MATH4406 Homework 4

Mitch Gooding

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Question 1

Method 1

Fortunately, A is already in the controller form, i.e $A = A_C$ and $P = I$, for $\mu_1 = 1$ and $\mu_2 = 3$. This means $\sigma_1 = \mu_1 = 1$ and $\sigma_2 = \mu_1 + \mu_2 = 4$. The matrices A_m and B_m are then the 1st and 4th rows of A_C and B_C, so

$$A_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -3 & 4 \end{bmatrix}$$

and

$$B_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If we want the system to have eigenvalues at $-1 \pm i, -2 \pm i$, then we want the characteristic equation of the system to be $((s+1)^2+1)((s+2)^2+1) = 10 + 18s + 15s^2 + 6s^3 + s^4$. A matrix with this as a characteristic equations is

$$A_d = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -18 & -15 & -6 \end{bmatrix}.$$

Thus,

$$A_{dm} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -10 & -18 & -15 & -6 \end{bmatrix}$$

We have that $F_C = B_m^{-1}(A_{dm} - A_m)$, so

$$F = F_C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -10 & -18 & -15 & -6 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -3 & 4 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -11 & -19 & -12 & -10 \end{bmatrix}.$$

This is the desired feedback control. to verify this,

$$A + BF = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -11 & -19 & -12 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -11 & -19 & -12 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -18 & -15 & -6 \end{bmatrix} = A_d$$

which has the desired eigenvalues.

Method 2

For each of the desired eigenvalues, we need to find the null space of the matrices $[\lambda_i I - A, B]$. Using Mathematica, we have

$$[(-1+i)I - A, B] = \begin{bmatrix} 4-5I & 57+62I \\ 1+9I & 4-5I \\ -10-8I & 1+9I \\ 18-2I & -10-8I \\ 0 & 123 \\ 123 & 0 \end{bmatrix} = \begin{bmatrix} M_1 \\ -D_1 \end{bmatrix}$$

and

$$[(-1+i)I - A, B] = \begin{bmatrix} -11-81I & 2661+1371I \\ 103+151I & -11-81I \\ -357-199I & 103+151I \\ 913+41I & -357-199I \\ 0 & 6682 \\ 6682 & 0 \end{bmatrix} = \begin{bmatrix} M_2 \\ -D_2 \end{bmatrix},$$

and

$$[(-1-i)I - A, B] = \begin{bmatrix} M_3 \\ -D_3 \end{bmatrix} = \begin{bmatrix} M_1^* \\ -D_1^* \end{bmatrix}$$

and

$$[(-2-i)I - A, B] = \begin{bmatrix} M_4 \\ -D_4 \end{bmatrix} = \begin{bmatrix} M_2^* \\ -D_2^* \end{bmatrix}.$$

We have that the eigenvectors satisfy $v_i = M_i a_i$, so each eigenvector is linearly dependent on M_i. Thus, we will choose eigenvectors from the columns of the M_i, remembering that $v_1 = v_3^*$ and $v_2 = v_4^*$. We will simply choose $a_i = [1, 0]^T$ for all i and which gives

$$V = \begin{bmatrix} 4-5I & 4+5I & -11-81I & -11+81I \\ 1+9I & 1-9I & 103+151I & 103-151I \\ -10-8I & -10+8I & -357-199I & -357+199I \\ 18-2I & 18+2I & 913+41I & 913-41I \end{bmatrix}.$$

This means

$$D = [D_i a_i] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -123 & -123 & -6682 & -6682 \end{bmatrix}.$$

Finally, we have that

$$F = DV^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -11 & -19 & -12 & -10 \end{bmatrix}.$$

This agrees with the previous method.

