Question 1

Give an example of a stable continuous system $\dot{x} = Ax$ such that $V(x) = x^T x$ is not a Lyapunov function.

We need to find a matrix $A$ such that its eigenvalues have negative real parts (for stability) and the function $V(x) > 0$ for some $x$ (i.e. not a Lyapunov function).

$$\dot{V}(x) = \dot{x}^T x + x^T \dot{x} = (Ax)^T x + x^T Ax = x^T (A^T + A)x$$

Thus we require $A^T + A$ to be NOT a negative definite matrix.

Choosing $A = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}$ gives eigenvalues of $\lambda = -1 \pm 2i$ (thus stable) and

$$A^T + A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix}$$

This is not a negative definite matrix. Consider the points $x = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}$

$$\begin{pmatrix} x_1 & -x_1 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = \begin{pmatrix} x_1 & -x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = 2x_1^2$$

which is positive, thus $V(x)$ is not a Lyapunov function.

Question 2

Give an example of a stable discrete system $x(k+1) = Ax(k)$ such that $V(x) = x^T x$ is not a Lyapunov function.

We need to find a matrix $A$ such that its eigenvalues are within the complex unit circle i.e. $\|\lambda_i\| \leq 1 \forall i$ (for stability) and the function $V(Ax) - V(x) > 0$ for some $x$ (i.e. not a Lyapunov function).

$$V(Ax) - V(x) = (Ax)^T Ax - x^T x = x^T (A^T A - I)x$$
Thus we require $A^T A - I$ to be NOT a negative definite matrix.

Choosing $A = \begin{pmatrix} 1/2 & -1/2 \\ 3/2 & -1/2 \end{pmatrix}$ gives eigenvalues of $\lambda = \pm 3/2$ (thus stable) and

$A^T A - I = \begin{pmatrix} 1/2 & 3/2 \\ 0 & -1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 3/2 & -1/2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3/2 & -3/4 \\ -3/2 & -3/4 \end{pmatrix}

This is not a negative definite matrix. Consider the points $x = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$

\[
\begin{pmatrix} x_1 & x_1 \\ \frac{33}{4} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{27}{4} x_1 & -\frac{9}{4} x_1 \\ x_1 & x_1 \end{pmatrix} = \frac{9}{2} x_1^2
\]

which is positive, thus $V(x)$ is not a Lyaponov function.

Question 3

Continuous

Consider the continuous system $\dot{x} = Ax$ with $V(x) = x^T P x$ and $P$ is a positive-definite matrix.

\[
\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = (Ax)^T P x + x^T P A x = x^T (A^T P + PA) x
\]

For stability we need $A^T P + AP \prec 0$ or $-(A^T P + PA) \succ 0$. From question 1 we have:

$A = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}

Which fails, however if we take $P = \begin{pmatrix} 22 & -3 \\ -3 & 7 \end{pmatrix}$ which is a symmetric matrix with positive eigenvalues $\frac{29 \pm 3\sqrt{29}}{2} \approx (22.5, 6.4)$ and thus is positive definite, then:

\[
A^T P + PA = \begin{pmatrix} -10 & -25 \\ 25 & -10 \end{pmatrix} + \begin{pmatrix} -10 & 25 \\ -25 & -10 \end{pmatrix} = \begin{pmatrix} -20 & 0 \\ 0 & -20 \end{pmatrix}
\]

\[
x^T (A^T P + PA) x = -20 x_1^2 - 20 x_2^2
\]

which is Lyapunov stable.

