MATH4406 - Assignment 5

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Question 1

Give an example of a stable continuous system $\dot{x} = Ax$ such that $V(x) = x^T x$ is not a Lyaponov function.

We need to find a matrix A such that it's eigenvalues have negative real parts (for stability) and the function $\dot{V}(x) > 0$ for some x (i.e. not a Lyaponov function).

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T x + x^T \dot{x} \\ &= (Ax)^T x + x^T Ax \\ &= x^T \left(A^T + A\right) x\end{aligned}$$

Thus we require $A^T + A$ to be NOT a negative definite matrix.

Choosing $A = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}$ gives eigenvalues of $\lambda = -1 \pm 2i$ (thus stable) and

$$A^T + A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix}$$

This is not a negative definite matrix. Consider the points $x = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}$

$$\begin{aligned}\begin{pmatrix} x_1 & -x_1 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} &= \begin{pmatrix} x_1 & -x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} \\ &= 2x_1^2\end{aligned}$$

which is positive, thus $V(x)$ is not a Lyaponov function.

Question 2

Give an example of a stable discrete system $x(k+1) = Ax(k)$ such that $V(x) = x^T x$ is not a Lyaponov function.

We need to find a matrix A such that it's eigenvalues are within the complex unit circle i.e. $\|\lambda_i\| \leq 1 \, \forall i$ (for stability) and the function $V(Ax) - V(x) > 0$ for some x (i.e. not a Lyaponov function).

$$\begin{aligned}V(Ax) - V(x) &= (Ax)^T Ax - x^T x \\ &= x^T \left(A^T A - I\right) x\end{aligned}$$

Thus we require $A^T A - I$ to be NOT a negative definite matrix.

Choosing $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 3 & -\frac{1}{2} \end{pmatrix}$ gives eigenvalues of $\lambda = \pm\frac{1}{2}$ (thus stable) and

$$A^T A - I = \begin{pmatrix} \frac{1}{2} & 3 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 3 & -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{33}{4} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{4} \end{pmatrix}$$

This is not a negative definite matrix. Consider the points $x = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$

$$\begin{aligned} \begin{pmatrix} x_1 & x_1 \end{pmatrix} \begin{pmatrix} \frac{33}{4} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} &= \begin{pmatrix} \frac{27}{4}x_1 & -\frac{9}{4}x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} \\ &= \frac{9}{2}x_1^2 \end{aligned}$$

which is positive, thus $V(x)$ is not a Lyaponov function.

Question 3

Continuous

Consider the continuous system $\dot{x} = Ax$ with $V(x) = x^T P x$ and P is a positive-definite matrix.

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P A x \\ &= x^T \left(A^T P + PA \right) x \end{aligned}$$

For stability we need $A^T P + AP \prec 0$ or $-\left(A^T P + PA \right) \succ 0$. From question 1 we have:

$$A = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Which fails, however if we take $P = \begin{pmatrix} 22 & -3 \\ -3 & 7 \end{pmatrix}$ which is a symmetric matrix with positive eigenvalues $\frac{29 \pm 3\sqrt{29}}{2} \approx (22.5, 6.4)$ and thus is positive definite, then:

$$\begin{aligned} A^T P + PA &= \begin{pmatrix} -10 & -25 \\ 25 & -10 \end{pmatrix} + \begin{pmatrix} -10 & 25 \\ -25 & -10 \end{pmatrix} \\ &= \begin{pmatrix} -20 & 0 \\ 0 & -20 \end{pmatrix} \\ x^T \left(A^T P + PA \right) x &= -20x_1^2 - 20x_2^2 \end{aligned}$$

Which is Lyaponov stable.

