{\omega_{0} e^{-t}}{\left(\omega_{0}^{2}+1\right)}+\frac{-\omega_{0}^{3}}{\omega_{0}^{4}+4} e^{-t} \cos t+\frac{2 \omega_{0}}{\omega_{0}^{4}+4} e^{-t} \sin t+2\left|\omega_{0}^{4}-4 \omega_{0}^{2}-3 i \omega_{0}^{3}+2 i \omega_{0}\right| \cos \left(\omega_{0} t+\phi\right)$
where

$$
\phi=\tan ^{-1}\left(\frac{\Im \alpha_{0}}{\Re \alpha_{0}}\right)=\tan ^{-1}\left(\frac{-3 \omega_{0}^{3}+2 \omega_{0}}{\omega_{0}^{4}-4 \omega_{0}^{2}}\right)=\tan ^{-1}\left(\frac{2-3 \omega_{0}^{2}}{\omega_{0}\left(\omega_{0}-2\right)\left(\omega_{0}+2\right)}\right)
$$



This plot shows that all but the sinusoidal term decays away almost immediately, for multiple frequencies. Furthermore, there is a trend that as the frequency increases, the amplitude of the graph increases. We can also observe from this graph that the phase shift of the response varies as with the frequency.

## Exercise 31



As observed above, the amplitude of the response increases as the frequency increases, which is what we observe in the top graph of this Bode plot. We also observed that the phase shift of the response depends on the frequency. The lower graph in this Bode plot makes this relationship clearer.

## Exercise 32

Omitted.

## Exercise 33

Let $\phi(t) \in C_{0}^{\infty}(\mathbb{R})$. Then

$$
\begin{gathered}
\int_{-\infty}^{\infty}\left(\frac{1}{\lambda} \dot{h}(t)+h(t)\right) \phi(t) \mathrm{d} t=\frac{-1}{\lambda} \int_{-\infty}^{\infty} h(t) \dot{\phi}(t) \mathrm{d} t+\int_{-\infty}^{\infty} h(t) \phi(t) \mathrm{d} t= \\
\frac{-1}{\lambda} \int_{-\infty}^{\infty} \lambda e^{-\lambda t} \mathbb{1}(t) \dot{\phi}(t) \mathrm{d} t+\int_{-\infty}^{\infty} \lambda e^{-\lambda t} \mathbb{1}(t) \phi(t) \mathrm{d} t=\int_{0}^{\infty}-e^{-\lambda t} \dot{\phi}(t) \mathrm{d} t+\int_{0}^{\infty} \lambda e^{-\lambda t} \phi(t) \mathrm{d} t .
\end{gathered}
$$

Integrating the first term by parts again gives
$=\left[-e^{-\lambda t} \phi(t)\right]_{0}^{\infty}+\int_{0}^{\infty} \lambda e^{-\lambda t} \phi(t) \mathrm{d} t+\int_{0}^{\infty} \lambda e^{-\lambda t} \phi(t) \mathrm{d} t=\left[-e^{-\lambda t} \phi(t)\right]_{0}^{\infty}=0--\phi(0)=\phi(0)$
since $\phi$ has compact support. By definition of the delta distribution, we have

$$
\int_{-\infty}^{\infty} \delta(t) \phi(t) \mathrm{d} t=\phi(0)=\int_{-\infty}^{\infty}\left(\frac{1}{\lambda} \dot{h}(t)+h(t)\right) \phi(t) \mathrm{d} t .
$$

Thus $\delta(t)=\frac{1}{\lambda} \dot{h}(t)+h(t)$ in the distributional sense.

## Exercise 34

Taking the Laplace transform gives:

$$
\begin{aligned}
\omega_{n}^{2}= & -\dot{h}(0)-s h(0)+s^{2} H(s)+2 \zeta \omega_{n}(-h(0)+s H(s))+\omega_{n}^{2} H(s) \\
& =H(s)\left(s^{2}+2 \zeta \omega_{n} s+\omega_{n}^{2}\right)-\dot{h}(0)-s h(0)-2 \zeta \omega_{n} h(0)
\end{aligned}
$$

which, after rearranging gives

$$
H(s)=\frac{\dot{\dot{h}}(0)+s h(0)+2 \zeta \omega_{n} h(0)+\omega_{n}^{2}}{s^{2}+2 \zeta \omega_{n} s+\omega_{n}^{2}} .
$$

If $\dot{h}(0)=h(0)=0$, then

$$
H(s)=\frac{\omega_{n}^{2}}{s^{2}+2 \zeta \omega_{n} s+\omega_{n}^{2}}
$$

## Exercise 35

Assuming $\zeta \neq 1$ we have $s^{2}+2 \zeta \omega_{n} s+\omega_{n}^{2}=\left(s-c_{1}\right)\left(s-c_{2}\right)$, where $c_{1}=-\zeta \omega_{n}+\omega_{n} \sqrt{\zeta^{2}-1}$ and $c_{1}=-\zeta \omega_{n}-\omega_{n} \sqrt{\zeta^{2}-1}$. So

$$
H(s)=\frac{\omega_{n}^{2}}{s^{2}+2 \zeta \omega_{n} s+\omega_{n}^{2}}=\frac{A}{s-c_{1}}+\frac{B}{s-c_{2}}=\frac{A s-A c_{1}+B s-B c_{2}}{\left(s-c_{1}\right)\left(s-c_{2}\right)}
$$

