MATH4406 Homework 3

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Question 1

Lemma 0.1. *For $j \in \mathbb{N}^+$,*

$$\int_0^t a(s)\ \mathrm{d}s\ + \ldots + \int_0^t a(s) \ldots \int_0^{s_{j-1}} a(u)\ \mathrm{d}u\ \ldots\ \mathrm{d}s\ = \frac{1}{j!}\left(\int_0^t a(u)\ \mathrm{d}u\ \right)^j.$$

Proof. Now, notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t a(s)\ \mathrm{d}s\ \right)^2 = 2a(t)\int_0^t a(s)\ \mathrm{d}s$$

by the product rule. Thus,

$$\int_0^t a(s)\int_0^s a(u)\ \mathrm{d}u\ \ \mathrm{d}s\ = \frac{1}{2}\left(\int_0^t a(u)\ \mathrm{d}u\ \right)^2.$$

Assume for induction that

$$\int_0^t a(s)\ \mathrm{d}s\ + \ldots + \int_0^t a(s) \ldots \int_0^{s_{j-1}} a(u)\ \mathrm{d}u\ \ldots\ \mathrm{d}s\ = \frac{1}{j!}\left(\int_0^t a(u)\ \mathrm{d}u\ \right)^j$$

and consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t a(s)\ \mathrm{d}s\ \right)^{j+1} = (j+1)a(t)\left(\int_0^t a(s)\ \mathrm{d}s\ \right)^j = (j+1)a(t)j!\int_0^t a(s)\ \mathrm{d}s + \ldots + \int_0^t a(s) \ldots \int_0^{s_{j-1}} a(u$$

by the assumption. Thus we have

$$\frac{1}{(j+1)!}\left(\int_0^t a(s)\ \mathrm{d}s\ \right)^{j+1} = \int_0^t a(s)\ \mathrm{d}s\ + \ldots + \int_0^t a(s) \ldots \int_0^{s_j} a(u)\ \mathrm{d}u\ \ldots\ \mathrm{d}s$$

so the assumption is true by induction and we have

$$\int_0^t a(s)\ \mathrm{d}s\ + \ldots + \int_0^t a(s) \ldots \int_0^{s_{j-1}} a(u)\ \mathrm{d}u\ \ldots\ \mathrm{d}s\ = \frac{1}{j!}\left(\int_0^t a(u)\ \mathrm{d}u\ \right)^j.$$

□

By Picard iterations we have

$$x_1(t) = x_0 + \int_0^t a(s)x_0 \ \mathrm{d}s \ = x_0 \left(1 + \int_0^t a(s) \ \mathrm{d}s \ \right)$$

$$x_2(t) = x_0 + \int_0^t a(s)x_0 \left(1 + \int_0^s a(u) \ \mathrm{d}u \ \right) \ \mathrm{d}s \ = x_0 \left(1 + \int_0^t a(s) \ \mathrm{d}s \ + \int_0^t a(s) \int_0^s a(u) \ \mathrm{d}u \ \ \mathrm{d}s \ \right)$$

Assume

$$x_j(t) = x_0 \left(1 + \int_0^t a(s) \ \mathrm{d}s \ + \ldots + \int_0^t a(s) \ldots \int_0^{s_{j-1}} a(u) \ \mathrm{d}u \ \ldots \ \mathrm{d}s \ \right).$$

Then

$$x_{j+1}(t) = x_0 + \int_0^t a(s)x_0 \left(1 + \int_0^s a(u) \ \mathrm{d}u \ + \ldots + \int_0^s a(s_2) \ldots \int_0^{s_j} a(u) \ \mathrm{d}u \ \ldots \ \mathrm{d}s_2 \ \right) \ \mathrm{d}s \ = x_0 \left(1 + \int_0\right.$$

So

$$x_j(t) = x_0 \left(1 + \int_0^t a(s) \ \mathrm{d}s \ + \ldots + \int_0^t a(s) \ldots \int_0^{s_{j-1}} a(u) \ \mathrm{d}u \ \ldots \ \mathrm{d}s \ \right)$$

is true for general $j \in \mathbb{N}$ by induction. By the Lemma we then have

$$x_j(t) = x_0 \left(1 + \left(\int_0^t a(s) \ \mathrm{d}s \ \right)^1 + \ldots + \frac{1}{j!} \left(\int_0^t a(s) \ \mathrm{d}s \ \right)^j \right) = x_0 \sum_{i=0}^{j} \frac{1}{j!} \left(\int_0^t a(s) \ \mathrm{d}s \ \right)^j.$$

A property of Picard iterations is that

$$\lim_{j \to \infty} x_j(t) = x(t)$$

but $\int_0^t a(s) \ \mathrm{d}s$ is just a real number, so by the Taylor expansion of the exponetial function, we have

$$x(t) = \lim_{j \to \infty} x_j(t) = x_0 \sum i = 0^\infty \frac{1}{j!} \left(\int_0^t a(s) \ \mathrm{d}s \ \right)^j = x_0 e^{\int_0^t a(s) \ \mathrm{d}s}$$

which is the desired result, assuming $x(0) = x_0 = a(0)$.