Discrete

Consider the discrete system $x(k+1) = Ax(k)$ with $V(x) = x^T P x$ and $P$ is a positive-definite matrix.
\[ D [V (x (k))] = V (x (k + 1)) - V (x (k)) \]
\[ = V (Ax (k)) - V (x (k)) \]
\[ = (Ax (k))^T PAx (k) - x (k)^T Px (k) \]
\[ = x^T (k) \left(A^T PA - P \right) x (k) \]

For stability we need \(D [V (x (k))] < 0\) for all \(x \neq 0\) thus we require \(A^T PA - P \prec 0\) or \(P - A^T PA \succ 0\). From question 2 we have:

\[ A = \begin{pmatrix} \frac{1}{2} & 0 \\ 3 & -\frac{1}{2} \end{pmatrix} \]
\[ P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

Which fails, however if we take \(P = \begin{pmatrix} 164 & -24 \\ -24 & 20 \end{pmatrix}\) which is a symmetric matrix with positive eigenvalues \(92 \pm 24\sqrt{10} \approx (167.9, 16.1)\) and thus is positive definite, then:

\[ A^T PA - P = \begin{pmatrix} \frac{1}{2} & 3 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 164 & -24 \\ -24 & 20 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 3 & -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} 164 & -24 \\ -24 & 20 \end{pmatrix} \]
\[ = \begin{pmatrix} 10 & 48 \\ 12 & -10 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 3 & -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} 164 & -24 \\ -24 & 20 \end{pmatrix} \]
\[ = \begin{pmatrix} 149 & -24 \\ -24 & 5 \end{pmatrix} - \begin{pmatrix} 164 & -24 \\ -24 & 20 \end{pmatrix} \]
\[ = \begin{pmatrix} -15 & 0 \\ 0 & -15 \end{pmatrix} \]

\[ x^T \left(A^T PA - P \right) x = -15x_1^2 - 15x_2^2 \]

Which is Lyapunov stable.

**Question 4**

Prove that if all eigenvalues of \(A\) have negative real part then there exists a matrix \(P\) such that \(V (x) = x^T Px\) is a Lyapunov function, for the system \(\dot{x} = Ax\).

In order for \(V (x)\) to be a Lyapunov function we require a positive definite \(P\) such that \(A^T P + PA = -Q\) is negative definite (or negative semi-definite), ie. \(\dot{V} (x) = -x^T Qx \leq 0\)

Thus given any suitable positive-definite \(Q\) we need to solve the Lyapunov equation \((A^T P + PA = -Q)\) for \(P\). Consider the definition:

\[ P = \int_0^\infty e^{A^T t}Qe^{At} \, dt \]

which is well defined (since \(A\) has eigenvalues with negative real parts), and also symmetric and positive-definite (from properties of \(Q\)). Then

\[ A^T P + PA = \int_0^\infty A^T e^{A^T t}Qe^{At} + e^{A^T t}Qe^{At} A \, dt \]
\[ = \int_0^\infty \frac{d}{dt} \left[e^{A^T t}Qe^{At}\right] \, dt \]
\[ = [e^{A^T t}Qe^{At}]_0^\infty \]
\[ = -Q \]
Thus $P$ exists and satisfies the Lyaponov equation. Note: to show $P$ is positive definite we consider

\[ x^T P x = \int_0^\infty x^T e^{A^T t} Q e^{A t} x \, dt = \int_0^\infty \| Q^{\frac{1}{2}} e^{A t} x \|^2 \, dt \geq 0 \]

Therefore if $A$ has eigenvalues with negative real parts, then we can choose any positive-definite $Q$, such that $P$ exists and $V(x) = x^T P x$ is a Lyaponov function.

**Question 5**

A rabbit is running on the $x$-axis at a constant velocity $R$, a hound is chasing at a constant velocity $H > R$, always facing directly towards (starting at any location). Show via Lyaponov that the hound catches the rabbit.

