# MATH4406 - Assignment 2 

Luke Marshall (41521209)
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## Question 1

A non-minimum phase is defined as a system with zeros in the RHP. These zeros are often called non-minimum phase zeros (or open-right-half-plane zeros), and can limit the response of the system.

A minimum phase system has all zeros in LHP (more specifically all zeros and poles within unit circle), so the term non-minimum clearly suggests that some but not necessarily all zeros are in the RHP.

The term phase refers to the angle of a complex number; phase for a system describes the sum of all angles (for poles) and opposite angles (for zeros) to a point as it moves around the unit circle. Obviously a non-minimum phase system has greater phase than a minimum phase system, however both systems may have the same magnitude.

## Question 2

Derive transfer function $G(s)=C(s I-A)^{-1} B(\mathrm{p} 46)$

$$
\begin{aligned}
\dot{x} & =A x+B u \\
y & =C x
\end{aligned}
$$

Take Laplace transform

$$
\begin{aligned}
s \hat{x}-x(0) & =A \hat{x}+B \hat{u} \\
(s I-A) \hat{x} & =B \hat{u}+x(0) \\
\hat{x} & =(s I-A)^{-1}(x(0)+B \hat{u})
\end{aligned}
$$

Substitute into $\hat{y}$ which gives

$$
\begin{aligned}
\hat{y} & =C \hat{x} \\
& =C(s I-A)^{-1}(x(0)+B \hat{u}) \\
& =C(s I-A)^{-1} x(0)+C(s I-A)^{-1} B \hat{u} \\
& =C(s I-A)^{-1} x(0)+G(s) \hat{u}
\end{aligned}
$$

where

$$
G(s)=C(s I-A)^{-1} B
$$

## Question 3



Figure 1

Figure 1 shows the bounded response to an unbounded exponential input $\left(u(t)=e^{t}\right)$, with the transfer function $G(s)=\frac{s-1}{(s+1)^{2}}$. The bounded response is due to the non-minimum phase zero ( $s=1$ )which "blocks" our unbounded input as shown below. We first note that $\mathcal{L}\left(e^{t}\right)=\frac{1}{s-1}$ so

$$
\begin{aligned}
\hat{y} & =\frac{G(s)}{s-1}=\frac{1}{(s+1)^{2}} \\
y(t) & =t e^{-t}
\end{aligned}
$$

Now the limit as $t \rightarrow \infty$ given by l'hopital

$$
\lim _{t \rightarrow \infty} \frac{t}{e^{t}}=0
$$

So this "blocking" is a cancelation of the input with our rational transfer function.

## Question 4

Reproduce Figure 2 with $G(s)=-\frac{s-1}{(s+1)^{2}}$, and $u(t)=\mathbf{1}(t)$

$$
\begin{aligned}
\hat{y} & =-\frac{s-1}{s(s+1)^{2}} \\
y(t) & =e^{-t}\left(e^{t}-2 t-1\right)
\end{aligned}
$$



Figure 2

## Question 5

Plot modified $G(s)$ to show several responses as zero moves into LHP


## Question 6

Proof of proposition (p51): The step response $y(t)$ exhibits initial undershoot iff $G(s)-G(\infty)$ has an odd number of positive zeros

Consider stable system with step response $\hat{y}(s)=\frac{G(s)}{s}$. Let $H(s)=G(s)-G(\infty)=\beta \frac{N(s)}{D(s)}$ be a rational function with $N, M$ monic polynomials (i.e. coefficient of highest order is 1 ) and $\beta \in \mathbb{R}$. We say $H(s)$ has relative degree $\rho \geq 1$.