Method 3

This is the direct method. Let

$$F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ f_5 & f_6 & f_7 & f_8 \end{bmatrix}.$$

$$A + BF = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -3 & 4 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f_5 & f_6 & f_7 & f_8 \end{bmatrix} = \begin{bmatrix} f_1 & f_2+1 & f_3 & f_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ f_5+1 & f_6+1 & f_7-3 & f_8+4 \end{bmatrix}.$$

Now,

$$\begin{aligned}
&\det(A + BF - sI) \\
&\quad =(f_1 - s)\begin{vmatrix} -s & 1 & 0 \\ 0 & -s & 1 \\ f_6+1 & f_7-3 & f_8+4-s \end{vmatrix} \\
&\qquad - (f_2+1)\begin{vmatrix} 0 & 1 & 0 \\ 0 & -s & 1 \\ f_5+1 & f_7-3 & f_8+4-s \end{vmatrix} + f_3\begin{vmatrix} 0 & -s & 0 \\ 0 & 0 & 1 \\ f_5+1 & f_6+1 & f_8+4-s \end{vmatrix} \\
&\qquad - f_4\begin{vmatrix} 0 & -s & 1 \\ 0 & 0 & -s \\ f_5+1 & f_6+1 & f_7-3 \end{vmatrix} \\
&\quad =(f_1 - s)\left(-s\begin{vmatrix} -s & 1 \\ f_7-3 & f_8+4-s \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ f_6+1 & f_8+4-s \end{vmatrix}\right) \\
&\qquad + (f_2+1)\begin{vmatrix} 0 & 1 \\ f_5+1 & f_8+4-s \end{vmatrix} + sf_3\begin{vmatrix} 0 & 1 \\ f_5+1 & f_8+4-s \end{vmatrix} - sf_4\begin{vmatrix} 0 & -s \\ f_5+1 & f_6+1 \end{vmatrix} \\
&\quad =(f_1 - s)\left(-s(-s(f_8+4-s) - (f_7-3)) + (f_6+1)\right) \\
&\qquad - (f_2+1)(f_5+1) - sf_3(f_5+1) - s^2 f_4(f_5+1) \\
&\quad =(f_8+4)f_1 s^2 - s^3(f_8+4) - f_1 s^3 + s^4 + sf_1(f_7-3) - s^2(f_7-3) - f_1(f_6+1) \\
&\qquad - s(f_6+1) - (f_2+1)(f_5+1) - sf_3(f_5+1) - s^2 f_4(f_5+1) \\
&\quad = - f_1(f_6+1) - (f_2+1)(f_5+1) + s(f_1(f_7-3) - (f_6+1) - f_3(f_5+1)) \\
&\qquad + s^2((f_8+4)f_1 - (f_7-3) - f_4(f_5+1)) + s^3(-(f_8+4) - f_1) + s^4
\end{aligned}$$

Equating coefficients of this equation with the characteristic equation of a matrix with the desired eigenvalues $(10 + 18s + 15s^2 + 6s^3 + s^4)$ gives 4 equations,

$$10 = -f_1(f_6 + 1) - (f_2 + 1)(f_5 + 1)$$

$$18 = f_1(f_7 - 3) - (f_6 + 1) - f_3(f_5 + 1)$$

$$15 = (f_8 + 4)f_1 - (f_7 - 3) - f_4(f_5 + 1)$$

$$6 = -(f_8 + 4) - f_1$$

We have 8 unknowns and only 4 equations. Initially, choose $f_1 = 0$. Then by the 4th equation, $f_8 = -10$. The remaining equations become:

$$-10 = (f_2 + 1)(f_5 + 1)$$

$$19 = -f_6 - f_3(f_5 + 1)$$

$$-12 = f_7 + f_4(f_5 + 1)$$

We still have more equations than unknowns, so choose $f_2 = 0$. The first of these equations gives $f_5 = -11$ and the remaining equations become:

$$19 = -f_6 + 10f_3$$

$$-12 = f_7 - 10f_4$$

These last two equations have no common unknowns, and we have two more equations than unknowns, so choose $f_3 = f_4 = 0$. The first of these equations gives $f_6 = -19$ and the remaining equation gives $f_7 = -12$. Thus, we have

$$F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ f_5 & f_6 & f_7 & f_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -11 & -19 & -12 & -10 \end{bmatrix}$$

as a solution. This agrees with the previous results.

