Question 1

Method 1

Fortunately, \( A \) is already in the controller form, i.e \( A = A_C \) and \( P = I \), for \( \mu_1 = 1 \) and \( \mu_2 = 3 \). This means \( \sigma_1 = \mu_1 = 1 \) and \( \sigma_2 = \mu_1 + \mu_2 = 4 \). The matrices \( A_m \) and \( B_m \) are then the 1st and 4th rows of \( A_C \) and \( B_C \), so

\[
A_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -3 & 4 \end{bmatrix}
\]

and

\[
B_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

If we want the system to have eigenvalues at \(-1 \pm i, -2 \pm i\), then we want the characteristic equation of the system to be \((s + 1)^2 + 1)((s + 2)^2 + 1) = 10 + 18s + 15s^2 + 6s^3 + s^4\). A matrix with this as a characteristic equations is

\[
A_d = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -18 & -15 & -6 \end{bmatrix}.
\]

Thus,

\[
A_{dm} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -10 & -18 & -15 & -6 \end{bmatrix}.
\]

We have that \( F_C = B_m^{-1}(A_{dm} - A_m) \), so

\[
F = F_C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix} 0 & 1 & 0 & 0 \\ -10 & -18 & -15 & -6 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -3 & 4 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -11 & -19 & -12 & -10 \end{bmatrix}.
\]

This is the desired feedback control. to verify this,

\[
A + BF = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -11 & -19 & -12 & -10 \end{bmatrix}.
\]
which has the desired eigenvalues.

**Method 2**

For each of the desired eigenvalues, we need to find the null space of the matrices \([\lambda_i I - A, B]\). Using Mathematica, we have

\[
((-1 + i)I - A, B) = \begin{bmatrix}
4 - 5I & 57 + 62I \\
1 + 9I & 4 - 5I \\
-10 - 8I & 1 + 9I \\
18 - 2I & -10 - 8I \\
0 & 123 \\
123 & 0
\end{bmatrix} = \begin{bmatrix} M_1 \\ -D_1 \end{bmatrix},
\]

and

\[
((-1 + i)I - A, B) = \begin{bmatrix}
-11 - 81I & 2661 + 1371I \\
103 + 151I & -11 - 81I \\
-357 - 199I & 103 + 151I \\
913 + 41I & -357 - 199I \\
0 & 6682 \\
6682 & 0
\end{bmatrix} = \begin{bmatrix} M_2 \\ -D_2 \end{bmatrix},
\]

and

\[
((-1 - i)I - A, B) = \begin{bmatrix} M_3 \\ -D_3 \end{bmatrix} = \begin{bmatrix} M_3^* \\ -D_3^* \end{bmatrix},
\]

and

\[
((-2 - i)I - A, B) = \begin{bmatrix} M_4 \\ -D_4 \end{bmatrix} = \begin{bmatrix} M_4^* \\ -D_4^* \end{bmatrix}.
\]

We have that the eigenvectors satisfy \(v_i = M_i a_i\), so each eigenvector is linearly dependent on \(M_i\). Thus, we will choose eigenvectors from the columns of the \(M_i\), remembering that \(v_1 = v_3^*\) and \(v_2 = v_4^*\). We will simply choose \(a_i = [1, 0]^T\) for all \(i\) and which gives

\[
V = \begin{bmatrix}
4 - 5I & 4 + 5I & -11 - 81I & -11 + 81I \\
1 + 9I & 1 - 9I & 103 + 151I & 103 - 151I \\
-10 - 8I & -10 + 8I & -357 - 199I & -357 + 199I \\
18 - 2I & 18 + 2I & 913 + 41I & 913 - 41I
\end{bmatrix}.
\]

This means

\[
D = [D_i a_i] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -123 & -123 & -6682 & -6682 \end{bmatrix}.
\]
Finally, we have that
\[
F = DV^{-1} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-11 & -19 & -12 & -10 
\end{bmatrix}.
\]
This agrees with the previous method.

