MATH4406 (Control Theory)
HW5 (Unit 5): Stability

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1. Find an example of a continuous time linear system $\dot{x} = Ax$ that is stable, yet for which the function, $V(x) = x'x$ is not a Lyapounov function.

2. Repeat exercise 1 for a discrete time system.

3. Explain (briefly) your results of the previous two exercises, in terms of the linear matrix inequality (LMI):

$$-(PA' + AP) > 0 \quad \text{and/or} \quad P - A'PA > 0.$$

   Specifically, explain where the second LMI comes from (it was not done in class).

4. Prove that if all eigenvalues of A have negative real part, then there exists a matrix P such that, $V(x) = x'Px$ is a Lyapounov function.

5. Suppose a hound is chasing a rabbit that is running at a straight line across the x-axis at a constant velocity R. The hound is running at a constant velocity H in a way that it is always pointing directly towards the rabbit. Show (using a Lyapounov function) that if $H > R$ then the hound always catches the rabbit (no matter where it starts). Note: For the purpose of the exercise it will suffice to show that the distance between the hound and the rabbit converges to zero - i.e. you are not expected to show that the hound catches the rabbit in finite time.

6. This problem shows that the concept of a Lyapounov function can be applied to show convergence of algorithms.

   Consider the problem of finding the square root of a positive number, α. I.e. finding a solution of the equation,

$$x^2 - \alpha = 0,$$

   or,

$$x = x + \alpha - x^2.$$

   This equation suggests to evaluate $\sqrt{\alpha}$ by means of the the dynamic system,

$$x(k+1) = x(k) + \alpha - x(k)^2.$$

   Explain the intution behind the above system.

   Further, find (prove your result) the range of values of $x(0)$ and α for which,

$$\lim_{k \to \infty} x(k) = \sqrt{\alpha}.$$

   Illustrate your results numerically if you see fit.

7. This question deals with the stochastic discrete event control systems presented in class – taken from "An Exploration of Random Processes for Engineers" by Bruce Hajek (preprint of book, 2008) – see link on course web page.

   Consider a system of n queues and the policy of routing to queue i w.p. u_i (in class we had $n = 2$ and took $u = u_1$ and $\bar{u} = 1 - u_1$).

   a) Write a necessary condition for stability - explain your result.

   b) For the case of $n = 2$ find a range of u's that stabilizes the system based on the parameters a, d_1 and d_2.

   Consider now the "join the shortest queue policy". Justify why this policy is as good as can be in terms of stability (i.e. it stabilizes the system under the necessary condition) – show your results formally - i.e. go through the derivation of the Lyapounuv function and show it is bounded by $-\epsilon$ where $\epsilon > 0$ is indepdenent of the state.