From initial value theorem we have

$$
\begin{aligned}
y\left(0^{+}\right) & =\lim _{s \rightarrow \infty} s \hat{y}(s) \\
& =\lim _{s \rightarrow \infty} s \frac{G(s)}{s} \\
& =G(\infty)
\end{aligned}
$$

similarly from final value theorem (and system stability)

$$
\begin{aligned}
y(\infty) & =\lim _{s \rightarrow 0} s \hat{y}(s) \\
& =\lim _{s \rightarrow 0} s \frac{G(s)}{s} \\
& =G(0)
\end{aligned}
$$

Since $\hat{y}$ has relative degree $\rho+1$ we have $y^{(\rho)}\left(0^{+}\right)$as the first nonzero derivative (at $t=0$ ), that is $y^{(1)}\left(0^{+}\right)=\cdots=y^{(\rho-1)}\left(0^{+}\right)=0$, thus the laplace transform gives

$$
\mathcal{L}\left(y^{\rho}\left(0^{+}\right)\right)=s^{\rho} \hat{y}(s)-s^{\rho-1} y\left(0^{+}\right)
$$

Now

$$
\begin{aligned}
y^{(\rho)}\left(0^{+}\right) & =\lim _{s \rightarrow \infty} s\left(s^{\rho} \hat{y}(s)-s^{\rho-1} y\left(0^{+}\right)\right) \\
& =\lim _{s \rightarrow \infty} s^{\rho} G(s)-s^{\rho} y\left(0^{+}\right) \\
& =\lim _{s \rightarrow \infty} s^{\rho}(G(s)-G(\infty)) \\
& =\lim _{s \rightarrow \infty} s^{\rho} H(s) \\
& =\lim _{s \rightarrow \infty} s^{\rho} \beta \frac{N(s)}{D(s)} \\
& =\beta \lim _{s \rightarrow \infty} s^{\rho} \frac{\prod_{i=1}^{n}\left(s-z_{i}\right)}{\prod_{i=1}^{n+\rho}\left(s-p_{i}\right)} \\
& =\beta
\end{aligned}
$$

We say that $y(t)$ exhibits initial undershoot if the system 'starts off in the wrong direction', that is the first nonzero derivative has an opposite sign to the steady state value (taking initial position into account). Algebraically this is

$$
y^{(\rho)}\left(0^{+}\right)\left[y(\infty)-y\left(0^{+}\right)\right]<0
$$

Note that

$$
\begin{aligned}
y(\infty)-y(0+) & =G(0)-G(\infty) \\
& =H(0) \\
& =\beta \frac{N(0)}{D(0)}
\end{aligned}
$$

Thus $y(t)$ has initial undershoot iff

$$
\beta^{2} \frac{N(0)}{D(0)}<0
$$

Since $D$ is Hurwitz (i.e. a polynomial with positive real coefficients, whose zeros are in LHP) we have $D(0)$ positive, thus $y(t)$ has initial undershoot iff $N(0)$ is negative.

Since $N(0)$ is the product of the (negative) roots of $N$, it is negative iff it has an odd number of positive roots.

## Question 7

Prove (p49) that asysmptotically stable transfer functions overshoot if $G(s)-G(0)$ has at least one positive zero.

Let $z>0$ be a positive zero of $G(s)-G(0)$. Taking a step function input $\hat{u}(s)=\frac{1}{s}$ then we have

$$
\hat{y}=\frac{G(s)}{s}
$$

When $s=z$ we have

$$
\begin{aligned}
G(z)-G(0) & =0 \\
z \frac{G(z)}{z}-\lim _{s \rightarrow 0} s \frac{G(s)}{s} & =0 \\
z \hat{y}(z)-\lim _{s \rightarrow 0} s \hat{y}(s) & =0 \\
z \hat{y}(z)-\lim _{t \rightarrow \infty} y(t) & =0 \\
\hat{y}(z)-\frac{y(\infty)}{z} & =0 \\
\int_{0}^{\infty} e^{-z t}[y(t)-y(\infty)] d t & =0
\end{aligned}
$$

Since $e^{-z t}$ is positive over $[0, \infty)$ then $y(t)-y(\infty)$ must change sign over this region. Considering that $y(\infty)$ is the equilibrium solution and $y(\infty)=\lim _{t \rightarrow \infty} y(t)$ we have that $y(t)$ has values below and above the steady-state solution - that is, it overshoots.

Thus asysmptotically stable transfer functions overshoot if $G(s)-G(0)$ have at least one positive zero.

## Question 8

Is above applicable to all systems that overshoot? Prove or give counterexample.