**Method 3**

This is the direct method. Let
\[
F = \begin{bmatrix}
f_1 & f_2 & f_3 & f_4 \\
f_5 & f_6 & f_7 & f_8
\end{bmatrix}.
\]

\[
A + BF = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 1 & -3 & 4
\end{bmatrix} + \begin{bmatrix}
f_1 & f_2 & f_3 & f_4 \\
0 & 0 & 0 & 0 \\
f_5 & f_6 & f_7 & f_8
\end{bmatrix} = \begin{bmatrix}
f_1 & f_2 + 1 & f_3 & f_4 \\
0 & 0 & 1 & 0 \\
f_5 + 1 & f_6 + 1 & f_7 - 3 & f_8 + 4
\end{bmatrix}.
\]

Now,
\[
\det(A + BF - sI) = (f_1 - s)(-s) \begin{vmatrix}
-s & 1 & 0 \\
0 & -s & 1 \\
f_6 + 1 & f_7 - 3 & f_8 + 4 - s
\end{vmatrix} - (f_2 + 1)(-s) \begin{vmatrix}
0 & 1 & 0 \\
0 & -s & 1 \\
f_5 + 1 & f_7 - 3 & f_8 + 4 - s
\end{vmatrix} + (f_3)(-s) \begin{vmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
f_5 + 1 & f_6 + 1 & f_8 + 4 - s
\end{vmatrix} \begin{vmatrix}
f_1 & f_2 + 1 & f_3 & f_4 \\
f_5 & f_6 & f_7 & f_8
\end{vmatrix} = (f_1 - s)(-s) \begin{vmatrix}
-s & 1 & 0 \\
f_7 - 3 & f_8 + 4 - s & -s \\
0 & f_6 + 1 & f_8 + 4 - s & 0
\end{vmatrix} + (f_2 + 1) \begin{vmatrix}
f_5 + 1 & f_8 + 4 - s & 0 \\
f_5 + 1 & f_8 + 4 - s & -s f_4
\end{vmatrix} + (f_3) \begin{vmatrix}
f_1 & f_2 + 1 & f_3 & f_4 \\
f_5 & f_6 & f_7 & f_8
\end{vmatrix} = (f_1 - s)(-s)\left(-s(f_8 + 4 - s) - (f_7 - 3) + (f_6 + 1)\right)
\]
\[
- (f_2 + 1)(f_5 + 1) - sf_3(f_5 + 1) - s^2 f_4(f_5 + 1)
\]
\[
= (f_8 + 4)f_1 s^2 - s^3(f_8 + 4) - f_1 s^3 + s^4 + sf_1(f_7 - 3) - s^2(f_7 - 3) - f_1(f_6 + 1)
\]
\[
- s(f_6 + 1) - (f_2 + 1)(f_5 + 1) - sf_3(f_5 + 1) - s^2 f_4(f_5 + 1)
\]
\[
= - f_1(f_6 + 1) - (f_2 + 1)(f_5 + 1) + s(f_1(f_7 - 3) - (f_6 + 1) - f_3(f_5 + 1))
\]
\[
+ s^2((f_8 + 4)f_1 - (f_7 - 3) - f_4(f_5 + 1)) + s^3(-f_8 + 4 - f_1) + s^4
\]
Equating coefficients of this equation with the characteristic equation of a matrix with the desired eigenvalues \((10 + 18s + 15s^2 + 6s^3 + s^4)\) gives 4 equations,

\[
10 = -f_1(f_6 + 1) - (f_2 + 1)(f_5 + 1)
\]
\[
18 = f_1(f_7 - 3) - (f_6 + 1) - f_3(f_5 + 1)
\]
\[
15 = (f_8 + 4)f_1 - (f_7 - 3) - f_4(f_5 + 1)
\]
\[
6 = -(f_8 + 4) - f_1
\]

We have 8 unknowns and only 4 equations. Initially, choose \(f_1 = 0\). Then by the 4th equation, \(f_8 = -10\). The remaining equations become:

\[-10 = (f_2 + 1)(f_5 + 1)\]
\[19 = -f_6 - f_3(f_5 + 1)\]
\[-12 = f_7 + f_4(f_5 + 1)\]

We still have more equations than unknowns, so choose \(f_2 = 0\). The first of these equations gives \(f_5 = -11\) and the remaining equations become:

\[19 = -f_6 + 10f_3\]
\[-12 = f_7 - 10f_4\]

These last two equations have no common unknowns, and we have two more equations than unknowns, so choose \(f_3 = f_4 = 0\). The first of these equations gives \(f_6 = -19\) and the remaining equation gives \(f_7 = -12\). Thus, we have

\[
F = \begin{bmatrix}
    f_1 & f_2 & f_3 & f_4 \\
    f_5 & f_6 & f_7 & f_8 
\end{bmatrix} = \begin{bmatrix}
    0 & 0 & 0 & 0 \\
    -11 & -19 & -12 & -10 
\end{bmatrix}
\]
as a solution. This agrees with the previous results.

