MATH4406 Homework 2

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Question 1

Definition 0.1. *A non-minimum phase, LTI, SISO, rational system, satisfies the following conditions:*

- *the system takes a scalar input and produces a scalar output;*
- *the system must be linear, i.e. if $u_1(t)$ and $u_2(t)$ are two input signals with respective output $y_1(t)$ and $y_2(t)$, then $\alpha_1 y_1(t) + \alpha_2 y_2(t)$ is the output for $\alpha_1 u_1(t) + \alpha_2 u_2(t)$ for any $\alpha_1, \alpha_2 \in \mathbb{R}$;*
- *the system must be time invariant, i.e. if $u(t)$ is an input signal with output $y(t)$, then $y(t-\tau)$ is the output for the input $u(t-\tau)$ for any $\tau \in \mathbb{R}$;*
- *the system is rational, i.e. the transfer function $G(s)$ of the system can be written in the form*
$$G(s) = \frac{N(s)}{D(s)},$$
where $N(s)$ and $D(s)$ are polynomial functions;
- *and finally, the system is nonminimum-phase, i.e. there exists at least one zero of the transfer function of the system in the open right hand plane.*

As a consequence of having no zeros in the right hand plane, a minimum phase system minimises the mass of the distribution of the impulse response, i.e. it concentrates the mass of the impulse response near 0. In other words, it minimises for all $m \in \mathbb{R}^+$
$$\int_m^\infty |h(t)|^2 \, \mathrm{d}t \, .$$
The Hilbert transform of the logarithm of the magnitude of the frequency response of a minimum-phase system is equal to the phase angle of the frequency response. Thus, when the magnitude of the impulse response is minimised, the phase is minimised also, hence the term 'minimum-phase'. A non-minimum phase system does not have no zeros in the right hand plane, so the minimum-phase system with an impulse response of equivalent magnitude will have a smaller phase contribution. Logically, this non-minimum phase system does not minimise the the phase of the system, which explains the name.ed also, hence the term 'minimum-phase'. A non-minimum phase system does not have no zeros in the right hand plane, so the minimum-phase system with an impulse response of equivalent magnitude will have a smaller phase contribution. Logically, this non-minimum phase system does not minimise the the phase of the system, which explains the name.

Question 2

$$y(t) = Cx(t) = CA^{-1}(\dot{x}(t) - Bu(t))$$

so taking the Laplace transform gives

$$Y(s) = CA^{-1}(-x(0) + sX(s) - BU(s)).$$

Recalling that $y(t) = Cx(t)$, and thus $Y(s) = CX(s)$ we have

$$AC^{-1}Y(s) = -x(0) + sC^{-1}Y(s) - BU(s)$$

and rearranging gives

$$(sC^{-1} - AC^{-1})Y(s) = x(0) + BU(s).$$

We may assume that $x(0) - 0$, so we have

$$(sI - A)C^{-1}Y(s) = BU(s)$$

$$C^{-1}Y(s) = (sI - A)^{-1}BU(s)$$

$$Y(s) = C(sI - A)^{-1}BU(s)$$

$$Y(s) = G(s)U(s).$$

Thus

$$G(s) = C(sI - A)^{-1}B$$

Question 3

Figure 1 in the paper plots the response to the input e^t over time of a system with a zero in the open right half plane. Specifically, the transfer function of the system is $G(s) = \frac{s-1}{(s+1)^2}$. As can be observed from the graph, the response of this system decays away to 0 as time approaches infinity. Consequently, the response of the system is bounded. This is unusual, as the input being applied is unbounded, and if this system did not have a zero at $s = 1$, then the response of the system would be unbounded also. The phenomenon of a signal which does not decay to 0 being applied to a system with a zero resulting in response which does decay to 0 is referred to in the paper as the blocking effect of that zero. The transfer function of the system in the graph $G(s) = \frac{s-1}{(s+1)^2}$ has a zero at 1, which is in the RHP. The paper states that a system with such a zero can block certain exponential input.