Since constant velocity we have the rabbit

\[
\begin{align*}
\dot{x}_r &= R \\
\dot{y}_r &= y_r = 0
\end{align*}
\]

We have the hound moving closer (directly) towards the rabbit, at an equally scaled proportion $c \in \mathbb{R}_+$ based on current position.

\[
\begin{align*}
\dot{x}_h &= -c (x_h - x_r) \\
\dot{y}_h &= -c (y_h - y_r) = -cy_h
\end{align*}
\]

Since the hound runs at constant speed $H$ we have

\[
\begin{align*}
\dot{x}_h^2 + \dot{y}_h^2 &= H^2 \\
c^2 (x_h - x_r)^2 + c^2 y_h^2 &= H^2 \\
c &= \frac{H}{\sqrt{(x_h - x_r)^2 + y_h^2}}
\end{align*}
\]

For simplicity we relabel variables and take the rabbit as the origin $x = x_h - x_r$, $y = y_h$

\[
\begin{align*}
\dot{x} &= \dot{x}_h - \dot{x}_r \\
&= -H \frac{\dot{x}_h}{\sqrt{(x_h - x_r)^2 + y_h^2}} (x_h - x_r) - R \\
&= -H \frac{x_r}{\sqrt{x^2 + y^2}} x - R \\
\dot{y} &= \dot{y}_h \\
&= -H \frac{y}{\sqrt{x^2 + y^2}} y
\end{align*}
\]

Now consider the Lyaponov function $v(x, y) = x^2 + y^2$ which is actually the distance squared between the rabbit and the hound.
\[ \dot{v}(x, y) = 2\dot{x}x + 2\dot{y}y \]
\[ = -2x^2 \frac{H}{\sqrt{x^2 + y^2}} - 2Rx - 2y^2 \frac{H}{\sqrt{x^2 + y^2}} \]
\[ = -2H \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} - 2Rx \]
\[ = -2H \sqrt{x^2 + y^2} - 2Rx \]

Clearly for \( y \neq 0 \) we have \( \dot{v}(0, y) < 0 \), and similarly for \( x > 0 \). When \( x < 0 \) we have

\[ -2H \sqrt{x^2 + y^2} - 2Rx \leq -2H \sqrt{x^2 - 2Rx} \]
\[ = -2H |x| + 2R |x| \]
\[ = 2|x|(R - H) \]
\[ < 0 \]

since \( H > R \). Since \( \dot{v}(x, y) < 0 \) except for origin \((0, 0)\) we know that \( v(x, y) \) (i.e. distance squared) is a Lyaponov function, hence the equilibrium solution of \( 0 \) (hound catches rabbit) is stable.

Thus we have shown that if \( H > R \) the hound always catches the rabbit (no matter where it starts).

**Question 6**

Consider the fixed point iteration to find the square root: \( x^2 = \alpha \) with \( \alpha > 0 \)

\[ x(k + 1) = f(x(k)) \]
\[ = x(k) + \alpha - x^2(k) \]

Clearly if \( x(k) = \sqrt{\alpha} \) we have \( x(k + 1) = \sqrt{\alpha} + \alpha - (\sqrt{\alpha})^2 = \sqrt{\alpha} \) which is a fixed point equilibrium \( \bar{x} \).

We can consider \( x(k) \) and \( x(k + 1) \) in terms of \( \bar{x} \)

\[ x(k) = \bar{x} - \varepsilon_k \]
\[ x(k + 1) = \bar{x} - \varepsilon_{k+1} \]

where convergence requires \( \lim_{k \to \infty} \varepsilon_k = 0 \)

The range of values for convergence can be found by analyzing for small \( \varepsilon \)

\[ x(k + 1) = f(\bar{x} - \varepsilon_k) \]
\[ \bar{x} - \varepsilon_{k+1} = f(\bar{x}) - \varepsilon_k f'(\bar{x}) + \varepsilon_k^2 f''(\bar{x}) + \cdots \]
\[ = \bar{x} - \varepsilon_k (1 - 2\bar{x}) + O(\varepsilon_k^2) \]
\[ \varepsilon_{k+1} = \varepsilon_k (1 - 2\bar{x}) + O(\varepsilon_k^2) \]

Thus we have approximately a geometric sequence
\[ \varepsilon_k \approx (1 - 2\bar{x})^{k-1} \varepsilon_1 \]

which has convergence when

\[
0 < |1 - 2\bar{x}| < 1 \\
0 < \bar{x} < 1
\]

Thus \(0 < \alpha < 1\) and \(0 < x_0 < 1\).