**Question 2**

Define \(\dot{x}(t) = e^{\alpha t}x(t)\) and \(\dot{u}(t) = e^{\alpha t}u(t)\). Then the performance index becomes

\[
\tilde{J}(u) = \int_0^\infty \dot{x}^TQ\dot{x} + \dot{u}^TR\dot{u} \, dx
\]

and the system becomes

\[
\dot{x} = \alpha e^{\alpha t}x(t) + e^{\alpha t}\dot{x}(t) = \alpha e^{\alpha t}x(t) + e^{\alpha t}Ax(t) + e^{\alpha t}Bu(t) = e^{\alpha t}(\alpha I + A)x(t) + e^{\alpha t}Bu(t)
\]
\[ (\alpha I + A)\ddot{x}(t) + B\ddot{u}(t) = \dot{A}\ddot{x}(t) + B\ddot{u}(t) \]

and
\[ \tilde{y}(t) = e^{\alpha t}y(t) = e^{\alpha t}Cu(t) = Cu(t). \]

Now, this system with transformed variables is now a system where \( Q \) and \( R \) are constant, so we may apply the results from the textbook. In particular we have that the optimal control for this system is
\[ \tilde{u}^* = F^*u = -R^{-1}B^TPc\dot{x}(t) \]
where \( P_c^* \) is the solution to the algebraic Riccati equation
\[ \tilde{A}^TPc + Pc\tilde{A} - PcBR^{-1}B^TPc + Q = 0. \]

This implies
\[ e^{\alpha t}u^* = e^{\alpha t}F^*u = -R^{-1}B^TPc\dot{x}(t)e^{\alpha t} \]
and as \( e^{\alpha t} \) for all \( t \in \mathbb{R} \), we can divide through by this and we have
\[ u^* = F^*u = -R^{-1}B^TP_c^*x(t). \]

Furthermore, the Riccati equation can be simplified to give
\[ (\alpha I + A)^TPc + Pc(\alpha I + A) - PcBR^{-1}B^TPc + Q = 0 \]
which implies
\[ 2\alpha P_c + A^TPc + PcA - PcBR^{-1}B^TP_c + Q = 0. \]

Clearly, the solution \( P_c^* \) to this equation is a constant matrix, as no term in the Riccati equation is a function of time. This means that our feedback control law \( F^*u(t) = -R^{-1}B^TP_c^*x(t) \) is a fixed control law.

**Question 3**

a) First notice that \( x(k + l) = A^l x(k) + \sum_{i=0}^{l-1} A^i Bu(k + i) + A^i Eq(k + i) \). We can prove this with induction. For \( i = 1 \) we have \( x(k + 1) = Ax(k) + Bu(k) + Eq(k) \) as expected, so assume \( x(k + j) = A^j x(k) + \sum_{i=0}^{j-1} A^i Bu(k + i) + A^i Eq(k + i) \) is true for \( l = j \), and consider \( l = j + 1 \). Then
\[
\begin{align*}
  x(k + j + 1) &= Ax(k + j) + Bu(k + j) + Eq(k + j) \\
  &= A \left( A^j x(k) + \sum_{i=0}^{j-1} A^i Bu(k + i) + A^i Eq(k + i) \right) + Bu(k + j) + Eq(k + j)
\end{align*}
\]
\[ A^{i+1}x(k) + \sum_{i=0}^{j+1} A^i Bu(k+i) + A^i Eq(k+i) \]

which finished the proof by induction. We then have that

\[ y(k+l) = CA^l x(k) + \sum_{i=0}^{l-1} CA^i Bu(k+i) + CA^i Eq(k+i), \]

so for \( y(k+l) \) to be not effected by \( q \), we must have \( 0 = \sum_{i=0}^{l-1} CA^i E \). For no \( y(k) \) to be not effected by \( q \) for any time until \( y(k+l) \), then we must have \( CA^i E = 0 \) for all \( i \in [0, ..., l-1] \).

