MATH4406 Homework 5

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Question 1

Firstly, $V(x) = x^T x = x_1^2 + x_2^2 > 0$ for all $x \neq 0$, and $DV(x) = (2x_1, 2x_2)$, so V(x) is continuous with continuous partial derivative and has a unique minimum in \mathbb{R}^2 at 0. So consider $A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$. Then

$$\dot{V}(x) = DV(x) \cdot Ax = (2x_1, 2x_2) \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1, x_2) \begin{bmatrix} x_1 + 2x_2 \\ -x_1 - x_2 \end{bmatrix} = x_1^2 - x_2^2 + x_1 x_2.$$

On the line $x_1 = x_2$, this becomes $\dot{V}(x) = x_1^2$ which is positive in every neighbourhood of 0, the unique minimum of V(x). Thus $V(x) = x^T x$ is not a Liapounov function for this system.

This system is stable. Consider

$$W(x) = x^{T} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x = x_{1}^{2} + 2x_{2}^{2} + 2x_{1}x_{2} = x_{2}^{2} + (x_{1} + x_{2})^{2} > 0$$

and $DW(x) = (2x_1+2x_2, 2x_1+4x_2)$ so W(x) is continuous with continuous partial derivative and has a unique minimum in \mathbb{R}^2 at 0. However,

$$\dot{W}(x) = DW(x) \cdot Ax = 2(x_1 + x_2, x_1 + 2x_2) \begin{bmatrix} x_1 + 2x_2 \\ -x_1 - x_2 \end{bmatrix}$$
$$= 2((x_1 + x_2)(x_1 + 2x_2) - (x_1 + 2x_1)(x_1 + x_2)) = 0 \le 0$$

for all $x \in \mathbb{R}$. So W(x) is a Liapounov function for this system, which means this system is stable (but not necessarily stable).

Question 2

As in Question 1, V(x) is continuous and has a unique minimum in \mathbb{R}^2 at x = 0. So consider $A = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$. Then

$$\Delta V(x) = V(Ax) - V(x) = x^T A^T A x - x^T x = (x_1, x_2) \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}^2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - (x_1^2 + x_2^2)$$

$$= (-x_1 - x_2, 0) \begin{bmatrix} -x_1 - x_2 \\ 0 \end{bmatrix} - (x_1^2 + x_2^2) = (x_1^2 + x_2^2)^2 - (x_1^2 + x_2^2) = x_1 x_2.$$

So $\Delta V(x)$ is greater than zero, and thus not a Liapounov function, in a neighbourhood of the function's unique minimum, as every neighbourhood of 0 contains a point such that $x_1, x_2 > 0.$

This system is stable. Consider $W(x) = x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x = x_1^2 + 2x_2^2 + 2x_1x_2 = x_2^2 + (x_1 + x_2)^2 > 0$ for all $x \neq 0$ so W(x) is continuous and has a unique minimum in \mathbb{R}^2 at 0. However,

$$\Delta W(x) = x^T A^T P A x - x P x = (-x_1 - x_2, 0) \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -x_1 - x_2 \\ 0 \end{bmatrix} - (x_1^2 + 2x_2^2 + 2x_1x_2)$$

$$= (-x_1 - x_2, 0) \begin{bmatrix} -x_1 - x_2 \\ -x_1 - x_2 \end{bmatrix} - x_1^2 - 2x_2^2 - 2x_1x_2 = (x_1 + x_2)^2 - x_1^2 - 2x_2^2 - 2x_1x_2 = -x_2^2 \le 0$$

for all $x \in \mathbb{R}$. So W(x) is a Liapounov function for this system, which means this system is stable.

Question 3

In the previous questions, the Liapounov function has been of the form $V(x) = x^T P x$ where P = I.

Consider a function of the form $V(x) = x^T P x$ for a discrete time system. For this function to be an asymptotically stable Liapounov function, $V(Ax) - V(x) = x^T A^T P A x - x^T P x =$ $x^{T}(A^{T}PA - P)x < 0$. Thus, for this function to be an asymptotically stable Liapounov function, $A^T P A - P < 0$, which is equivalent to $P - A^T P A > 0$. In the previous questions, the Liapounov function has been of the form $V(x) = x^T P x$ where P = I. The matrix A was chosen so that this LMI, $I - A^T A > 0$, does not hold. $I - A^T A = I - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ has eigenvalues ± 1 , which means $I - A^T A$ is not positive definite.

Similarly, for the continuous case, for this function to be an asymptotically stable Liapounov function, $\dot{V}(x) = x^T A^T P x + x^T P A x = x^T (A^T P + P A) x \le 0$ which is equivalent to $-(A^TP + PA) \ge 0.$ $A + A^T = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$ has eigenvalues $\pm \sqrt{5}$, which means $-(A^T + A)$ is not positive definite.