Question 2

By Cayley-Hamilton we have that $0 = d_0 I + d_1 A + d_2 A^2 + \ldots + d_{n-1} A^{n-1} + A^n$, where $s^n + \sum_{i=0}^{n-1} d_i s^i$ is the characteristic polynomial of A. So $A^n = -\sum_{i=0}^{n-1} d_i A^i$. Assume for induction that $A^{n+k} = \sum_{i=0}^{n-1} c_{ik} A^i$ for some $k \in \mathbb{N}^+$ and constants $c_{ik} \in \mathbb{R}$. Then

$$A^{n+k+1} = AA^{n+k} = \sum_{i=0}^{n-1} c_{ik} A^{i+1} = c_{n-1,k} A^n + \sum_{i=0}^{n-2} c_{ik} A^{i+1} = -c_{n-1,k} \sum_{i=0}^{n-1} d_i A^i + \sum_{i=1}^{n-1} c_{i-1,k} A^i$$

$$= -c_{n-1,k}d_0 I + \sum_{i=1}^{n-1}(c_{n-1,k}d_i - c_{i-1,k})A^i = \sum_{i=0}^{n-1} c_{i,k+1}A^i$$

for some $c_{i,k+1} \in \mathbb{R}$, so by induction, $A^{n+k} = \sum_{i=0}^{n-1} c_{ik}A^i$ for all $k \in \mathbb{N}$.

By definition of matrix exponential we have

$$e^{At} = \sum_{k=0}^{\infty}\frac{t^k}{k!}A^i = \sum_{k=0}^{n-1}\frac{t^k}{k!}A^k + \sum_{k=0}^{\infty}\frac{t^{k+n}}{(k+n)!}A^{k+n} = \sum_{k=0}^{n-1}\frac{t^k}{k!}A^k + \sum_{k=0}^{\infty}\sum_{i=0}^{n-1}\frac{c_{ik}t^{k+n}}{(k+n)!}A^i = \sum_{k=0}^{n-1}\frac{t^k}{k!}A^k + \sum_{i=0}^{n-1}\sum_{k=0}^{\infty}\frac{c_{ik}t}{(k-}$$

which was to be proven.

Question 3

(a) Recall that

$$\dot{X} = AX + BU \qquad \text{and} \qquad Y = CX + DU$$

and taking the Laplace transform of this and assuming $x(0) = 0$ gives

$$s\hat{X} = A\hat{X} + B\hat{U} \qquad \text{and} \qquad \hat{Y} = C\hat{X} + D\hat{U}$$

which can be rearranged to give

$$\hat{Y} = (C(sI - A)^{-1}B + D)\hat{U}.$$

Subbing values gives

$$\begin{aligned}
H(s) =& C(sI - A)^{-1}B + D \\
=& \begin{bmatrix} 1 & -10 \end{bmatrix}\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -3 & -4 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1 \\
=& \begin{bmatrix} 1 & -10 \end{bmatrix}\left(\begin{bmatrix} s & -2 \\ 3 & s+4 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1 \\
=& \frac{1}{s^2 + 4s + 6}\begin{bmatrix} 1 & -10 \end{bmatrix}\begin{bmatrix} s+4 & 2 \\ -3 & s \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1 \\
=& \frac{1}{s^2 + 4s + 6}\begin{bmatrix} 1 & -10 \end{bmatrix}\begin{bmatrix} 4 \\ 2s \end{bmatrix} + 1 \\
=& \frac{4 - 20s}{s^2 + 4s + 6} + 1 \\
=& \frac{s^2 - 16s + 10}{s^2 + 4s + 6}.
\end{aligned}$$

(b) The impulse response is given by

$$Y(s) = H(s)U(s) = \frac{s^2 - 16s + 10}{s^2 + 4s + 6} \cdot 1 = 1 + \frac{-20(s+2) + 44}{(s+2)^2 + \sqrt{2}^2}.$$

Transforming back (using the laplace transform tables) gives:

$$y(t) = \delta(t) + e^{-2t}\left(-20\cos(\sqrt{2}t) + 22\sqrt{2}\sin(\sqrt{2}t)\right).$$

This is the impulse response matrix.

(c) X has dimension $n = 2$ so the controlability matrix is

$$\mathrm{Con}(A,B) = [B\ AB] = \left[\begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} 0 & 4 \\ 2 & -8 \end{bmatrix}.$$

The determinant of this matrix is $|\mathrm{Con}(A,B)| = 0 \cdot -8 - 4 \cdot 2 = -8$, so $\mathrm{Con}(A,B)$ is full rank, thus the system is controllable.

(d) Again, X has dimension $n = 2$ so the observability matrix is

$$\mathrm{Obs}(A,B) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & -10 \end{bmatrix} \\ \begin{bmatrix} 1 & -10 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -3 & -4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & -10 \\ 30 & 42 \end{bmatrix}.$$

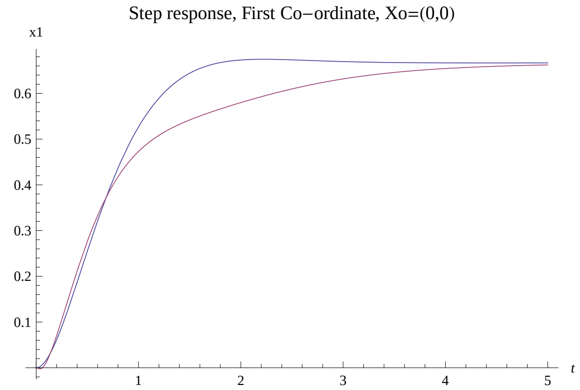
The determinant of this matrix is $|\mathrm{Obs}(A,B)| = 1 \cdot 42 + 10 \cdot 30 = 342$, so $\mathrm{Con}(A,B)$ is full rank, thus the system is observable.