By the Cayley Hamilton Theorem, for any \( l \geq n \), \( CA^l E \) is a linear combination of \( CA^i E \) for \( i \in [0, ..., l-1] \). Thus, \( y \) will not be effected by \( q \) for all time if it is not effected at any time step from \( k \) until \( k + n - 1 \), which will happen if \( CA^i E = 0 \) for all \( i \in [0, ..., n-1] \).

If we arrange these equations into one matrix equation we have

\[
0 = \begin{bmatrix} CE \\ \vdots \\ CA^{n-1}E \end{bmatrix} = \text{Obs}(A,C)E.
\]

Thus, the effects of \( q \) will be completely eliminated if \( \text{Obs}(A,C)E = 0 \).

b) For \( A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) and \( C = [1, 1] \), we have \( \text{Obs}(A,C) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \). As \( q \in \mathbb{R}^r \), \( E \in \mathbb{R}^{2 \times r} \).

However, \( \text{Obs}(A,C)E = 0_{2 \times r} \) implies \( \text{Obs}(A,C)E_i = 0_{2 \times 1} \), where \( E_i \) is the \( i^{th} \) column of \( E \). This means that each column of \( E \) must be a scalar multiple of the eigenvector of \( \text{Obs}(A,C) \) for the eigenvalue 0, which is the vector \( [-1, 1]^T \). So \( E \) is characterised by \( [x^T, -x^T]^T \), where \( x \) is any element in \( \mathbb{R}^r \).

c) The z-transform of \( x(k) \) is

\[ zX(x) = AX(z) + BU(z) + EQ(z). \]

Rearranging for \( X(z) \) gives

\[ X(z) = (BU(z) + EQ(z))(zI - A)^{-1} \]

and recalling that the z-transform of a step function is \( Q(z) = \frac{z}{z-1} \) gives

\[ X(z) = (BU(z)(z-1) + Ez)(z-1)^{-1}(zI - A)^{-1}. \]

Thus,

\[ Y(z) = C(BU(z)(z-1) + Ez)(z-1)^{-1}(zI - A)^{-1}. \]
Now, if \( q \) was absent, then the output would be \( \bar{Y}(z) = C(BU(z))(zI - A)^{-1} \). We want the error introduced by \( q \) to go to 0 asymptotically, so consider

\[
Y(z) - \bar{Y}(z) = C(BU(z)(z-1) + E\tilde{z})(z-1)^{-1}(zI-A)^{-1} - C(BU(z)(z-1))(z-1)^{-1}(zI-A)^{-1}
\]

\[= CEz(z-1)^{-1}(zI-A)^{-1}.
\]

By the final value theorem for the z-transform, if the eigenvalues of \( A \) are have magnitude strictly less than 1,

\[y(\infty) - \bar{y}(\infty) = \lim_{z \to 1} (z - 1)(Y(z) - \bar{Y}(z))\]

\[= \lim_{z \to 1} CEz(zI-A)^{-1} = CE(I-A)^{-1}.
\]

Thus, if the eigenvalues of \( A \) are strictly less than 1, there will be a constant asymptotic error of \( CE(I-A)^{-1} \).

**Question 4**

This example aims to find the Kalman decomposition of the system described by

\[
\dot{x} = Ax + Bu \quad \text{and} \quad y = Cx
\]

with

\[
A = \begin{bmatrix}
0 & -1 & 1 \\
1 & -2 & 1 \\
0 & 1 & -1
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
1 & 0 \\
1 & 1 \\
1 & 2
\end{bmatrix},
\]

and

\[C = [0, 1, 0].\]

First, we must find the controllability and observability matrices for this system. \( n = 3 \), so we have that

\[
\text{Con}(A, B) = [B, AB, A^2B] = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & -1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & -1 & 0 & 1
\end{bmatrix}.
\]