Question 4

We have $G(s) = \frac{-s+1}{(s+1)^2}$ and the input is a step function, so $U(s) = \frac{1}{s}$. By partial fractions we have

$$Y(s) = G(s)U(s) = \frac{-s+1}{s(s+1)^2} = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s} = \frac{As + Bs^2 + Bs + Cs^2 + 2Cs + C}{s(s+1)^2}.$$

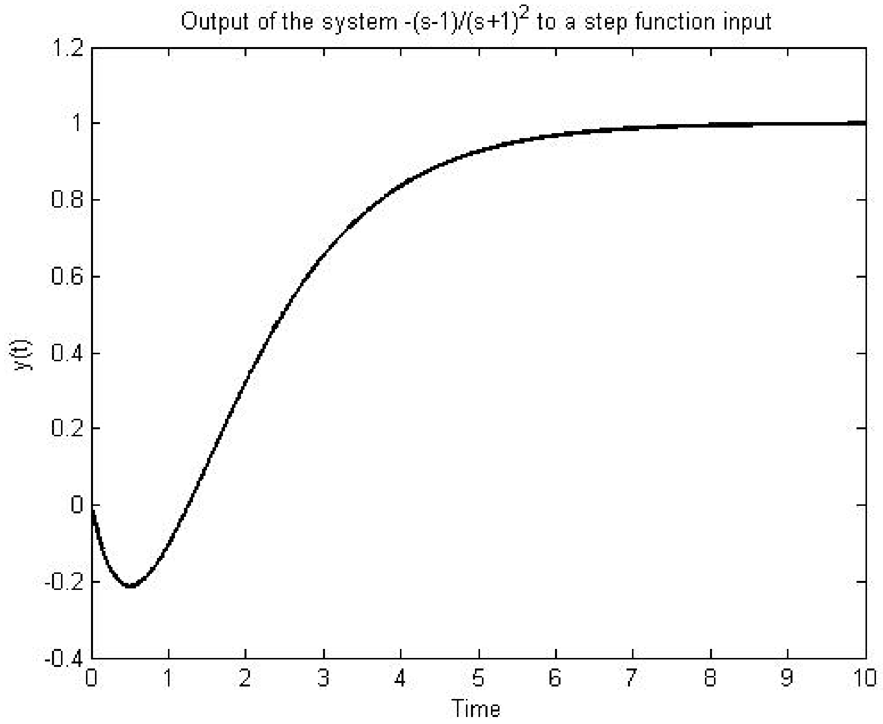
Thus $C = 1$, $B = -1$ and $A = -2$ and we have

$$Y(s) = \frac{-2}{(s+1)^2} - \frac{1}{(s+1)} + \frac{1}{s}.$$

Transforming this back into the time domain (using the transform tables) gives for $t > 0$

$$y(t) = -2te^{-t} - e^{-t} + 1.$$

This function is plotted here:



Question 5

Modifying the above working for the transfer function $G(s) = \frac{-s+a}{(s+1)^2}$ and the input is the same step function, $U(s) = \frac{1}{s}$. By partial fractions we have

$$Y(s) = G(s)U(s) = \frac{-s+a}{s(s+1)^2} = \frac{A}{(s+1)^2} + \frac{B}{s+a} + \frac{C}{s} = \frac{As + Bs^2 + Bs + Cs^2 + 2Cs + C}{s(s+1)^2}.$$