Discrete

Consider the discrete system $x(k+1) = Ax(k)$ with $V(x) = x^T P x$ and P is a positive-definite matrix.

$$
\begin{aligned}
D\left[V\left(x\left(k\right)\right)\right] &= V\left(x\left(k+1\right)\right)-V\left(x\left(k\right)\right)\\
&= V\left(Ax\left(k\right)\right)-V\left(x\left(k\right)\right)\\
&= \left(Ax\left(k\right)\right)^{T}PAx\left(k\right)-x\left(k\right)^{T}Px\left(k\right)\\
&= x^{T}\left(k\right)\left(A^{T}PA-P\right)x\left(k\right)
\end{aligned}
$$

For stability we need $D\left[V\left(x\left(k\right)\right)\right]<0$ for all $x\neq 0$ thus we require $A^{T}PA-P\prec 0$ or $P-A^{T}PA\succ 0$. From question 2 we have:

$$
A=\begin{pmatrix}\frac{1}{2} & 0\\ 3 & -\frac{1}{2}\end{pmatrix},\ P=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}
$$

Which fails, however if we take $P=\begin{pmatrix}164 & -24\\ -24 & 20\end{pmatrix}$ which is a symmetric matrix with positive eigenvalues $92\pm 24\sqrt{10}\approx(167.9,\,16.1)$ and thus is positive definite, then:

$$
\begin{aligned}
A^{T}PA-P &= \begin{pmatrix}\frac{1}{2} & 3\\ 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}164 & -24\\ -24 & 20\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ 3 & -\frac{1}{2}\end{pmatrix}-\begin{pmatrix}164 & -24\\ -24 & 20\end{pmatrix}\\
&= \begin{pmatrix}10 & 48\\ 12 & -10\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ 3 & -\frac{1}{2}\end{pmatrix}-\begin{pmatrix}164 & -24\\ -24 & 20\end{pmatrix}\\
&= \begin{pmatrix}149 & -24\\ -24 & 5\end{pmatrix}-\begin{pmatrix}164 & -24\\ -24 & 20\end{pmatrix}\\
&= \begin{pmatrix}-15 & 0\\ 0 & -15\end{pmatrix}\\
x^{T}\left(A^{T}PA-P\right)x &= -15x_1^2-15x_2^2
\end{aligned}
$$

Which is Lyaponov stable.

Question 4

Prove that if all eigenvalues of A have negative real part then there exists a matrix P such that $V\left(x\right)=x^{T}Px$ is a Lyaponov function, for the system $\dot{x}=Ax$.

In order for $V\left(x\right)$ to be a Lyaponov function we require a positive definite P such that $A^{T}P+PA=-Q$ is negative definite (or negative semi-definite), ie. $\dot{V}\left(x\right)=-x^{T}Qx\leq 0$

Thus given any suitable positive-definite Q we need to solve the Lyaponov equation $\left(A^{T}P+PA=-Q\right)$ for P. Consider the definition:

$$
P=\int_{0}^{\infty}e^{A^{T}t}Qe^{At}\,dt
$$

which is well defined (since A has eigenvalues with negative real parts), and also symmetric and positive-definite (from properties of Q). Then

$$
\begin{aligned}
A^{T}P+PA &= \int_{0}^{\infty}A^{T}e^{A^{T}t}Qe^{At}+e^{A^{T}t}Qe^{At}A\,dt\\
&= \int_{0}^{\infty}\frac{d}{dt}\left[e^{A^{T}t}Qe^{At}\right]dt\\
&= \left[e^{A^{T}t}Qe^{At}\right]_{0}^{\infty}\\
&= -Q
\end{aligned}
$$

Thus P exists and satisfies the Lyaponov equation. Note: to show P is positive definite we consider

$$
\begin{aligned}
x^T P x &= \int_0^\infty x^T e^{A^T t} Q e^{At} x \, dt \\
&= \int_0^\infty \left\| Q^{\frac{1}{2}} e^{At} x \right\|^2 \, dt \geq 0
\end{aligned}
$$

Therefore if A has eigenvalues with negative real parts, then we can choose any positive-definite Q, such that P exists and $V(x) = x^T P x$ is a Lyaponov function.

Question 5

A rabbit is running on the x-axis at a constant velocity R, a hound is chasing at a constant velocity $H > R$, always facing directly towards (starting at any location). Show via Lyaponov that the hound catches the rabbit.