We also have

\[
\text{Obs}(A, C) = \begin{bmatrix}
C \\
CA \\
CA^2
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
1 & -2 & 1 \\
-2 & 4 & -2
\end{bmatrix}.
\]
$n_o$ is the dimension of the observable subspace $R_o$ so we have
\[ n_o = \text{rank}(Obs) = 2 \]
and thus the dimension of the uncontrollable subspace $R_{\bar{o}}$ is $n_{\bar{o}} = n - n_o = 1$. The null space of $Obs(A, B)$ is $[1, 0, -1]$, which is the unobservable subspace. $n_r$ is the dimension of the controllable subspace $R_r$ so we have
\[ n_r = \text{rank}(Con) = 2. \]
The null space of the controllability matrix is $[1, -2, 1]$, which means the two vectors mutually perpendicular to this form $R_r$. In particular, $[1, 0, -1]$ is perpendicular to $[1, -2, 1]$, so we choose the vectors to describe $R_r$ as $R_r = \{[1, 0, -1], [1, 1, 1]\}$. Thus, it is clear what the observable and controllable subspace is, $[1, 1, 1]$, the dimension of which is $n_{r\bar{o}}$. Also, the only vector in the unobservable and uncontrollable subspace is the 0 vector, as the unobservable subspace is a subset of the controllable subspace. This means $A_{24}, A_{43}, A_{44}$ have dimensions of 0, i.e. they do not exist.

Now, $Q$ is defined as
\[ Q = [v_1, \ldots v_{n_r}, Q_N, \hat{v}_1, \ldots, \hat{v}_{n_o-n_{r\bar{o}}}] \]
where the first $n_r - n_{r\bar{o}} = 1$ columns are the basis of the controllable and observable space, $[1, 1, 1]$, the next $n_{\bar{o}} = 1$ columns are the basis of the controllable but unobservable subspace $[1, 0, -1]$, the last $n_o - n_{r\bar{o}} = 0$ vectors are the basis of the unobservable and uncontrollable subspace (which is empty in this question). $Q_N$ is chosen so that $Q$ is nonsingular. In this example $Q_N$ was chosen to be $[0, 0, 1]$, which gave
\[ Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}. \]
$Q$ is non singular, as $\det Q = -1(1 \cdot 1 - 0 \cdot 1) = -1$. From here we can calculate
\[ \hat{A} = Q^{-1}AQ = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \]
\[ \hat{B} = Q^{-1}BQ = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} \]
and
\[ \hat{C} = Q^{-1}CQ = [1, 0, 0] = [C_1, 0, C_3]. \]
Thus, the matrix $A_{11} = 0$, which corresponds to the eigenvalue 0, is controllable and observable. We also have $(A_c, B_c)$, where

$$A_c = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

and

$$B_c = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

is controllable, which implies the matrix $A_{22} = -1$, which corresponds to the eigenvalue -1, is controllable but unobservable. Finally, we also have $(A_o, C_o)$, where

$$A_o = \begin{bmatrix} A_{11} & A_{13} \\ A_{21} & 0 & A_{33} \end{bmatrix}$$

and

$$C_o = [C_1, C_3]$$

is observable, which implies the matrix $A_{33} = -2$, which corresponds to the eigenvalue -2, is observable but uncontrollable. As pointed out above, there are no eigenvalues which are neither controllable nor observable.

**Question 5**

a) Before proving this, note that if $v$ is a left eigenvector of $A$ with eigenvalue $\lambda$, then $v$ is a left eigenvector of $A$ with eigenvalue $e^{\lambda T}$. To see this calculate

$$vA = ve^{\lambda T} = v \sum_{m=0}^{\infty} \frac{1}{m!}(AT)^m = \sum_{m=0}^{\infty} \frac{1}{m!}vA^mT^m = \sum_{m=0}^{\infty} \frac{1}{m!}v\lambda^mT^m = ve^{\lambda T}.$$ 

The same holds for right eigenvectors.

Another theorem that will be used in this proof is one of the PHB tests, specifically that

$$\text{rank}([\lambda I - A, B]) < n$$

for an eigenvalue $\lambda$ of $A$ if and only if $(A, B)$ is uncontrollable, and

$$\text{rank} \left( \begin{bmatrix} c\lambda I - A \\ C \end{bmatrix} \right) < n$$