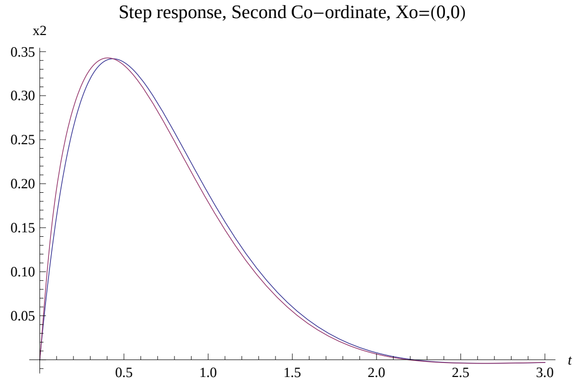
(e) Let $K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$. From lectures,

$$\dot{e}(t) = (A - KC)e(t) = \left(\begin{bmatrix} 0 & 2 \\ -3 & -4 \end{bmatrix} - \begin{bmatrix} k_1 & -10k_1 \\ k_2 & -10k_2 \end{bmatrix}\right)e(t) = \begin{bmatrix} -k_1 & 2 + 10k_1 \\ -3 - k_2 & -4 + 10k_2 \end{bmatrix} e(t) = Qe($$

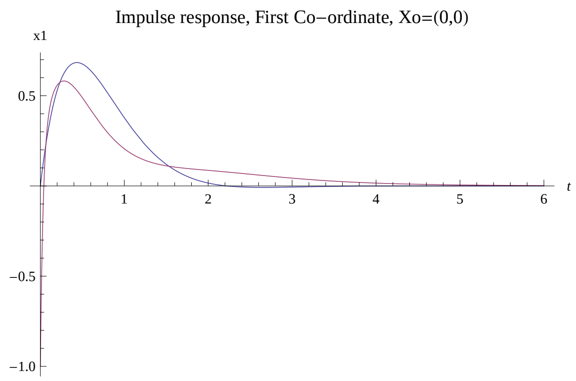
We want a Q that has negative eigenvalues (so that the estimation error converges to 0) and no imaginary part (so there is no oscillation). The characteristic equation for Q is $(k_1+\lambda)(\lambda+4-10k_2)+2(1+5k_1)(3+k_2) = \lambda^2+(k_1+4-10k_2)\lambda+4k_1-10k_1k_2+2(1+5k_1)(3+k_2)$ so we have eigenvalues $\lambda = \frac{-(k_1+4-10k_2)\pm\sqrt{(k_1+4-10k_2)^2-4(4k_1-10k_1k_2+2(1+5k_1)(3+k_2))}}{2} = \frac{10k_2-4-k_1\pm\sqrt{k_1^2+100k_2^2-88k_2-20k_1k_2-8-128k_1}}{2}$. The larger eigenvalue was plotted and $k_1 = 1$ and $k_2 = -3$ were chosen so that the larger eigenvalue was negative. This gave eigenvalues of $\frac{-35\pm\sqrt{1089}}{2} = -1, -34$. This observer should cause the error in estimation to go to 0. This is demonstrated in the graphs below.

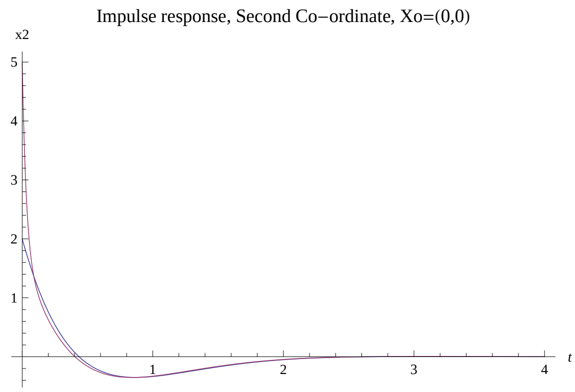
(f) The following graphs compare the step response of $\hat{x}(t)$ and $x(t)$ for initial conditions of $X_0 = (0, 0)$.



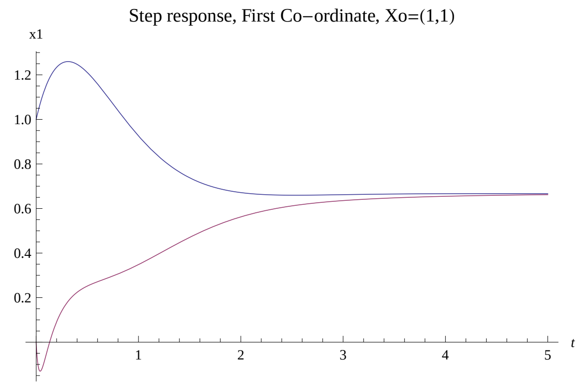


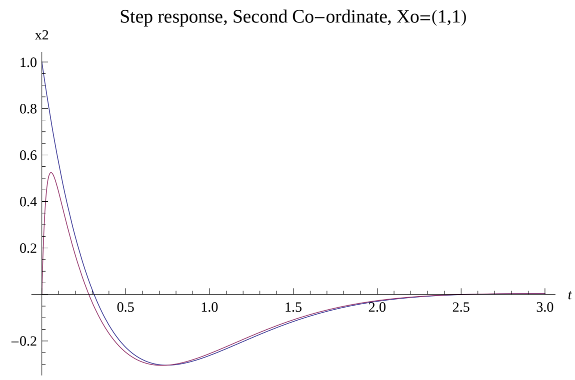
And these graphs compare the impulse response of $\hat{x}(t)$ and $x(t)$ for the same initial conditions.





The following graphs compare the step response of $\hat{x}(t)$ and $x(t)$ for initial conditions of $X_0 = (1, 1)$.