Question 2

Define $\tilde{x}(t) = e^{\alpha t}x(t)$ and $\tilde{u}(t) = e^{\alpha t}u(t)$. Then the performance index becomes

$$\tilde{J}(u) = \int_0^\infty \tilde{x}^T Q\tilde{x} + \tilde{u}^T R\tilde{u} \, \mathrm{d}x$$

and the system becomes

$$\dot{\tilde{x}} = \alpha e^{\alpha t}x(t) + e^{\alpha t}\dot{x}(t) = \alpha e^{\alpha t}x(t) + e^{\alpha t}Ax(t) + e^{\alpha t}Bu(t) = e^{\alpha t}(\alpha I + A)x(t) + e^{\alpha t}Bu(t)$$

$$= (\alpha I + A)\tilde{x}(t) + B\tilde{u}(t) = \tilde{A}\tilde{x}(t) + B\tilde{u}(t)$$

and

$$\tilde{y}(t) = e^{\alpha t}y(t) = e^{\alpha t}Cu(t) = C\tilde{u}(t).$$

Now, this system with transformed variables is now a system where Q and R are constant, so we may apply the results from the textbook. In particular we have that the optimal control for this system is

$$\tilde{u}^* = F^*\tilde{u} = -R^{-1}B^T P_c * \tilde{x}(t)$$

where P_c^* is the solution to the algebraic Riccati equation

$$\tilde{A}^T P_c + P_c\tilde{A} - P_c B R^{-1} B^T P_c + Q = 0.$$

This implies

$$e^{\alpha t}u^* = e^{\alpha t}F^* u = -R^{-1}B^T P_c^* x(t) e^{\alpha t}$$

and as $e^{\alpha t}$ for all $t \in \mathbb{R}$, we can divide through by this and we have

$$u^* = F^* u = -R^{-1}B^T P_c^* x(t).$$

Furthermore, the Riccati equation can be simplified to give

$$(\alpha I + A)^T P_c + P_c(\alpha I + A) - P_c B R^{-1} B^T P_c + Q = 0$$

which implies

$$2\alpha P_c + A^T P_c + P_c A - P_c B R^{-1} B^T P_c + Q = 0.$$

Clearly, the solution P_c^* to this equation is a constant matrix, as no term in the Riccati equation is a function of time. This means that our feedback control law $F^* u(t) = -R^{-1}B^T P_c^* x(t)$ is a fixed control law.

Question 3

a) First notice that $x(k+l) = A^l x(k) + \sum_{i=0}^{l-1} A^i Bu(k+i) + A^i Eq(k+i)$. We can prove this with induction. For $i = 1$ we have $x(k+1) = Ax(k) + Bu(k) + Eq(k)$ as expected, so assume $x(k+l) = A^l x(k) + \sum_{i=0}^{l-1} A^i Bu(k+i) + A^i Eq(k+i)$ is true for $l = j$, and consider $l = j+1$. Then

$$x(k+j+1) = Ax(k+j) + Bu(k+j) + Eq(k+j)$$

$$= A\left(A^j x(k) + \sum_{i=0}^{j-1} A^i Bu(k+i) + A^i Eq(k+i)\right) + Bu(k+j) + Eq(k+j)$$

$$= A^{j+1}x(k) + \sum_{i=0}^{j+1-1} A^i Bu(k+i) + A^i Eq(k+i)$$

which finished the proof by induction. We then have that

$$y(k+l) = CA^l x(k) + \sum_{i=0}^{l-1} CA^i Bu(k+i) + CA^i Eq(k+i),$$

so for $y(k+l)$ to be not effected by q, we must have $0 = \sum_{i=0}^{l-1} CA^i E$. For no $y(k)$ to be not effected by q for any time until $y(k+l)$, then we must have $CA^i E = 0$ for all $i \in [0, ..., l]$. By the Cayley Hamilton Theorem, for any $l \geq n$, $CA^l E$ is a linear combination of $CA^i E$ for $i \in [0, ..., l-1]$. Thus, y will not be effected by q for all time if it is not effected at any time step from k until $k+n-1$, which will happen if $CA^i E = 0$ for all $i \in [0, ..., n-1]$. If we arrange these equations into one matrix equation we have