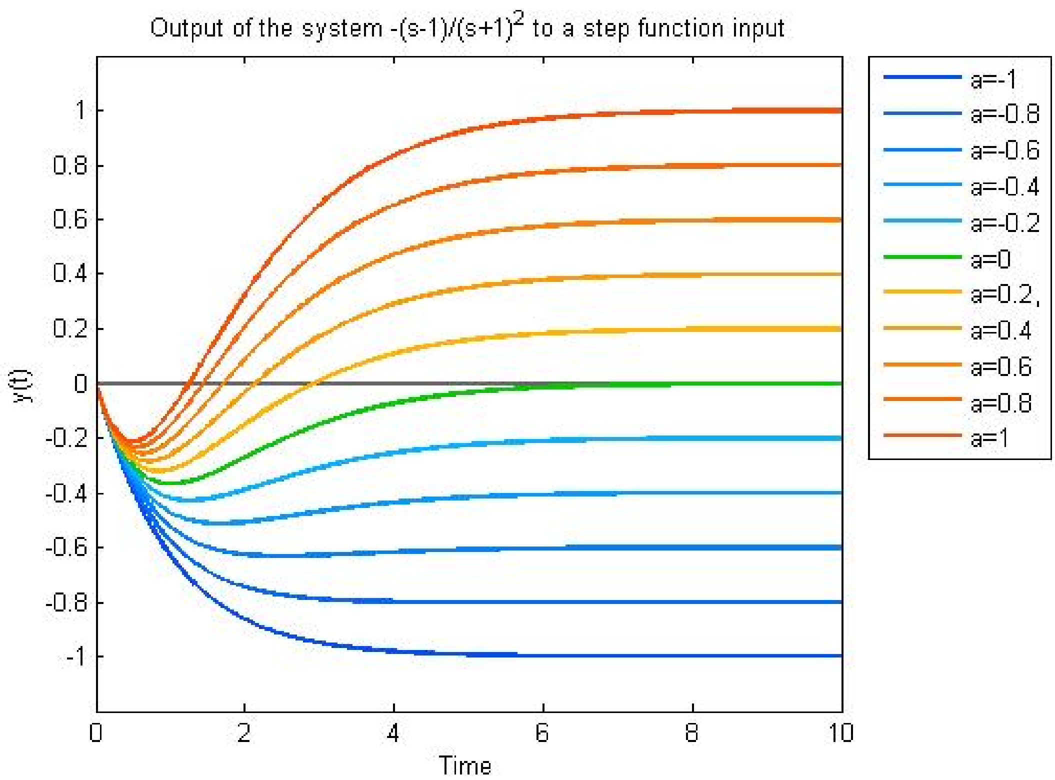
Thus $C = a$, $B = -a$ and $A = -1 - a$ and we have

$$Y(s) = \frac{-1-a}{(s+1)^2} - \frac{a}{(s+1)} + \frac{a}{s}.$$

Transforming this back into the time domain (using the transform tables) gives for $t > 0$

$$y(t) = -(1+a)te^{-t} - ae^{-t} + a.$$

This function is plotted here:



This graph tells an interesting story. When $a = 1$ we see the previous figure (although in red in this graph). This corresponds to a zero at $s = 1$ in the transition function. As the zero moves towards the left hand plane, the response behaves in a similar way to when $a = 1$, but the equilibrium value of the response decreases. This is not surprising, as the equilibrium value is is the limit of the output as $t \to \infty$, and inspecting the output functions reveals that their limit is just a.

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} -(1+a)te^{-t} - \lim_{t\to\infty} ae^{-t} + \lim_{t\to\infty} a = 0 + 0 + a = a$$

The other thing that changes is that the response takes longer to return to the equilibrium value as a increases.

The green line on the graph corresponds to when $a = 0$. As mentioned in the article, for a step function input, if 'the number 0 is a zero of the transfer function G, that is, $G(0) = 0$, then the steady state response of G is zero, that is, the dc gain of the system is zero'. This is what we see on the graph; for $a = 0$, which corresponds to a zero if G at 0, the equilibrium position of the response is 0.

Question 6

From the paper, we define $y(t)$ exhibits initial undershoot iff $y^{(\rho)}(0^+)[y(\infty) - y(0^+)] < 0$.