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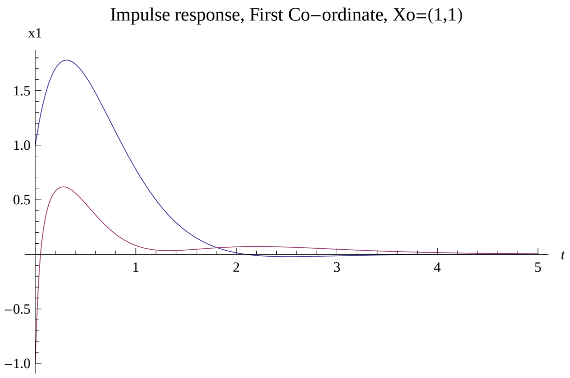
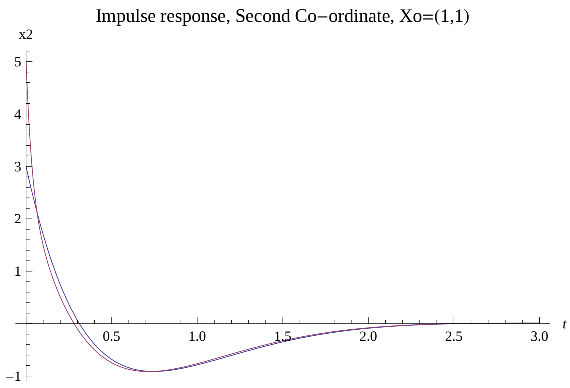




Question 4

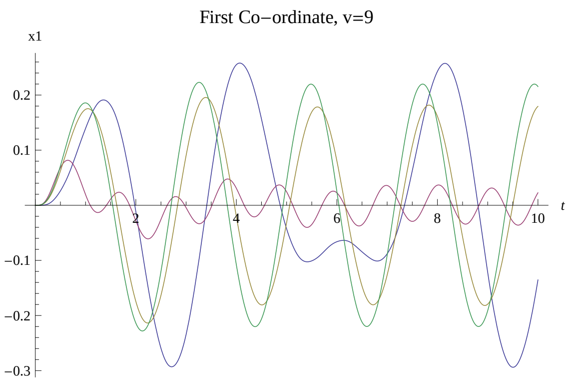
(a) The eigenvalues of A are

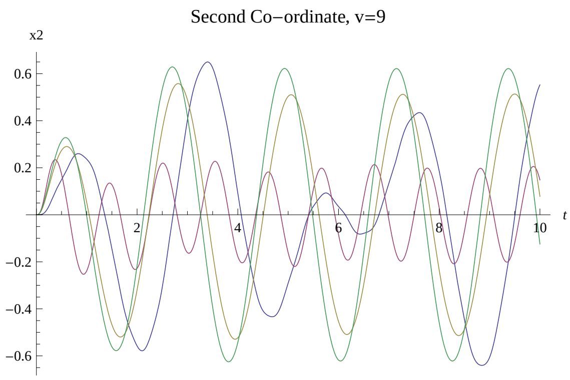
c	*eigenvalues*			
0	$16.353i$	$-16.353i$	$1.8002i$	$-1.8002i$
375	$-6.4143 + 14.750i$	$-6.4143 - 14.750i$	$-0.33569 + 1.7993i$	$-0.33569 - 1.7993i$
750	$-12.766 + 8.1505i$	$-12.766 - 8.1505i$	$-0.73427 + 1.7997i$	$-0.73427 - 1.7997i$
1125	-32.835	-4.8820	$-1.3913 + 1.8629i$	$-1.3914 - 1.8629i$

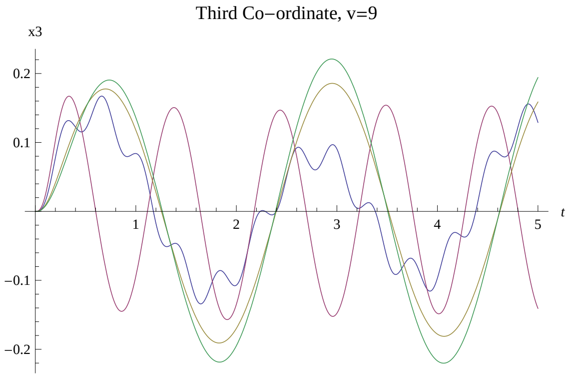
(b) The solution to this equation is

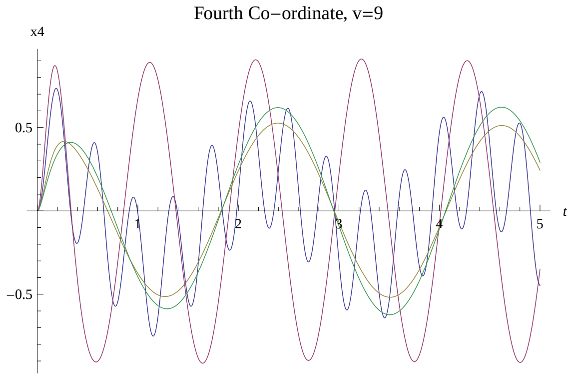
$$x(t) = e^{(A+FB)t}X_0 + \int_0^t e^{A(t-s)}Bu(t)\ \mathrm{d}s\ = \frac{1}{6}\int_0^t e^{A(t-s)}B\sin(\frac{2\pi vt}{20})\ \mathrm{d}s\ .$$

Solving this and plotting on Mathematica gives the following graphs. First, four graphs (one for each co-ordinate of $x(t)$) showing the response to $\frac{1}{6}\sin(\frac{\pi vt}{10})$ with $v = 9$ for different dampening constants c.