Since constant velocity we have the rabbit

$$
\begin{aligned}
\dot{x}_r &= R \\
\dot{y}_r &= y_r = 0
\end{aligned}
$$

We have the hound moving closer (directly) towards the rabbit, at an equally scaled proportion $c \in \mathbb{R}_+$ based on current position.

$$
\begin{aligned}
\dot{x}_h &= -c(x_h - x_r) \\
\dot{y}_h &= -c(y_h - y_r) = -c y_h
\end{aligned}
$$

Since the hound runs at constant speed H we have

$$
\begin{aligned}
\dot{x}_h^2 + \dot{y}_h^2 &= H^2 \\
c^2 (x_h - x_r)^2 + c^2 y_h^2 &= H^2 \\
c &= \frac{H}{\sqrt{(x_h - x_r)^2 + y_h^2}}
\end{aligned}
$$

For simplicity we relabel variables and take the rabbit as the origin $x = x_h - x_r$, $y = y_h$

$$
\begin{aligned}
\dot{x} &= \dot{x}_h - \dot{x}_r \\
&= -\frac{H}{\sqrt{(x_h - x_r)^2 + y_h^2}} (x_h - x_r) - R \\
&= -\frac{H}{\sqrt{x^2 + y^2}} x - R \\
\\
\dot{y} &= \dot{y}_h \\
&= -\frac{H}{\sqrt{x^2 + y^2}} y
\end{aligned}
$$

Now consider the Lyaponov function $v(x, y) = x^2 + y^2$ which is actually the distance squared between the rabbit and the hound

$$
\begin{aligned}
\dot{v}(x,y) &= 2\dot{x}x + 2\dot{y}y \\
&= -2x^2\frac{H}{\sqrt{x^2+y^2}} - 2Rx - 2y^2\frac{H}{\sqrt{x^2+y^2}} \\
&= -\frac{2H}{\sqrt{x^2+y^2}}\left(x^2+y^2\right) - 2Rx \\
&= -2H\sqrt{x^2+y^2} - 2Rx
\end{aligned}
$$

Clearly for $y \neq 0$ we have $\dot{v}(0,y) < 0$, and similarly for $x > 0$. When $x < 0$ we have

$$
\begin{aligned}
-2H\sqrt{x^2+y^2} - 2Rx &\leq -2H\sqrt{x^2} - 2Rx \\
&= -2H|x| + 2R|x| \\
&= 2|x|(R-H) \\
&< 0
\end{aligned}
$$

since $H > R$. Since $\dot{v}(x,y) < 0$ except for origin $(0,0)$ we know that $v(x,y)$ (i.e. distance squared) is a Lyaponov function, hence the equilibrium solution of 0 (hound catches rabbit) is stable.

Thus we have shown that if $H > R$ the hound always catches the rabbit (no matter where it starts).

Question 6

Consider the fixed point iteration to find the square root: $x^2 = \alpha$ with $\alpha > 0$

$$
\begin{aligned}
x(k+1) &= f(x(k)) \\
&= x(k) + \alpha - x^2(k)
\end{aligned}
$$

Clearly if $x(k) = \sqrt{\alpha}$ we have $x(k+1) = \sqrt{\alpha} + \alpha - (\sqrt{\alpha})^2 = \sqrt{\alpha}$ which is a fixed point equilibrium $\bar{x}$.

We can consider $x(k)$ and $x(k+1)$ in terms of $\bar{x}$

$$
\begin{aligned}
x(k) &= \bar{x} - \varepsilon_k \\
x(k+1) &= \bar{x} - \varepsilon_{k+1}
\end{aligned}
$$

where convergence requires $\lim_{k\to\infty} \varepsilon_k = 0$

The range of values for convergence can be found by analyzing for small ε

$$
\begin{aligned}
x(k+1) &= f(\bar{x} - \varepsilon_k) \\
\bar{x} - \varepsilon_{k+1} &= f(\bar{x}) - \varepsilon_k f'(\bar{x}) + \varepsilon_k^2\frac{f''(\bar{x})}{2} + \cdots \\
&= \bar{x} - \varepsilon_k(1-2\bar{x}) + \mathcal{O}\left(\varepsilon_k^2\right) \\
\varepsilon_{k+1} &= \varepsilon_k(1-2\bar{x}) + \mathcal{O}\left(\varepsilon_k^2\right)
\end{aligned}
$$

Thus we have approximately a geometric sequence

$$\varepsilon_k \approx (1-2\bar{x})^{k-1}\,\varepsilon_1$$

which has convergence when

$$\begin{array}{rcl} 0< & |1-2\bar{x}| & <1 \\ 0< & \bar{x} & <1 \end{array}$$

Thus $0<\alpha<1$ and $0<x_0<1$.