As in the paper, let $G(s)$ be an asymptotically stable transfer function with with relative degree $d \geq 0$. We are assuming $G(s)$ is of the form $\frac{N_G(s)}{D_G(s)}$. *A function of the form* $\frac{N_G(s)}{D_G(s)}$ *has relative degree* d *if* $d = m - n$*, where* n *is the order* $N_G(s)$ *and* m *is the order of* $D_G(s)$*.* Also as in the question, let $y(t)$ be the step response of G. By the initial value theorem, the step response has the initial value $y(0+) = \lim_{s\to\infty} G(s) = G(\infty) < \infty$ as $G(s)$ is asymptotically stable. By the final value theorem, the step response has the final value $y(\infty) = \lim_{s\to 0} G(0) = G(0)$.

Since $G(s)$ is a rational polynomial, then $G(s) - G(\infty)$ can also be represented as a rational polynomial, so we can define $H(s) = G(s) - G(\infty) = \frac{\beta N(s)}{D(s)}$, where $\beta \ni \mathbb{R}$ is the number such that N and D are polynomials with their highest order coefficient equal to 1 (i.e. they are monic polynomials).

I will prove the statement in two cases, for $d \geq 1$ and for $d = 0$. Firstly, assume $d \geq 1$.

Lemma 0.2. *For* $d \geq 1$*,*

$$y^{(d)}(0^+) = \lim_{s\to\infty} s^d H(s).$$

Proof. Notice that

$$y(0^+) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} s\frac{G(s)}{s} = \lim_{s\to\infty} G(s) = G(\infty),$$

but d, the relative degree of G is greater than or equal to 1, so $G(\infty) = 0$.

Assume that $y^m(0^+) = 0$ for all $m < n$ for some $n < d$, and consider $y^n(0^+)$. By the initial value theorem

$$y^{(n)}(0^+) = \lim_{s\to\infty} s^{n+1}Y(s) - \sum_{k=1}^{d} s^k y^{n-k}(0^+) = \lim_{s\to\infty} s^n G(s) - \sum_{k=1}^{n} s^k 0 = \lim_{s\to\infty} s^n G(s).$$

The relative degree of $s^n G(s) = d - n > 0$, so $\lim_{s\to\infty} s^n G(s) = 0$. So by induction we have that for all $n < d$, $y^n(0^+) = 0$.

Now consider $y^d(0^+) = 0$. By the initial value theorem and the above again we have,

$$y^{(d)}(0^+) = \lim_{s\to\infty} s^{d+1}Y(s) - \sum_{n=1}^{d} s^n y^{d-n}(0^+) = \lim_{s\to\infty} s^d G(s) - \sum_{n=1}^{d} 0 = \lim_{s\to\infty} s^d (G(s) - 0) = \lim_{s\to\infty} s^d (G(s) - G(\infty$$

$\square$

Furthermore, if $d \geq 1$, then $G(\infty) = 0$, so $H(s) = G(s)$ which means the relative degree of H is d. Then the relative degree of $s^d H(s)$ is 0, which means

$$\lim_{s\to\infty} s^d H(s) = \lim_{s\to\infty} \frac{\beta N(s)}{D(s)s^d} = \beta$$

and thus, by the lemma

$$y^{(d)}(0^+) = \beta.$$

Now assume in addition that $y(t)$ exhibits initial undershoot. Then $y^{(d)}(0^+)[y(\infty) - y(0^+)] < 0$. Then we have

$$0 > y^{(d)}(0^+)[G(0) - G(\infty)] = y^{(d)}(0^+)H(s) = y^{(d)}(0^+)\frac{\beta N(0)}{D(0)} = \frac{\beta^2 N(0)}{D(0)}.$$