$$0 = \begin{bmatrix} CE \\ \vdots \\ CA^{n-1}E \end{bmatrix} = Obs(A, C)E.$$

Thus, the effects of q will be completely eliminated if $Obs(A, C)E = 0$.

b) For $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $C = [1, 1]$, we have $Obs(A, C) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. As $q \in \mathbb{R}^r$, $E \in \mathbb{R}^{2\times r}$. However, $Obs(A, C)E = 0_{2\times r0}$ implies $Obs(A, C)E_i = 0_{2\times 1}$, where E_i is the ith column of E. This means that each column of E must be a scalar multiple of the eigenvector of $Obs(A, C)$ for the eigenvalue 0, which is the vector $[-1, 1]^T$. So E is characterised by $[x^T, -x^T]^T$, where x is any element in R^r.

c) The z-transform of $x(k)$ is

$$zX(x) = AX(z) + BU(z) + EQ(z).$$

Rearranging for $X(z)$ gives

$$X(z) = (BU(z) + EQ(z))(zI - A)^{-1}$$

and recalling that the z-transform of a step function is $Q(z) = \frac{z}{z-1}$ gives

$$X(z) = (BU(z)(z-1) + Ez)(z-1)^{-1}(zI - A)^{-1}.$$

Thus,

$$Y(z) = C(BU(z)(z-1) + Ez)(z-1)^{-1}(zI - A)^{-1}.$$

Now, if q was absent, then the output would be $\bar{Y}(z) = C(BU(z))(zI - A)^{-1}$. We want the error introduced by q to go to 0 asymptotically, so consider

$$Y(z) - \bar{Y}(z) = C(BU(z)(z-1) + Ez)(z-1)^{-1}(zI-A)^{-1} - C(BU(z)(z-1))(z-1)^{-1}(zI-A)^{-1}$$
$$= CEz(z-1)^{-1}(zI-A)^{-1}.$$

By the final value theorem for the z-transform, if the eigenvalues of A are have magnitude strictly less than 1,

$$y(\infty) - \bar{y}(\infty) = \lim_{z \to 1}(z-1)(Y(z) - \bar{Y}(z))$$
$$= \lim_{z \to 1} CEz(zI - A)^{-1} = CE(I - A)^{-1}.$$

Thus, if the eigenvalues of A are strictly less than 1, there will be a constant asymptotic error of $CE(I - A)^{-1}$.

Question 4

This example aims to find the Kalman decomposition of the system described by

$$\dot{x} = Ax + Bu \quad \text{and} \quad y = Cx$$

with

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix},$$

and

$$C = [0, 1, 0].$$

First, we must find the controllability and observability matrices for this system. $n = 3$, so we have that

$$Con(A, B) = [B, AB, A^2B] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

We also have

$$Obs(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ -2 & 4 & -2 \end{bmatrix}.$$

n_o is the dimension of the observable subspace R_o so we have

$$n_o = \text{rank}(Obs) = 2$$

and thus the dimension of the uncontrollable subspace $R_{\bar{o}}$ is $n_{\bar{o}} = n - n_o = 1$. The null space of $Obs(A, B)$ is $[1, 0, -1]$, which is the unobservable subspace. n_r is the dimension of the controllable subspace R_r so we have

$$n_r = \text{rank}(Con) = 2.$$

The null space of the controllability matrix is $[1, -2, 1]$, which means the two vectors mutually perpendicular to this form R_r. In particular, $[1, 0, -1]$ is perpendicular to $[1, -2, 1]$, so we choose the vectors to describe R_r as $R_r = \{[1, 0, -1], [1, 1, 1]\}$. Thus, it is clear what the observable and controllable subspace is, $[1, 1, 1]$, the dimension of which is $n_{r\bar{o}}$. Also, the only vector in the unobservable and uncontrollable subspace is the 0 vector, as the unobservable subspace is a subset of the controllable subspace. This means A_{24}, A_{43}, A_{44} have dimensions of 0, i.e. they do not exist.