We consider the system with transfer function

$$
\begin{aligned}
G(s) & =\mathcal{L}\left(\frac{\sin t}{t}\right) \\
& =\arctan \frac{1}{s}
\end{aligned}
$$

Feeding the system with the unit step function we see (below left) that we overshoot the steadystate solution $\left(\frac{\pi}{2}\right)$, also we note (shown below right) that there are no positive $>0$ zeros of $G(s)-G(0)=\arctan \frac{1}{s}-\frac{\pi}{2}$



Thus not all systems that overshoot (with asysmptotically stable transfer functions) need $G(s)-$ $G(0)$ to have at least one positive zero. That is, it is sufficient but not necessary.

## Question 9

Servo-mechanism: Derive transfer functions $S(s), T(s)(\mathrm{p} 50,51)$
We have the loop transfer function for the closed loop with proper feedback controller $C$ and plant $G$, and we assume $L$ is strictly proper (i.e. $L(\infty)=0$ ).

$$
\begin{aligned}
L(s) & =C(s) G(s) \\
& =\frac{N(s)}{D(s)}
\end{aligned}
$$

Such that the response

$$
\begin{aligned}
Y & =R C G-Y C G \\
& =R \frac{C G}{1+C G} \\
& =R \frac{L}{1+L} \\
& =R \frac{N}{D+N}
\end{aligned}
$$

Looking at the error

$$
\begin{aligned}
E & =R-Y \\
& =R-R \frac{N}{D+N} \\
& =R\left(1-\frac{N}{D+N}\right) \\
& =R\left(\frac{D}{D+N}\right) \\
& =R\left(\frac{1}{1+\frac{N}{D}}\right) \\
& =R\left(\frac{1}{1+L}\right)
\end{aligned}
$$

We define the sensitivity transfer function such that $E=R S$, to be

$$
S(s)=\frac{1}{1+L(s)}=\frac{D(s)}{N(s)+D(s)}
$$

Considering this as a transfer function, $S$ has undershoot (i.e. larger error value than step difference) if the following has odd number of positive zeros

$$
\begin{aligned}
S(s)-S(\infty) & =S(s)-\frac{1}{1+L(\infty)} \\
& =S(s)-1
\end{aligned}
$$

We define the complementary sensitivity transfer function as

$$
\begin{aligned}
T(s) & =1-S(s) \\
& =1-\frac{1}{1+L(s)} \\
& =\frac{L(s)}{1+L(s)} \\
& =\frac{\frac{N(s)}{D(s)}}{1+\frac{N(s)}{D(s)}} \\
& =\frac{N(s)}{N(s)+D(s)}
\end{aligned}
$$

which looking at the response equations above

$$
Y=R T
$$

Since $T(\infty)=0$ (strictly proper), both $S(s)$ and $T(s)$ have undershoot if $T(s)$ (and hence $L(s)$ ) has an odd number of positive zeros.

## Question 10

Prove zero-crossing statement ( p 51 ): The error $e(t)$ changes sign and thus has at least one zero crossing when $L$ has at least one positive pole.

Let $p$ be a positive pole for $L$, then we have

$$
S(p)=\frac{1}{1+L(p)}=\frac{D(p)}{N(p)+D(p)}=\frac{0}{1}=0
$$

Consider output response (of the error) from a step function at the pole $p$

$$
\begin{aligned}
E(p) & =\frac{S(p)}{p}=\frac{0}{p}=0 \\
\int_{0}^{\infty} e^{-p t} e(t) d t & =0
\end{aligned}
$$

Thus $e(t)$ must cross zero (at least once).

## Question 11

Discuss the "Bicycle Countersteering Revisited" box (p53)
Assuming a constant speed such that countersteer occurs (e.g. push the left handlebar to turn left) we consider the linearization (ignoring nonlinear terms) of a closed-loop bicycle turning system.

When turning (say left), we issue a step command; initial undershoot says we will move to the right briefly before turning left.

Clearly in the physical system a suitably large (but bounded) input will cause the system to 'crash' i.e. it is unstable - thus we have a positive pole in our system and hence a positive zero of the sensitivity transfer function (Q9) which experiences overshoot (Q7), and the error $e(t)$ has a zero-crossing (Q10).

Typically the controller exhibits odd number of positive zeros giving rise to nonminimum-phase, limiting response maneuverability; alternatively a LQG control can be used which avoids initial undershoot due to an even number ( $>0$ ) of positive zeros - however zero-crossing cannot be avoided since we have at least one positive zero.