for an eigenvalue $\lambda$ of $A$ if and only if $(A, C)$ is unobservable.
First, we prove that if \( \Im(\lambda_i - \lambda_j) = \frac{2\pi k}{T} \) for any eigenvalues \( \Re(\lambda_i - \lambda_j) = 0 \), then \((\bar{A}, \bar{B})\) is uncontrollable and \( \bar{A}, \bar{B} \) is unobservable. If the antecedent is true, then let \( \lambda_i = \lambda_j + \frac{2\pi k}{T} \) and consider the (left) eigenvectors \( v_i \) and \( v_j \) of \( A \). By the above argument, \( v_i \) and \( v_j \) are an eigenvectors of \( \bar{A} \) with eigenvalues \( e^{\lambda_i T} \) and \( e^{\lambda_j T} \). However, due to the fact that \( e^z = e^{z + 2\pi ki} \) for any \( z \in \mathbb{C} \) and \( k \in \mathbb{Z} \), \( e^{\lambda_i T} = e^{\lambda_j T} \), i.e. the eigenvalue \( e^{\lambda_i T} \) has arithmetic multiplicity of at least 2. But this means \( e^{\lambda_i T} I - \bar{A} \) has 0 as an eigenvalue with multiplicity of at least 2, so \( \text{rank}(e^{\lambda_i T} I - \bar{A}) \geq n - 2 \). Adding another column to this matrix can increase the rank by no more than 1, so \( \text{rank}([e^{\lambda_i T} I - \bar{A}, \bar{B}]) \leq n - 1 < n \). Thus, by the PHB test, \((\bar{A}, \bar{B})\) is uncontrollable.

Similarly if \( v_i \) and \( v_j \) are eigenvectors of \( A \), then by the above argument, \( v_i \) and \( v_j \) are an eigenvectors of \( \bar{A} \) with eigenvalues \( e^{\lambda_i T} \) and \( e^{\lambda_j T} \). Again, the eigenvalue \( e^{\lambda_i T} \) has arithmetic multiplicity of at least 2 and \( \text{rank}(e^{\lambda_i T} I - \bar{A}) \leq n - 2 \). Adding another row to this matrix can increase the rank by no more than 1, so \( \text{rank}([e^{\lambda_i T} I - \bar{A}, \bar{B}]) \leq n - 1 < n \). Thus, again, by the PHB test, \((\bar{A}, \bar{C})\) is unobservable.

Now assume if \( \Im(\lambda_i - \lambda_j) \neq \frac{2\pi k}{T} \) for any eigenvalues such that \( \Re(\lambda_i - \lambda_j) = 0 \). We know that \((A, B)\) is controllable, so by the PHB test, for every eigenvalue \( \lambda_i \) of \( A \), \( \text{rank}([\lambda_i I - A, B]) = n \). However, as \( \lambda_i \) is an eigenvector, \( \text{rank}(\lambda_i I - A) < n \), and as adding another column to this matrix can increase the rank by no more than 1, then \( \text{rank}(\lambda_i I - A) = n - 1 \). Thus, the multiplicity of every eigenvalue of \( A \) can be no more than 1, i.e. every eigenvalue of \( A \) is unique.

Furthermore, \( B \) is linearly independent of \( \lambda_i I - A \) for each \( i \), which means for every \( n - 1 \) set of eigenvectors of \( A, B \) is linearly independent of that set. As \( v_j(\lambda_i I - A) = \lambda_i v_i - \lambda_j v_j \), the eigenvectors of \( \lambda_i I - A \) are the eigenvectors of \( A \), however they now correspond to the eigenvalues \( \lambda_i - \lambda_j \). As \( A \) and \( \bar{A} \) have the same eigenvectors, for every \( n - 1 \) set of eigenvectors of \( \bar{A} \), \( B \) is linearly independent of that set. Thus we may define \( \alpha_i \neq 0 \) such that

\[
B = \sum_{i=1}^{n} \alpha_i v_i.
\]

Also, as \( \int_{0}^{T} e^{At} \, dt \) is a non singular matrix for \( T > 0 \) with the same eigenvectors as \( A \), then

\[
\bar{B} = \int_{0}^{T} e^{At} \, dt \, B = \sum_{i=1}^{n} \alpha_i \int_{0}^{T} e^{At} \, dt \, v_i = \sum_{i=1}^{n} \mu_i \alpha_i v_i = \sum_{i=1}^{n} \beta_i v_i
\]

where \( \mu_i \) are the eigenvalues of \( \int_{0}^{T} e^{At} \, dt \). Note that \( \mu_i \alpha_i = \beta_i \neq 0 \), as \( \alpha_i \neq 0 \) and \( \mu_i \neq 0 \), as \( \int_{0}^{T} e^{At} \, dt \) is nonsingular. Thus, \( \bar{B} \) is also has the property that for every \( n - 1 \) set of
eigenvectors of $\hat{A}$, $\hat{B}$ is linearly independent of that set.