Now, Q is defined as

$$Q = [v_1, \dots v_{n_r}, Q_N, \hat{v}_1, \dots, \hat{v}_{n_o - n_{r\bar{o}}}]$$

where the first $n_r - n_{r\bar{o}} = 1$ columns are the basis of the controllable and observable space, $[1, 1, 1]$, the next $n_{\bar{o}} = 1$ columns are the basis of the controllable but unobservable subspace $[1, 0, -1]$, the last $n_o - n_{r\bar{o}} = 0$ vectors are the basis of the unobservable and uncontrollable subspace (which is empty in this question). Q_N is chosen so that Q is nonsingular. In this example Q_N was chosen to be $[0, 0, 1]$, which gave

$$Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

Q is non singular, as $\det Q = -1(1 \cdot 1 - 0 \dot{-} 1) = -1$. From here we can calculate

$$\hat{A} = Q^{-1}AQ = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}$$

$$\hat{B} = Q^{-1}BQ = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}$$

and

$$\hat{C} = Q^{-1}CQ = [1, 0, 0] = [C_1, 0, C_3].$$

Thus, the matrix $A_{11} = 0$, which corresponds to the eigenvalue 0, is controllable and observable. We also have (A_c, B_c), where

$$A_c = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

and

$$B_c = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

is controllable, which implies the matrix $A_{22} = -1$, which corresponds to the eigenvalue -1, is controllable but unobservable. Finally, we also have (A_o, C_o), where

$$A_o = \begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix}$$

and

$$C_o = [C_1, C_3]$$

is observable, which implies the matrix $A_{33} = -2$, which corresponds to the eigenvalue -2, is observable but uncontrollable. As pointed out above, there are no eigenvalues which are neither controllable nor observable.

Question 5

a) Before proving this, not that if v is a left eigenvector of A with eigenvalue λ, then v is a left eigenvector of $\bar{A}$ with eigenvalue $e^{\lambda T}$. To see this calculate

$$v\bar{A} = ve^{AT} = v\sum_{m=0}^{\infty}\frac{1}{m!}(AT)^m = \sum_{m=0}^{\infty}\frac{1}{m!}vA^mT^m = \sum_{m=0}^{\infty}\frac{1}{m!}v\lambda^mT^m = ve^{\lambda T}.$$

The same holds for right eigenvectors.

Another theorem that will be used in this proof is one of the PHB tests, specifically that

$$\text{rank}([\lambda I - A, B]) < n$$

for an eigenvalue λ of A if and only if (A, B) is uncontrollable, and

$$\text{rank}\left(\begin{bmatrix} c\lambda I - A \\ C \end{bmatrix}\right) < n$$

for an eigenvalue λ of A if and only if (A, C) is unobservable.

First, we prove that if $\Im(\lambda_i - \lambda_j) = \frac{2\pi k}{T}$ for any eigenvalues $\Re(\lambda_i - \lambda_j) = 0$, then $(\bar{A}, \bar{B})$ is uncontrollable and $\bar{A}, \bar{B}$ is unobservable. If the antecedent is true, then let $\lambda_i = \lambda_j + \frac{2\pi k}{T}$ and consider the (left) eigenvectors v_i and v_j of A. By the above argument, v_i and v_j are an eigenvectors of $\bar{A}$ with eigenvalues $e^{\lambda_i T}$ and $e^{\lambda_j T}$. However, due to the fact that $e^z = e^{z+2k\pi i}$ for any $z \in \mathbb{C}$ and $k \in \mathbb{Z}$, $e^{\lambda_i T} = e^{\lambda_j T}$, i.e. the eigenvalue $e^{\lambda_i T}$ has arithmetic multiplicity of at least 2. But this means $e^{\lambda_i T} I - \bar{A}$ has 0 as an eigenvalue with multiplicity of at least 2, so $\text{rank}(e^{\lambda_i T} I - \bar{A}) \leq n - 2$. Adding another column to this matrix can increase the rank by no more than 1, so $\text{rank}([e^{\lambda_i T} I - \bar{A}, \bar{B}]) \leq n - 1 < n$. Thus, by the PHB test, $(\bar{A}, \bar{B})$ is uncontrollable.