Question 4

 $A^T P + P A$ looks like it is the result of the product rule, so assume P = BC where $\dot{B} = A^T B$ and $\dot{C} = CA$. Fortunately these are one of the easier matrix equations to solve, and we have

 $B = e^{A^T t} D$ and $C = D e^{At}$, for some arbitrary constant matrix D. We shall choose D = I for simplicity, which means we have $P = e^{A^T t} e^{At}$, and then

$$A^T P + PA = A^T e^{A^T t} e^{At} + e^{A^T t} e^{At} A = \frac{\mathrm{d}}{\mathrm{d}t} \left(e^{A^T t} e^{At} \right).$$

Then we have

$$\int_{a}^{b} A^{T} P + PA \, \mathrm{d}t = A^{T} \int_{a}^{b} P \, \mathrm{d}t + \int_{a}^{b} P \, \mathrm{d}t A = e^{A^{T} a} D e^{Aa} - e^{A^{T} b} D e^{Ab}$$

Choosing a = 0 and $b = \infty$ we have

$$A^{T} \int_{0}^{\infty} P \, \mathrm{d}t \, + \int_{0}^{\infty} P \, \mathrm{d}t \, A = \lim_{t \to \infty} e^{A^{T}t} e^{At} - e^{0} e^{0}.$$

Now as all the eigenvalues of A are negative, $\lim_{t\to\infty} e^{A^T t} = 0$, and $\lim_{t\to\infty} e^{At} = 0$. So we have

$$A^{T} \int_{0}^{\infty} e^{A^{T}t} e^{At} dt + \int_{0}^{\infty} e^{A^{T}t} e^{At} dt A = -I.$$

Thus, if we take $P = \int_0^\infty e^{A^T t} e^{At} dt$, then $A^T P + PA = -I < 0$. Thus in the continuous case, if $P = \int_0^\infty e^{A^T t} e^{At} dt$, then $A^T P + PA < 0$, which means $V(x) = x^T Px$ is a Liapounov function.

This theorem does not hold for the discrete case. Consider A = -2I. Then $A^T P A - P = 4P - P = 3P$, so for $A^T P A - P$ to be nonpositive definite, then P must be nonpositive definite. But then $V(x) = x^T P x$ has a unique local maximum, rather than a local minimum, so V(x) cannot be a Liapounov function. As P was arbitrary, so then there is no P which makes $V(x) = x^T P x$ a Liapounov function for the system described by A = -2I.

Question 5

Let $h = (h_1, h_2) \in \mathbb{R}$ be the position of the hound in the plane, and $r = (r_1, r_2) \in \mathbb{R}$ be the position of the rabbit in the plane. We have two constraints, the hound's speed $H^2 = \dot{h}_1^2 + \dot{h}_2^2$ and the hound's direction, which is that $D_t h = (\dot{h}_1, \dot{h}_2)$ is antiparallel to $h - r = (h_1 - r_1, h_2)$ i.e. $\frac{\dot{h}_1}{\dot{h}_2} = \frac{h_1 - r_1}{h_2}$. Equating these equations for \dot{h}_1^2 gives $H^2 - \dot{h}_2^2 = \dot{h}_1^2 = \dot{h}_2^2 \frac{(r_1 - h_1)^2}{h_2^2}$ which we can rearrange to give $\dot{h}_2^2 = H^2 h_2^2 ((r_1 - h_1)^2 + h_2^2)^{-1}$. \dot{h}_2 should have the opposite sign to h_2^2 , as the hound is always moving towards the x-axis. So we choose the negative root, which gives

$$\dot{h}_2 = -Hh_2((r_1 - h_1)^2 + h_2^2)^{-\frac{1}{2}}.$$

Using this and the direction constraint we can calculate h_1 :

$$\dot{h}_1 = H(r_1 - h_1)((r_1 - h_1)^2 + h_2^2)^{-\frac{1}{2}}.$$

So the system of equations is

$$\begin{bmatrix} \dot{h_1} \\ \dot{h_2} \\ \dot{r_1} \\ \dot{r_2} \end{bmatrix} = \begin{bmatrix} H(r_1 - h_1)((r_1 - h_1)^2 + h_2^2)^{-\frac{1}{2}} \\ -Hh_2((r_1 - h_1)^2 + h_2^2)^{-\frac{1}{2}} \\ R \\ 0 \end{bmatrix}.$$

We can change co-ordinates so that we fix the rabbit at the origin. Define these co-ordinates as $x = r_1 - h_1$ and $y = h_2$. Then the system of equations becomes

$$\begin{bmatrix} x\\ y \end{bmatrix} = \begin{bmatrix} \dot{r_1} - \dot{h_1}\\ \dot{h_2} \end{bmatrix} = \begin{bmatrix} R - Hx(x^2 + y^2)^{-\frac{1}{2}}\\ -Hy(x^2 + y^2)^{-\frac{1}{2}} \end{bmatrix}$$