A polynomial is called Hurwitz if its coefficients are positive real numbers and its zeros are located in the left half-plane. Since $D(s)$ is monic we can write $D(s)$ as

$$N(s) = \prod_i (s + p_i)^{q_i}$$

for some pairs $p_i \in \mathbb{R}$ and $q_i \in \mathbb{N}_0$, and by assumption our system is non-minimum phase, so for all p_i, $p_i > 0$, as all the zeros of $D(s)$ are in the left hand plane. If this product is expanded out, since all the $p_i > 0$, all the coefficients of $D(s)$ will be positive, which means $D(s)$ is Hurwitz. As all the coefficients of D are positive, $D(0) = \sum_{i=1}^{m} 0^i d_i = d_0 > 0$, which means $N(0) < 0$. Now, as N is monic, N can be written as

$$N(s) = \prod_i (s - a_i)^{b_i}$$

for some pairs $a_i \in \mathbb{R}$ and $b_i \in \mathbb{N}_0$. So $N(0)$ will be less than 0 if there are an odd number of positive zeros with odd multiplicity. If the multiplicity of a zero at a is b, then we say there are b roots at a. So the total number of positive roots of zeros with odd multiplicity is an odd number of odd numbers, which itself is an odd number. Any root that has an even multiplicity contributes an even number of roots, and an even number plus an odd number is odd, so e total number of positive roots is odd. The roots of $N(s)$ are the roots of $H(s) = G(s) - G(\infty)$, so if $d \geq 1$ and $y(t)$ exhibits initial undershoot then $G(s) - G(\infty)$ has an odd number of positive zeros.

Now, assume $d \geq 1$ and $G(s) - G(\infty)$ has an odd number of positive zeros. Using a similar argument as above, $G(s) - G(\infty) = H(s) = \frac{\beta N(s)}{D(s)}$, so if $G(s) - G(\infty)$ has an odd number of positive zeros, $N(0) < 0$. $D(s)$ is still Hurwitz, so $D(0) = \sum_{i=1}^{m} 0^i d_i = d_0 > 0$ which means $\frac{N(0)}{D(0)} < 0$. Again, since $d \geq 1$, then

$$y^{(d)}(0^+) = \beta.$$

So we have

$$0 > \frac{\beta^2 N(0)}{D(0)} = y^{(d)}(0^+)\frac{\beta N(0)}{D(0)} = y^{(d)}(0^+)H(0) = y^{(d)}(0^+)(G(0)-G(\infty)) = y^{(d)}(0^+)(y(\infty)-y(0^+))$$

which is the condition for overshoot. Thus if $G(s) - G(\infty)$ has an odd number of positive zeros and $d \geq 1$, then $y(t)$ exhibits initial undershoot.

So the statement has been proven for $d \geq 1$. Now assume $d = 0$.

As for $H(s)$, define $G(s) = \frac{\alpha \bar{N}(s)}{\bar{D}(s)}$ where $\alpha \ni \mathbb{R}$ is the number such that N and D are monic polynomials. As the relative degree of $G(s)$ is 0, repeated application of L'Hopital's rule gives that

$$G(\infty) = \lim_{s\to\infty} G(s) = \alpha \lim_{s\to\infty} \frac{\bar{N}(s)}{\bar{D}(s)} = \alpha \lim_{s\to\infty} \frac{\lim_{s\to\infty} 1}{\lim_{s\to\infty} 1} = \alpha.$$

So

$$H(s) = \frac{\alpha \bar{N}(s)}{\bar{D}(s)} - \alpha = \frac{\alpha \bar{N}(s) - \alpha \bar{D}(s)}{\bar{D}(s)}.$$

Now, $\bar{D}$ and $\bar{N}$ are monic polynomials, so their highest order terms cancel each other in this fraction. Thus the degree of $N(s) = \alpha \bar{N}(s) - \alpha \bar{D}(s)$ is at least one less than the degree of $D(s) = \bar{D}(s)$ which means the relative degree of H, which we will call r is greater than or equal to 1.