Similarly if v_i and v_j are eigenvectors of A, then by the above argument, v_i and v_j are an eigenvectors of $\bar{A}$ with eigenvalues $e^{\lambda_i T}$ and $e^{\lambda_j T}$. Again, the eigenvalue $e^{\lambda_i T}$ has arithmetic multiplicity of at least 2 and $\text{rank}(e^{\lambda_i T} I - \bar{A}) \leq n - 2$. Adding another row to this matrix can increase the rank by no more than 1, so $\text{rank}([e^{\lambda_i T} I - \bar{A}, \bar{B}]) \leq n - 1 < n$. Thus, again, by the PHB test, $(\bar{A}, \bar{C})$ is unobservable.

Now assume if $\Im(\lambda_i - \lambda_j) \neq \frac{2\pi k}{T}$ for any eigenvalues such that $\Re(\lambda_i - \lambda_j) = 0$. We know that (A, B) is controllable, so by the PHB test, for every eigenvalue λ_i of A, $\text{rank}([\lambda_i I - A, B]) = n$. However, as λ_i is an eigenvector, $\text{rank}(\lambda_i I - A) < n$, and as adding another column to this matrix can increase the rank by no more than 1, then $\text{rank}(\lambda_i I - A) = n - 1$. Thus, the multiplicity of every eigenvalue of A can be no more than 1, i.e. every eigenvalue of A is unique.

Furthermore, B is linearly independent of $\lambda_i I - A$ for each i, which means for every $n-1$ set of eigenvectors of A, B is linearly independent of that set. As $v_j(\lambda_i I - A) = \lambda_i vi - \lambda_j v_j$, the eigenvectors of $\lambda_i I - A$ are the eigenvectors of A, however they now correspond to the eigenvalues $\lambda_i - \lambda_j$. As A and $\bar{A}$ have the same eigenvectors, for every $n - 1$ set of eigenvectors of $\bar{A}$, B is linearly independent of that set. Thus we may define $\alpha_i \neq 0$ such that

$$B = \sum_{i=1}^{n} \alpha_i v_i.$$

Also, as $\int_0^T e^{A\tau}\ \mathrm{d}t$ is a non singular matrix for $T > 0$ with the same eigenvectors as A, then

$$\bar{B} = \int_0^T e^{A\tau}\ \mathrm{d}t\ B = \sum_{i=1}^{n} \alpha_i \int_0^T e^{A\tau}\ \mathrm{d}t\ v_i = \sum_{i=1}^{n} \mu_i \alpha_i v_i = \sum_{i=1}^{n} \beta_i v_i$$

where μ_i are the eigenvalues of $\int_0^T e^{A\tau}\ \mathrm{d}t$. Note that $\mu_i \alpha_i = \beta_i \neq 0$, as $\alpha_i \neq 0$ and $\mu_i \neq 0$, as $\int_0^T e^{A\tau}\ \mathrm{d}t$ is nonsingular. Thus, $\bar{B}$ is also has the property that for every $n - 1$ set of

eigenvectors of $\bar{A}$, $\bar{B}$ is linearly independent of that set.