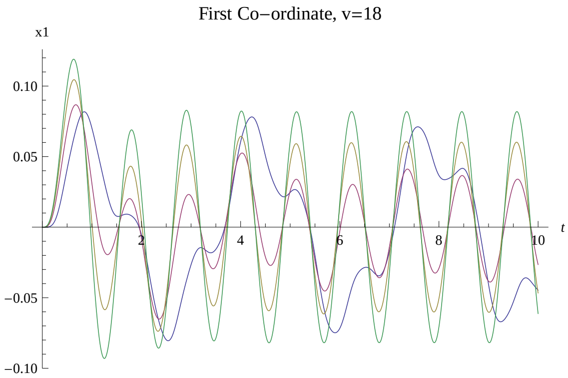


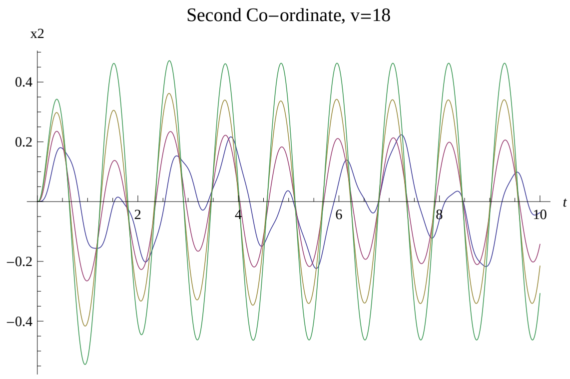


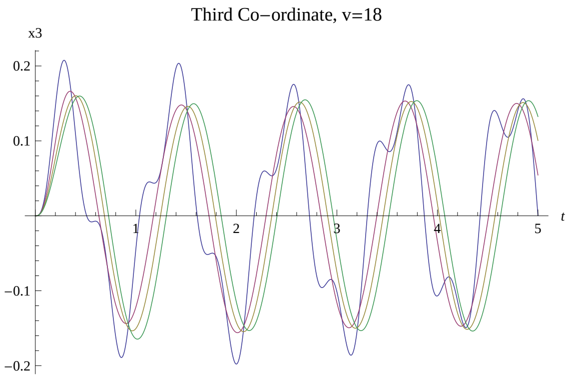


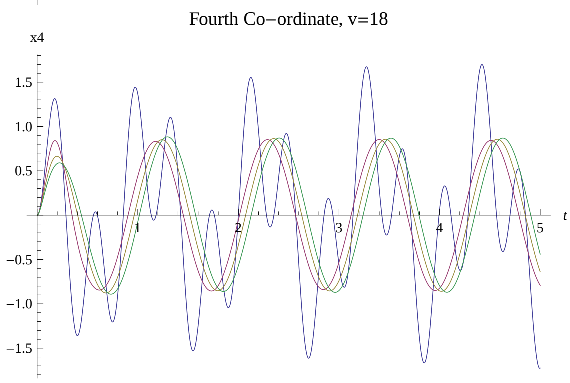


Next are a set of four graphs for $v = 18$.