Now assume in addition that $y(t)$ exhibits initial undershoot. Then $y'(0^+)[y(\infty) - y(0^+)] < 0$. Recall that $y(0+) = G(\infty) < \infty$ and notice that

$$y'(0^+) = \lim_{s\to\infty} s(-y(0^+)+sY(s)) = \lim_{s\to\infty} -sG(\infty)+s^{\frac{G(s)}{s}} = \lim_{s\to\infty} s(G(s)-G(\infty)) = H(\infty) = \beta.$$

Then we have

$$0 > y^{(d)}(0^+)[G(0) - G(\infty)] = y^{(d)}(0^+)H(s) = y^{(d)}(0^+)\frac{\beta N(0)}{D(0)} = \frac{\beta^2 N(0)}{D(0)}$$

and from here we can follow the previous reasoning to show that $G(s) - G(\infty)$ has an odd number of positive zeros.

Finally, assume $G(s) - G(\infty)$ has an odd number of positive zeros and $d = 0$. Again, like the previous case we have that if $G(s) - G(\infty)$ has an odd number of positive zeros, $N(0) < 0$. $D(s)$ is still Hurwitz, so $D(0) = \sum_{i=1}^{m} 0^i d_i = d_0 > 0$ which means $\frac{N(0)}{D(0)} < 0$ and thus $\beta^2 \frac{N(0)}{D(0)} < 0$. In this case the reasoning

$$y'(0^+) = \lim_{s\to\infty} s(-y(0^+) + sY(s)) = \lim_{s\to\infty} -sG(\infty) + s^{\frac{G(s)}{s}} = \lim_{s\to\infty} s(G(s) - G(\infty)) = H(\infty) = \beta$$

is still valid, so we have

$$0 > y'(0^+)\frac{\beta N(0)}{D(0)} = y'(0^+)H(0) = y'(0^+)(G(0) - G(\infty)) = y'(0^+)(y(\infty) - y(0^+))$$

which is the condition for overshoot when $d = 0$. Thus if $G(s) - G(\infty)$ has an odd number of positive zeros and $d = 0$, then $y(t)$ exhibits initial undershoot.

Thus for all $d \geq 0$, $G(s) - G(\infty)$ has an odd number of positive zeros iff $y(t)$ exhibits initial undershoot.

Question 7

Assume $G(s)$ is the step ($u(t) = \mathbb{1}(t)$) response of an asymptotically stable transfer function of a system with output $Y(s)$, and let $G(s) - G(0)$ have a positive zero at z. For an LTI system we have

$$Y(s) = U(s)G(s) = \frac{G(s)}{s}.$$

Now consider the output at $s = z$,

$$Y(z) = \frac{G(z)}{z} = \frac{G(z) - G(0)}{z} + \frac{G(0)}{z} = \frac{G(0)}{z}$$

by definition of z.

We will abuse notation and define $\lim_{t\to\infty} y(t) = y(\infty)$. The final value theorem says that if $y(\infty) < \infty$, then

$$y(\infty) = \lim_{t\to 0} sY(s).$$

By assumption, this system is asymptotically stable in response to a step function, so we have $y(\infty) < \infty$. Applying this theorem to the definition of $Y(s)$ we have

$$y(\infty) = \lim_{t\to 0} s\frac{G(s)}{s} = \lim_{t\to 0} G(s) = G(0).$$

Thus we have

$$Y(z) = \frac{G(0)}{z} = \frac{y(\infty)}{z}\cdot 1 = \frac{y(\infty)}{z}\cdot z\int_0^\infty e^{-zt}\ \mathrm{d}t\ = \int_0^\infty y(\infty)e^{-zt}\ \mathrm{d}t\ .$$