Now, by assumption, whenever $\Re(\lambda_i - \lambda_j) = 0$, then $\Im(\lambda_i - \lambda_j) \neq \frac{2\pi k}{T}$, so there are no eigenvalues of A such that $\lambda_i = \lambda_j + \frac{2k\pi i}{T}$ for any $k \in \mathbb{Z} \setminus \{0\}$, and hence for every pair of eigenvalue such that $i \neq j$, $e^{\lambda_i T} \neq e^{\lambda_j T}$. Thus, every eigenvalue of $\bar{A}$ has multiplicity of 1. Thus, for every eigenvalue, $\text{rank}(e^{\lambda_i T} I - A) = n - 1$. Now, as $\bar{B}$ is has the property that for every $n-1$ set of eigenvectors of $\bar{A}$, $\bar{B}$ is linearly independent of that set, then for any eigenvalue,

$$\text{rank}([e^{\lambda_i T} I - A, B]) = \text{rank}(e^{\lambda_i T} I - A) + 1 = n.$$

As this holds for any eigenvalue, by the PHB test, (A, B) is controllable.

Similarly, we know that (A, C) is observable, so by the PHB test, for every eigenvalue λ_i of A,

$$\text{rank}\left(\begin{bmatrix} c\lambda I - A \\ C \end{bmatrix}\right) = n.$$

However, as λ_i is an eigenvector, $\text{rank}(\lambda_i I - A) < n$ and as adding another column to this matrix can increase the rank by no more than 1, then $\text{rank}(\lambda_i I - A) = n - 1$. Thus, C is linearly independent of $\lambda_i I - A$ for each i, which means for every $n-1$ set of eigenvectors of A, C is linearly independent of that set. As $v_j(\lambda_i I - A) = \lambda_i vi - \lambda_j v_j$, the eigenvectors of $\lambda_i I - A$ are the eigenvectors of A, however they now correspond to the eigenvalues $\lambda_i - \lambda_j$. As A and $\bar{A}$ have the same eigenvectors, for every $n-1$ set of eigenvectors of $\bar{A}$, C is linearly independent of that set. Furthermore, $C = \bar{C}$ so the same property holds for $\bar{C}$

We still have that every eigenvalue of $\bar{A}$ has multiplicity of 1. Thus, for every eigenvalue, $\text{rank}(e^{\lambda_i T} I - A) = n - 1$. Now, as $\bar{C}$ is has the property that for every $n-1$ set of eigenvectors of $\bar{A}$, $\bar{C}$ is linearly independent of that set, then for any eigenvalue,

$$\text{rank}\left(\begin{bmatrix} c\lambda I - A \\ C \end{bmatrix}\right) = \text{rank}(e^{\lambda_i T} I - A) + 1 = n.$$

As this holds for any eigenvalue, by the PHB test, (A, C) is observable.

b) First, note that the double integrator is exactly the system described in the first part of this question, so the results of that part can be applied here. The eigenvalues of A in the first case, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ are $\lambda_1 = \lambda_2 = 0$. Thus for any $\lambda_i - \lambda_j$, $\Re(\lambda_i - \lambda_j) = 0$ but $\Im(\lambda_i - \lambda_j) = 0$, which is not equal to $\frac{2\pi k}{T}$ for any $k \in \mathbb{Z} \setminus \{0\}$. Thus, by the above

theorem, (A, B) is controllable and (A, C) is observable for any $T > 0$.

In first case, where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, the eigenvalues of A are $\lambda_1 = i$ and $\lambda_2 = -i$. Again, for any $i, j \in \{1, 2\}$ we have that $\Re(\lambda_1 - \lambda_2) = 0$, but if $i = j$ then $\Im(\lambda_i - \lambda_j) = 0$, which again is not equal to $\frac{2\pi k}{T}$ for any $k \in \mathbb{Z} \setminus \{0\}$. However, if $i \neq j$ then $\Im(\lambda_i - \lambda_j) = \pm 2$. Thus, by the above theorem, (A, B) is uncontrollable and (A, C) is unobservable for $T = k\pi$ for any $k \in \mathbb{N}$.