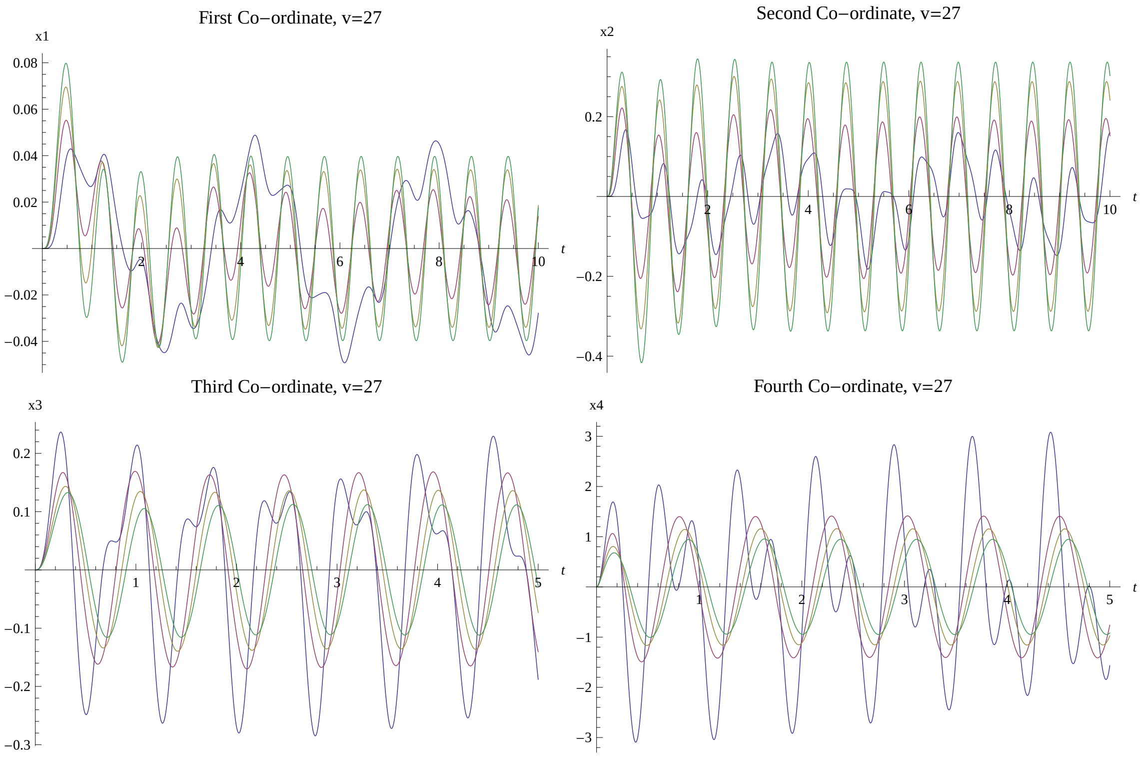




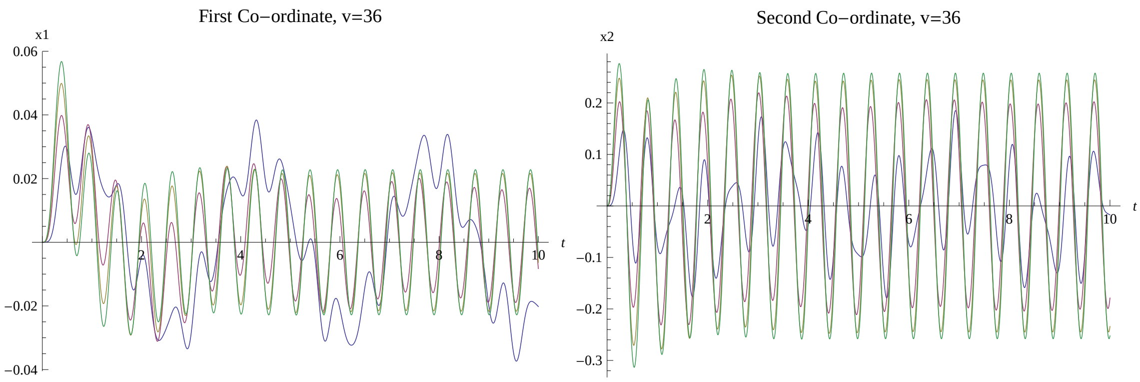


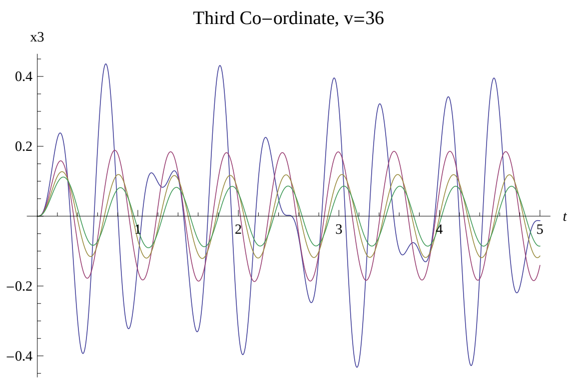


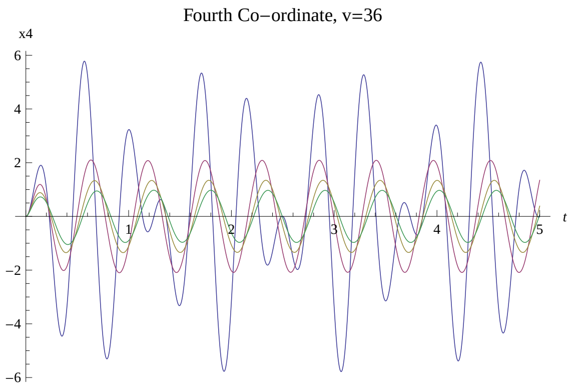
Now the same thing for $v = 27$.



And finally for $v = 36$.







From inspecting individual graphs, the variying the dampening constant does not necessarily reduce the magnitude of the oscilations in each co-ordinate. However, increasing the dampening constant causes the oscilations to become more regular. When the dampening constant is 0, the graph has multiple local maxima, which are not global maxima. This means the system is very sensitive to the changing input and changes direction frequenctly. However as the damping constant increases, the oscilation becomes more sinusodal, having local maxima and minima which appear also to be global maxima and minima.

By inspecting the graphs as a series, the main effect of increasing the velocity v is to increase the frequency of oscilations in the graph. This is expected, as the faster the car travels, the more frequently the system must respond.

Question 5

We want the system to be stable but not assymptotically stable, so choose poles at $\pm i$ and a double pole at -1. This gives a desired character equation of $s^4 + d_3s^3 + d_2s^2 + d_1s^1 + d_0 = 1 + 2s + 2s^2 + 2s^3 + s^4$. The control matrix of this system is

$$\mathrm{Con}(A, B) = [B, AB, A^2B, A^3B] = \begin{bmatrix} 0 & \frac{1}{M} & \frac{-F}{M^2} & \frac{F^2}{M^3} \\ \frac{1}{M} & \frac{-F}{M^2} & \frac{F^2}{M^3} & \frac{-F^3}{M^4} \\ 0 & 0 & 0 & \frac{-g}{LM} \\ 0 & 0 & \frac{-g}{LM} & \frac{Fg}{LM^2} \end{bmatrix}.$$

This has inverse

$$\begin{bmatrix} F & M & 0 & 0 \\ M & 0 & 0 & \frac{-FL}{g} \\ 0 & 0 & \frac{-FL}{g} & -\frac{-LM}{g} \\ 0 & 0 & \frac{-LM}{g} & 0 \end{bmatrix}$$

so define $q = \left[0, 0, \frac{-LM}{g}, 0\right]$. From classes, we showed that the appropriate co-ordinate trasformation is

$$P = \begin{bmatrix} q \\ qA \\ qA^2 \\ qA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{LM}{g} & 0 \\ 0 & 0 & 0 & -\frac{LM}{g} \\ M & 0 & -M & 0 \\ 0 & M & 0 & -M \end{bmatrix}.$$

Now we can calculate

$$A_C = PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{Fg}{LM} & \frac{g}{L} & \frac{-F}{M} \end{bmatrix},$$

and thus, if $F_C = [f_0, f_1, f_2, f_3]$

$$A_{CF} = A_C + [0, 0, 0, 1]^T F_C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(0 - f_0) & -\left(-\frac{Fg}{LM} - f_1\right) & -\left(-\frac{g}{L} - f_2\right) & -\left(\frac{F}{M} - f_3\right) \end{bmatrix}.$$