Additionally, observing that $Y(z) = \int_0^\infty y(t)e^{-zt}\ \mathrm{d}t$, we have

$$\int_0^\infty y(t)e^{-zt}\ \mathrm{d}t\ = \int_0^\infty y(\infty)e^{-zt}\ \mathrm{d}t$$

which implies

$$0 = \int_0^\infty y(t)e^{-zt}\ \mathrm{d}t\ - \int_0^\infty y(\infty)e^{-zt}\ \mathrm{d}t\ = \int_0^\infty e^{-zt}[y(t) - y(\infty)]\ \mathrm{d}t\ .$$

e^{-zt} is nonnegative on $[0,\infty)$, so for this integral to be 0, either $y(t) - y(\infty) \equiv 0$ or there must be at least one region interval were the area under the graph on that interval is positive (i.e the graph is above the x-axis), and at least one region where the area under the graph on that interval is negative (i.e the graph is below the x-axis). As the output is continuous, then by the intermediate value theorem, there exists some value $c \in [0,\infty)$ where $y(c) - y(\infty)$ changes sign. Thus some time before or after $t = c$ $y(c)$ overshoots $y(\infty)$, which is the steady state value of the system.

Question 8

Consider the transfer function $G(s) = \frac{6}{s^2+2s+3}$. $G(0) = 2$, so

$$G(s) - G(0) = \frac{6}{s^2+2s+3} - 2 = \frac{-2s^2-4s+6-6}{s^2+2s+3} = \frac{-2s(s+2)}{s^2+2s+3}.$$

Thus, the zeros of $G(s) - G(0)$ are 0 and -2, i.e. $G(s) - G(0)$ has no positive zeros. Now we find $y(t)$.

$$Y(S) = U(s)G(s) = \frac{1}{s}\frac{1}{(s+1)^2+2} = \frac{A}{s} + \frac{Bs+C}{(s+1)^2+2} = \frac{As^2+2As+3A+Bs^2+Cs}{s^2+2s+3}$$

Equating co-efficients gives $A = \frac{1}{3}$, $B = \frac{-1}{3}$ and $C = \frac{-2}{3}$ and we have $Y(S) = \frac{1}{3s} - \frac{1}{3}\frac{s-\sqrt{2}^2}{(s+1)^2+\sqrt{2}^2}$. The inverse Laplace transform of the function is $y(t) = \frac{1}{3}(1-e^{-t}(\cos\sqrt{2}t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t)$, which has

$$y(\infty) = \lim_{t\to\infty}\frac{1}{3} - e^{-t}(\cos\sqrt{2}t + \sin\sqrt{2}t) = \frac{1}{3}.$$

However,

$$y(2) = \frac{1}{3}(1 - e^{-2}(\cos 2\sqrt{2} + \frac{1}{\sqrt{2}} \sin 2\sqrt{2})) = 0.3861 > \frac{1}{3} = y(\infty).$$

Thus this function has overshoot, but does not have a positive pole. We have a counter example, and the result from the above question does not generalise to all functions that express overshoot.

Question 9

The sensitivity function is defined in the paper as the transfer function that transforms the reference input to the error $E(s)$, i.e. it is the function $S(s)$ such that $E(s) = R(s)S(s)$. The complimentary sensitivity transfer function is defined in the paper as $T(s) = 1 - S(s)$.

We have that $Y(s) = U(s)G(s)$, but the input to the plant G is the output of the plant $C(s)$ with input $E(s)$, which is the transform of the error signal. This gives $Y(s) = E(s)C(s)G(s) = E(s)L(s)$, as the paper defines $L(s)$ as $C(s)G(s)$.

We also have that the input to the plant C is the reference minus the feedback term i.e. $E(s) = R(s) - Y(s)$. Substituting the previous equation in for $Y(s)$ gives $E(s) = R(s) - E(s)L(s)$ and rearranging for $E(s)$ gives

$$E(s) = \frac{R(s)}{1 + L(s)}.$$

Thus, the function $S(s)$ which satisfies $E(s) = R(s)S(s)$ is $S(s) = \frac{1}{1+L(s)}$.