Equating co-efficients of $s$ we get $A=-B$, so define $A=M=-B$. Equating $s^{0}$ co-efficients we get

$$
\omega_{n}^{2}=M c_{2}-M c_{1}=M \zeta \omega_{n}+M \omega_{n} \sqrt{\zeta^{2}-1}-M \zeta \omega_{n}+M \omega_{n} \sqrt{\zeta^{2}-1}=2 M \omega_{n} \sqrt{\zeta^{2}-1}
$$

So

$$
M=\frac{\omega_{n}^{2}}{2 \omega_{n} \sqrt{\zeta^{2}-1}}=\frac{\omega_{n}}{2 \sqrt{\zeta^{2}-1}}
$$

## Exercise 36

If

$$
H(s)=\frac{M}{s-c_{1}}-\frac{M}{s-c_{2}}
$$

then (from Laplace transform tables) $\mathcal{L}\left(e^{a t} \mathbb{1}(t)\right)=\frac{1}{s-a}$, then by linearity of the inverse Laplace transform, we have

$$
\begin{aligned}
h(t)=\mathcal{L}^{-1}(H(s)) & =\mathcal{L}^{-1}\left(\frac{M}{s-c_{1}}-\frac{M}{s-c_{2}}\right)=M \mathcal{L}^{-1}\left(\frac{1}{s-c_{1}}\right)-M \mathcal{L}^{-1}\left(\frac{1}{s-c_{2}}\right) \\
& =M e^{c_{1} t} \mathbb{1}(t)-M e^{c_{2} t} \mathbb{1}(t)=M\left(e^{c_{1} t}-e^{c_{2} t}\right) \mathbb{1}(t)
\end{aligned}
$$

## Exercise 37

Assuming $\zeta=1$ we have $s^{2}+2 \omega_{n} s+\omega_{n}^{2}=\left(s+\omega_{n}\right)^{2}$. So

$$
H(s)=\frac{\omega_{n}^{2}}{\left(s+\omega_{n}\right)^{2}}
$$

From Laplace transform tables $\mathcal{L}\left(t e^{a t}\right)=\frac{1}{s-a}^{2}$, so

$$
h(t)=\mathcal{L}^{-1}(H(s))=\mathcal{L}^{-1}\left(\frac{\omega_{n}^{2}}{\left(s+\omega_{n}\right)^{2}}\right)=\omega^{2} t e^{\omega t} .
$$

## Exercise 38

The poles of $H(s)$ are simple to investigate. As seen above, if $\zeta=1$, then there is only one pole of $H(s)$, at $s=\omega$, and if $\zeta \neq 1$, then there are exactly two poles, at $c_{1}=$ $-\zeta \omega_{n}+\omega_{n} \sqrt{\zeta^{2}-1}$ and $c_{1}=-\zeta \omega_{n}-\omega_{n} \sqrt{\zeta^{2}-1}$.

The following graphs (figure 5 and 6 ) show $h(t)$ for various parameters.
Figure 5 shows that the peak in the transfer function increases as $\omega_{n}$. We can think of an impulse as a brief exposure of the system to an extremely high frequency signal. When an input signal is close to the resonant frequency of the system, system begins to resonate. So if a system has a high resonant frequency, then we would expect it to react more to an impulse than a system with a lower resonant frequency. The transfer function is the response to an impulse, so since the peak in the transfer function correlates with $\omega_{n}$, it is natural to think of the parameter $\omega_{n}$ as the natural frequency of the system.

Figure 6 shows that as $\zeta$ increases, the time it takes for disturbances in the system to decay away increases. After the impulse is applied to the system, we would expect the reaction of the system to decay as time goes to infinity. If the system is heavily dampened,


Figure 5: $h(t)$ for different values of $\zeta$. Red indicates $\omega_{n}<1$, blue indicates $\omega_{n}>1$ and green indicates $\omega_{n}=1$. For these graphs, $\zeta=2$ was kept constant.


Figure 6: $h(t)$ for different values of $\zeta$. Red indicates $\zeta<1$, blue indicates $\zeta>1$ and green is the special case where $\zeta=1$. For these graphs, $\omega_{n}=1$ was kept constant.
any disturbance caused by the impulse would decay away very quickly, which is what we see for large $\zeta$. Furthermore, if there is very little dampening, then the oscillations cause by the impulse do not decay quickly, which is what we see for small $\zeta$. So this parameter is clearly related to the dampening of the system. It is called the dampening ratio as it,...?

## Exercise 39

Like in the calculation in the notes,

$$
Y(s)=U(s) H(s)=E(s) G_{1}(s) H(s)
$$

but now the equation would become

$$
Y(s)=\left(R(s)+Y_{m}(s)\right) G_{1}(s) H(s)=\left(R(s)+G_{2}(s) Y(s)\right) G_{1}(s) H(s)
$$

which would yield the equation

$$
Y(s)=\frac{R(s) G_{1}(s) H(s)}{1-G_{2}(s) G_{1}(s) H(s)}
$$

when rearranged for $Y(s)$. So the feedback system would be

$$
\tilde{H}(s)=\frac{G_{1}(s) H(s)}{1-G_{2}(s) G_{1}(s) H(s)}
$$

since $Y(s)=R(s) \tilde{H}(s)$.

## Exercise 40

Omitted