Now, by assumption, whenever $\Re(\lambda_i - \lambda_j) = 0$, then $\Im(\lambda_i - \lambda_j) \neq \frac{2\pi k}{T}$, so there are no eigenvalues of $A$ such that $\lambda_i = \lambda_j + \frac{2\pi k}{T}$ for any $k \in \mathbb{Z} \setminus \{0\}$, and hence for every pair of eigenvalue such that $i \neq j$, $e^{\lambda_i T} \neq e^{\lambda_j T}$. Thus, every eigenvalue of $A$ has multiplicity of 1. Thus, for every eigenvalue, $\text{rank}(e^{\lambda_i T} - A) = n - 1$. Now, as $\hat{B}$ is has the property that for every $n - 1$ set of eigenvectors of $\hat{A}$, $\hat{B}$ is linearly independent of that set, then for any eigenvalue,

$$\text{rank}([e^{\lambda_i T} I - A, B]) = \text{rank}(e^{\lambda_i T} I - A) + 1 = n.$$ 

As this holds for any eigenvalue, by the PHB test, $(A, B)$ is controllable.

Similarly, we know that $(A, C)$ is observable, so by the PHB test, for every eigenvalue $\lambda_i$ of $A$,

$$\text{rank} \left( \begin{bmatrix} cI - A \\ C \end{bmatrix} \right) = n.$$ 

However, as $\lambda_i$ is an eigenvector, $\text{rank}(\lambda_i I - A) < n$ and as adding another column to this matrix can increase the rank by no more than 1, then $\text{rank}(\lambda_i I - A) = n - 1$. Thus, $C$ is linearly independent of $\lambda_i I - A$ for each $i$, which means for every $n - 1$ set of eigenvectors of $A$, $C$ is linearly independent of that set. As $v_j(\lambda_i I - A) = \lambda_i v_i - \lambda_j v_j$, the eigenvectors of $\lambda_i I - A$ are the eigenvectors of $A$, however they now correspond to the eigenvalues $\lambda_i - \lambda_j$. As $A$ and $\hat{A}$ have the same eigenvectors, for every $n - 1$ set of eigenvectors of $\hat{A}$, $C$ is linearly independent of that set. Furthermore, $C = \hat{C}$ so the same property holds for $\hat{C}$.

We still have that every eigenvalue of $\hat{A}$ has multiplicity of 1. Thus, for every eigenvalue, $\text{rank}(e^{\lambda_i T} I - A) = n - 1$. Now, as $\hat{C}$ is has the property that for every $n - 1$ set of eigenvectors of $\hat{A}$, $\hat{C}$ is linearly independent of that set, then for any eigenvalue,

$$\text{rank} \left( \begin{bmatrix} cI - A \\ C \end{bmatrix} \right) = \text{rank}(e^{\lambda_i T} I - A) + 1 = n.$$ 

As this holds for any eigenvalue, by the PHB test, $(A, C)$ is observable.

b) First, note that the double integrator is exactly the system described in the first part of this question, so the results of that part can be applied here. The eigenvalues of $A$ in the first case, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ are $\lambda_1 = \lambda_2 = 0$. Thus for any $\lambda_i - \lambda_j$, $\Re(\lambda_i - \lambda_j) = 0$ but $\Im(\lambda_i - \lambda_j) = 0$, which is not equal to $\frac{2\pi k}{T}$ for any $k \in \mathbb{Z} \setminus \{0\}$. Thus, by the above
theorem, \((A, B)\) is controllable and \((A, C)\) is observable for any \(T > 0\).

In first case, where \(A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), the eigenvalues of \(A\) are \(\lambda_1 = i\) and \(\lambda_2 = -i\). Again, for any \(i, j \in \{1, 2\}\) we have that \(\Re(\lambda_1 - \lambda_2) = 0\), but if \(i = j\) then \(\Im(\lambda_i - \lambda_j) = 0\), which again is not equal to \(\frac{2\pi k}{T}\) for any \(k \in \mathbb{Z} \setminus \{0\}\). However, if \(i \neq j\) then \(\Im(\lambda_i - \lambda_j) = \pm 2\). Thus, by the above theorem, \((A, B)\) is uncontrollable and \((A, C)\) is unobservable for \(T = k\pi\) for any \(k \in \mathbb{N}\).