**Question 7**

Consider a system with \(n\) queues and a policy to route to queue \(i\) with probability \(u_i\) (\(\sum_i u_i = 1\)). We have arrival probability \(a\), and departure probability \(d_i\).

\[
X_i(k+1) = X_i(k) + A_i(k) - D_i(k) + L_i(k)
\]

\[
A(k) = \begin{cases} 
  e_1 & \text{w.p } au_1 \\
  \vdots & \\
  e_n & \text{w.p } au_n \\
  0 & \text{w.p } 1 - a 
\end{cases}
\]

\[
D_i(k) = \begin{cases} 
  1 & \text{w.p } d_i \\
  0 & \text{w.p } 1 - d_i 
\end{cases}
\]

\[
L_i(k) = (D_i - (X_i + A_i))^+ 
\]

a) Clearly a necessary (but not necessarily sufficient) condition of stability is that \(a < \sum_{i=1}^{n} d_i\) otherwise our total arrivals will be more than our total departures and the queue will continue to grow which is unstable.

b) For \(n = 2\) find range of \(u\)'s that stabilizes the system based on the parameters \(a\), \(d_1\) and \(d_2\). We require \(au < d_1\) and \(a(1-u) < d_2\) thus we have

\[
a u + a\bar{u} < d_1 + d_2 \\
a u + a - au < d_1 + d_2 \\
a < d_1 + d_2
\]

which is our necessary condition for stability. Then our values for \(u\) (rearranging \(au < d_1\) and \(a(1-u) < d_2\)) are

\[
\max \left\{ 0, 1 - \frac{d_2}{a} \right\} < u < \min \left\{ 1, \frac{d_1}{a} \right\}
\]

since \(0 < u < 1\)

c) Consider “join shortest queue” derive Lyaponov function and show it is bounded by \(-\varepsilon\) with \(\varepsilon > 0\)

Here we have

\[
A(k) = \begin{cases} 
  I_{\{x_1 \leq x_2\}} & \text{w.p } a \\
  I_{\{x_1 > x_2\}} & \\
  0 & \text{w.p } 1 - a
\end{cases}
\]

Intuitively this is good for stability as it distributes work appropriately across the system. Evaluating this with Foster-Lyaponov criterion gives:
\[ PV - V = \mathbb{E} [V (X(t+1)) \mid X(t) = x] - V(x) \]
\[ \leq \mathbb{E} \left[ \sum_{i=1}^{2} \frac{1}{2} (X_i + A_i(t) - D_i(t))^2 - \left(x_1^2 + x_2^2\right) \mid X(t) = x \right] \]
\[ = \sum_{i=1}^{2} x_i \mathbb{E} [A_i(t) - D_i(t) \mid X(t) = x] + \frac{1}{2} \mathbb{E} [(A_i(t) - D_i(t))^2 \mid X(t) = x] \]
\[ \leq 1 + \sum_{i=1}^{2} x_i \mathbb{E} [A_i(t) - D_i(t) \mid X(t) = x] \]
\[ = 1 + \left( x_1 \left(aI_{\{x_1 \leq x_2\}} - d_1\right) + x_2 \left(aI_{\{x_1 > x_2\}} - d_2\right) \right) \]

Consider when \( u = I_{\{x_1 \leq x_2\}} \) then we have \( 1 - u = I_{\{x_1 > x_2\}} \) and stability follows from our previous question.

Since we have that \( x_1 I_{\{x_1 \leq x_2\}} + x_2 I_{\{x_1 > x_2\}} \leq ux_1 + (1 - u)x_2 \) for any \( u \in [0,1] \) we have

\[ PV - V \leq 1 + (x_1 (au - d_1) + x_2 (a(1-u) - d_2)) \]
\[ \leq -\varepsilon + 1 \]

When \( au < d_1 \) and \( a(1-u) < d_2 \). Note: we can ignore the bounded constant 1 in Foster-Lyaponov.

Thus the 'join the shortest queue' policy is just as good as random allocation in terms of stability.