Finally,

$$T(s) = 1 - S(s) = \frac{1 + L(s)}{1 + L(s)} - \frac{1}{1 + L(s)} = \frac{L(s)}{1 + L(s)}.$$

This is how these functions were derived.

Question 10

The statement to be proven is that the error term $e(t)$ changes sign and has at least one zero crossing when L has at least one positive pole p.

Assume L has at least one positive pole p. Then $S(p) = \frac{1}{1+L(p)} = \frac{(L(p))^{-1}}{(L(p))^{-1}+1} = \frac{0}{0+1} = 0$. Thus, as we can the reference function to be $R(s) < \infty$, we have $E(p) = S(p)R(p) = 0$. By definition of Laplace transform we then have

$$0 = E(p) = \int_0^\infty e^{-pt} e(t) \, \mathrm{d}t$$

(note that, as in the paper, e the constant is Euler's constant, and $e(t)$ the function is the error signal). e^{-zt} is nonnegative on $[0, \infty)$, so for this integral to be 0, either $e(t) \equiv 0$ or there must be at least one region interval were the area under the graph on that interval is positive (i.e the graph is above the x-axis), and at least one region where the area under the graph on that interval is negative (i.e the graph is below the x-axis).

As the output is continuous, then by the intermediate value theorem, there exists some value $c \in [0, \infty)$ where $e(c)$ changes sign. Thus$e(c)$ has a zero crossing.

Question 11

A bicyclist or motorcyclist will generally briefly turn the front wheel of their bike in the opposite direction to the direction they want to turn. This action is called counter steering. The initial turn to the opposite direction has the effect of making the bicycle 'fall' in the direction the cyclist actually wants to turn. The cyclist then uses gravity to temporarily increase their centripetal force, allowing the cyclist to turn more sharply.

The phenomenon of counter-steering can be modeled with control theory. Consider a cyclist riding at constant speed. The cyclist and vehicle form the plant, and the output, among other things, is the direction the plant is traveling, specifically, define y as the angle between the velocity and acceleration vectors of the plant (let a negative angle represent a left turn, and a positive angle represent a right turn). If $\theta \in \mathbb{R}$, then a signal of $\theta\mathbb{1}(t)$, which is just a scalar multiple of the step response, would represent a right turn of angle θ. The output of such a system displaying countersteering would see a brief drop in y, corresponding to the brief left turn, before the system begins turning back toward the right, eventually reaching the vicinity of θ. This is an excellent example of undershoot.

From experience, to ride a bike at a constant speed, the cyclist must maintain an unstable equilibrium, otherwise the bike would fall to the left or right. According to the paper, the instability of the linearization of this open-loop system is due to a positive pole. Let $L(S)$ be the transfer function for this open system. As the sensitivity function $S(s)$ is given by $S(s) = \frac{1}{1+L(s)}$, then this pole is a positive zero of the sensitivity transfer function.

As countersteering is caused itself by steering, countersteering can be performed on the initial turn involved in countersteering. This is an example of a different controller which the cyclist could employ to achieve the goal of turning a particular direction. If the cyclist wants to turn left, they can momentarily steer left, causing the bike to lean to the right, at which point the cyclist continues to countersteer as described above. This controller has an advantage because it removes the initial undershoot of the basic countersteering controller. According to the paper, this controller now has an even (nonzero) number of positive zeros in the loop transfer function. The downside of this controller is that, as illustrated in figure 9 of the paper, it takes longer for the plant to reach the desired output.

This second controller is less common than the basic countersteering technique, but the second controller could be useful in certain situations. For example, if the vehicle must slow down rapidly while making the turn, but wants to complete the turn as quick as possible, initial undershoot would cause the plant to travel in the wrong direction while it is traveling fastest, and thus have to travel in the right direction for a longer amount of time. Eliminating the initial undershoot could allow the turn to be performed much quicker. This could be relevant in trick and racing motor and bicycling.