We want $\left[-(0 - f_0), -\left(-\frac{Fg}{LM} - f_1\right), -\left(-\frac{g}{L} - f_2\right), -\left(\frac{F}{M} - f_3\right)\right] = [-d_0, -d_1, -d_2, -d_3]$. Thus, we choose

$$F_C = [-1, \frac{-Fg}{LM} - 2, \frac{-g}{L} - 2, \frac{F}{M} - 2].$$

Transforming co-ordinates back gives

$$F = F_c P = \left[\frac{-M(g + 2L)}{L}, F - 2M, \frac{(g + L)^2 M}{gL}, \frac{2(g + L)M}{g}\right].$$

The system with feedback control is

$$\dot{X}(t) = (A + BF)X + Bu(t),$$

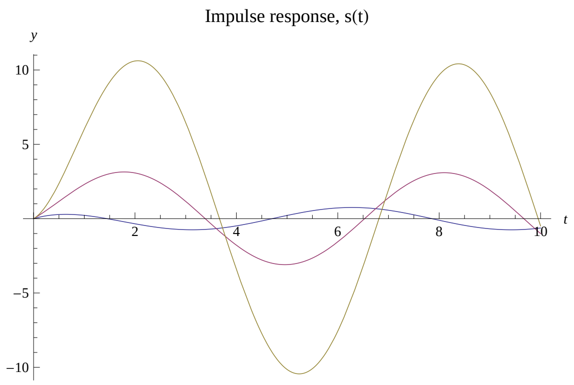
so using mathematica we have

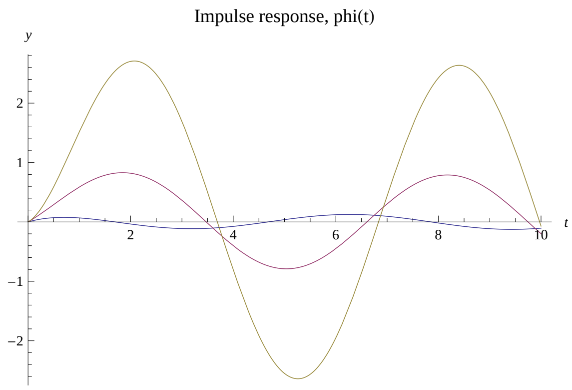
$$A + FB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{g+2L}{L} & -\frac{F}{M} + \frac{F-2M}{M} & \frac{(g+L)^2}{gL} & \frac{2(g+L)}{g} \\ 0 & 0 & 0 & 1 \\ -\frac{g}{L} & 0 & \frac{g}{L} & 0 \end{bmatrix}.$$

The solution to this equation is

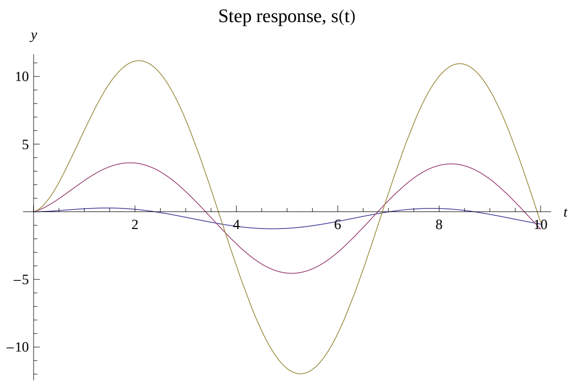
$$X(t) = e^{(A+FB)t} X_0 + \int_0^t e^{(A+FB)(t-s)} Bu(t) \; \mathrm{d}s \, .$$

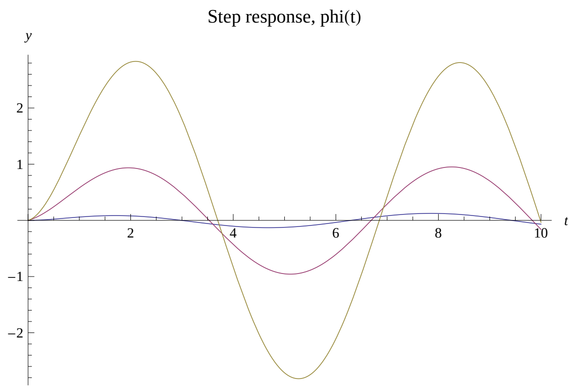
From the model, $s(t) = x_1(t)$ and $x_3(t) = s(t) + L\phi(t)$, which means $\phi(t) = \frac{x_3(t)-x_1(t)}{L}$. Numbers have been substituted from here to generate the graphs. I have taken $g = F = M = 1$ and $L = 2$. I will look at 3 initial conditions, $X_0^a = [0,0,0,0]^T$ which corresponds to $s(0) = 0$, $\dot{s}(0) = 0$, $\phi(0) = 0$ and $\dot{\phi}(0) = 0$, $X_0^b = [0,1,0,1]^T$ which corresponds to $s(0) = 0$, $\dot{s}(0) = 1$, $\phi(0) = 0$ and $\dot{\phi}(0) = 0$ and $X_0^c = [0,1,0,3]^T$ which corresponds to $s(0) = 0$, $\dot{s}(0) = 1$, $\phi(0) = 0$ and $\dot{\phi}(0) = 1$, and I will look at the impulse, step and periodic responses ($u(t) = \sin(t)$) of these systems. First the impulse responses:



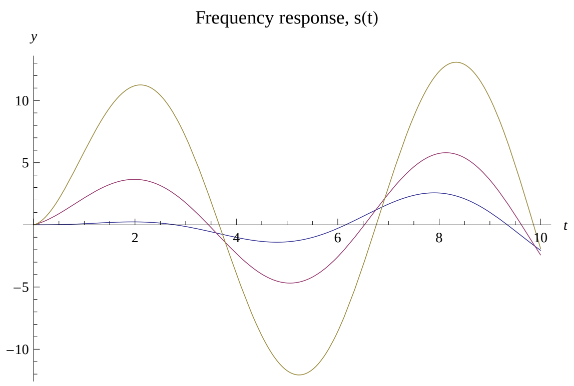


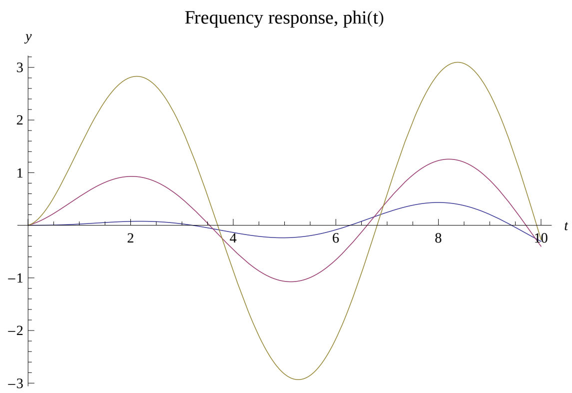
These are the step responses:





And these are the periodic responces:





These graphs look very similar, but careful inspection will reveal that they are not identical. This would be expected, since the system is being controlled so that the output behaves in a very particular way (the poles of the system are in the same positions).

Question 6

Let $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$. First, assume (A, C) is observable. Then there exists some T such that if for two initial conditions of the system x_0 and $\bar{x}_0$,

$$(y(t) =)\, Ce^{At}x_0 + \int_0^t e^{t-s}Bu(s)\ \mathrm{d}s\ = Ce^{At}\bar{x}_0 + \int_0^t e^{t-s}Bu(s)\ \mathrm{d}s\ (= \bar{y})$$

for all $t \in [0, T]$, then $x_0 = \bar{x}_0$. Thus, taking $u(t) = 0$ for all $t \in [0, T]$ and $\bar{x}_0 = 0$, then if $y(t) = Ce^{At}x_0 = 0$ for all $t \in [0, T]$, then $x_0 = 0$. By Question 2 we have

$$Ce^{At}x_0 = \sum_{i=0}^{n-1} \alpha_i(t)CA^i x_0,$$

so if $\sum_{i=0}^{n-1} \alpha_i(t)CA^i x_0 = 0$ for all $t \in [0, T]$, then $x_0 = 0$.

Now assume $\mathrm{Obs}(A, C)$ does not have full rank. Then there exists nonzero $a \in \mathbb{R}^n$ such that $\mathrm{Obs}(A, C)a = 0_{np\times 1}$, which implies $CA^i a = 0_{p\times 1}$ for all $i \in [0, \dots, n-1$. But then $\sum_{i=0}^{n-1} \alpha_i(t)CA^i a = \sum_{i=0}^{n-1} \alpha_i(t)0_{p\times 1} = 0_{p\times 1}$, so by the above reasoning, $a = 0$, which is a contradiction, as we assumed $a \neq 0$. So $\mathrm{Obs}(A, C)$ must have full rank. Next, assume

$\mathrm{Obs}(A, C)$ has full rank, and assume we know $y(t)$ and $u(t)$ for all $t \in [0, T]$. Rearranging and integrating $y(t) = Ce^{At}x_0 + \int_0^t e^{t-s}Bu(s)\ \mathrm{d}s$ over the known time and multiplying by $e^{A't}C'$ gives

$$\int_0^T e^{A't}C'Ce^{At}\ \mathrm{d}t\ x_0 = \int_0^T e^{A't}C'y(t) - e^{A't}C' \int_0^t e^{t-s}Bu(s)\ \mathrm{d}s\ \ \mathrm{d}t\ .$$

To show (A, C) is observable, we just need to show $\int_0^T e^{A't}C'Ce^{At}\ \mathrm{d}t$ is invertable, as if that is true, we can calculate x_0 by

$$x_0 = \left(\int_0^T e^{A't}C'Ce^{At}\ \mathrm{d}t\right)^{-1} \int_0^T e^{A't}C'y(t) - e^{A't}C' \int_0^t e^{t-s}Bu(s)\ \mathrm{d}s\ \ \mathrm{d}t\ .$$

For a contradiction, assume there exists nonzero $a \in \mathbb{R}^n$ such that $\int_0^T e^{A't}C'Ce^{At}\ \mathrm{d}t\ a = 0$ and define $b(t) = Ce^{At}a \in \mathbb{R}^p$. Then

$$0 = \int_0^T a'e^{A't}C'Ce^{At}a\ \mathrm{d}t\ = \int_0^T b(t)'b(t)\ \mathrm{d}t\ = \int_0^T \|b(t)\|^2\ \ \mathrm{d}t\ .$$

For this integral to be 0, $b(t) = Ce^{At}a$ must be 0 for all $t \in [0, T]$. Thus differentiating $b(t)$ $n-1$ times and evaluating at $t = 0$ gives n equations

$$Ca = 0$$

$$CAa = 0$$

$$\dots$$

$$CA^{n-1}a = 0.$$

Writing this in one matrix equation gives

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^{n-1} \end{bmatrix} a = \mathrm{Obs}(A, C)a = 0.$$

But $\mathrm{Obs}(A, C)$ has full rank, so this is only true if $a = 0$. We have a contradiction, so $\int_0^T e^{A't}C'Ce^{At}\, \mathrm{d}t\ a$ is invertable, and thus (A, C) is observable.