Question 7

Consider a system with n queues and a policy to route to queue i with probability u_i ($\sum_i u_i = 1$). We have arrival probability a, and departure probability d_i.

$$X_i(k+1) = X_i(k) + A_i(k) - D_i(k) + L_i(k)$$

$$A(k) = \begin{cases} e_1 & \text{w.p } au_1 \\ \vdots & \\ e_n & \text{w.p } au_n \\ \mathbf{0} & \text{w.p } 1-a \end{cases}, \quad D_i(k) = \begin{cases} 1 & \text{w.p } d_i \\ 0 & \text{w.p } 1-d_i \end{cases}, \quad L_i(k) = (D_i - (X_i + A_i))^+$$

a) Clearly a necessary (but not necessarily sufficient) condition of stability is that $a < \sum_{i=1}^n d_i$ otherwise our total arrivals will be more than our total departures and the queue will continue to grow which is unstable.

b) For $n=2$ find range of u's that stabilizes the system based on the parameters a, d_1 and d_2. We require $au < d_1$ and $a(1-u) < d_2$ thus we have

$$\begin{array}{rcl} au + a\bar{u} & < & d_1 + d_2 \\ au + a - au & < & d_1 + d_2 \\ a & < & d_1 + d_2 \end{array}$$

which is our necessary condition for stability. Then our values for u (rearranging $au < d_1$ and $a(1-u) < d_2$) are

$$\max\left\{0,\, 1 - \frac{d_2}{a}\right\} < u < \min\left\{1,\, \frac{d_1}{a}\right\}$$

since $0<u<1$

c) Consider "join shortest queue" derive Lyaponov function and show it is bounded by $-\varepsilon$ with $\varepsilon > 0$

Here we have

$$A(k) = \begin{cases} I_{\{x_1 \le x_2\}} & \text{w.p } a \\ I_{\{x_1 > x_2\}} & \\ & \\ \mathbf{0} & \text{w.p } 1-a \end{cases}$$

Intuitively this is good for stability as it distributes work appropriately across the system. Evaluating this with Foster-Lyaponov criterion gives:

$$
\begin{aligned}
PV - V &= \mathbb{E}\left[V\left(X\left(t+1\right)\right) \mid X\left(t\right) = x\right] - V\left(x\right) \\
&\leq \mathbb{E}\left[\sum_{i=1}^{2} \frac{1}{2}\left(X_i + A_i\left(t\right) - D_i\left(t\right)\right)^2 - \left(x_1^2 + x_2^2\right) \mid X\left(t\right) = x\right] \\
&= \sum_{i=1}^{2} x_i \mathbb{E}\left[A_i\left(t\right) - D_i\left(t\right) \mid X\left(t\right) = x\right] + \frac{1}{2}\mathbb{E}\left[\left(A_i\left(t\right) - D_i\left(t\right)\right)^2 \mid X\left(t\right) = x\right] \\
&\leq 1 + \sum_{i=1}^{2} x_i \mathbb{E}\left[A_i\left(t\right) - D_i\left(t\right) \mid X\left(t\right) = x\right] \\
&= 1 + \left(x_1\left(aI_{\{x_1 \leq x_2\}} - d_1\right) + x_2\left(aI_{\{x_1 > x_2\}} - d_2\right)\right)
\end{aligned}
$$

Consider when $u = I_{\{x_1 \leq x_2\}}$ then we have $1 - u = I_{\{x_1 > x_2\}}$ and stability follows from our previous question.

Since we have that $x_1 I_{\{x_1 \leq x_2\}} + x_2 I_{\{x_1 > x_2\}} \leq u x_1 + (1-u) x_2$ for any $u \in [0,1]$ we have

$$
\begin{aligned}
PV - V &\leq 1 + \left(x_1\left(au - d_1\right) + x_2\left(a\left(1-u\right) - d_2\right)\right) \\
&\leq -\varepsilon + 1
\end{aligned}
$$

When $au < d_1$ and $a\left(1-u\right) < d_2$. Note: we can ignore the bounded constant 1 in Foster-Lyaponov.

Thus the 'join the shortest queue' policy is just as good as random allocation in terms of stability.