Now consider $V(x,y) = \sqrt{x^2 + y^2}$. This function is continuous and has a unique minimum at 0, and $DV(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$ so it has continuous partial derivatives. Then we have

$$\dot{V} = x\dot{x} + y\dot{y} = -Hx^2(x^2 + y^2)^{-1} + R\frac{x}{\sqrt{x^2 + y^2}} - Hy^y(x^2 + y^2)^{-1} = -H + R\frac{x}{\sqrt{x^2 + y^2}}.$$

 $\dot{V} < 0$ if H > R since $\frac{x}{\sqrt{x^2+y^2}} \leq 1$. Thus $V(x,y) = \sqrt{x^2+y^2}$ is a Liapounov function for this system. Since $\dot{V} < 0$, this is an asymptotic Liapounov function. Thus, the origin is asymptotically stable, which means $\lim_{t\to\infty} (x(t), y(t)) = (0,0)$. Switching back to the original co-ordinates gives $\lim_{t\to\infty} (r_1 - h_1, h_2) = \lim_{t\to\infty} h - r = 0$, i.e. the distance between the rabbit and the hound approaches 0. So the hound catches the rabbit (but possibly in infinite time).

Question 6

Under certain conditions, fixed points of recursive equations can be found by repeated application of the function to a sufficiently good initial guess. There are several fixed point theorems that make this idea more explicit, such as the Banach or Brouwer fixed point theorems. This is an intuition that can motivate this method.

Consider $V(x) = (x - \sqrt{\alpha})^2$. This function is continuous and has a unique minimum at

 $\sqrt{\alpha}$.

$$\begin{aligned} \Delta V(x) = V(x + \alpha - x^2) - V(x) \\ = (x + \alpha - x^2 - \sqrt{\alpha})^2 - (x - \sqrt{\alpha})^2 \\ = x^2 + 2x(\alpha - x^2 - \sqrt{\alpha}) + (\alpha - x^2 - \sqrt{\alpha})^2 - x^2 + 2\sqrt{\alpha}x - \alpha \\ = 2x(\alpha - x^2) + (\alpha - x^2 - \sqrt{\alpha})^2 - \alpha \\ = 2x(\sqrt{\alpha} - x)(\sqrt{\alpha} + x) - 2\sqrt{\alpha}(\sqrt{\alpha} - x)(\sqrt{\alpha} + x) + (\alpha - x^2)^2 \\ = (\sqrt{\alpha} + x)(2x(\sqrt{\alpha} - x) - 2\sqrt{\alpha}(\sqrt{\alpha} - x) + (\sqrt{\alpha} + x)(\sqrt{\alpha} - x)^2) \\ = (\sqrt{\alpha} + x)(\sqrt{\alpha} - x)^2(\sqrt{\alpha} + x - 2). \end{aligned}$$

This is less than 0 if $\sqrt{\alpha} + x - 2 < 0$ and $\sqrt{\alpha} + x > 0$ or $2 + \sqrt{\alpha} + x > 0$ and $\sqrt{\alpha} + x < 0$, but as $\sqrt{\alpha} - 2 < \sqrt{\alpha}$, the second case is not possible. So V will be a Liapounov function on the domain $-\sqrt{\alpha} < x < 2 - \sqrt{\alpha}$ if the minimum of the Liapounov function is also in this domain, i.e. $-\sqrt{\alpha} < \sqrt{\alpha} < 2 - \sqrt{\alpha}$. This will occur if $\sqrt{\alpha} < 1$, which implies $0 < \alpha < 1$.

So V(x) is a Liapounov function for this system if 0 < a < 1 on the domain $x \in (-1,1)$. The system will remain in this domain if $x(0) \in (-1,1)$. Thus, as this is a Liapounov function, and under these conditions it is an asymptotic Liapounov function, then $\lim_{k\to\infty} x(k) = \sqrt{\alpha}$. This system may be stable under other conditions, but these conditions guarantee stability. Other Liapounov functions could be used, such as the more general $\sum_{i=1}^{n} \beta_i (x - \sqrt{\alpha})^{2i}$, which may result in less strict conditions on stability.

Question 7

a) The most obvious requirement for stability is that the total rate of (potential) departures from the queues must be greater than the arrival rate. This can be expressed by

$$a < \sum_{i=1}^{n} d_i.$$

b) Each server must be stable. Thus $au_i < d_i$. We are also subject to the constraint that $\sum_{i=1}^{n} u_i = 1$. Assuming $a \neq 0$, for n = 2, we then have

$$u_1 < \frac{d_1}{a}$$
 and $u_2 < \frac{d_2}{a}$.

But as $u_1 = 1 - u_2$ and $u_2 = 1 - u_1$ we then have

$$1 - \frac{d_2}{a} < u_1 < \frac{d_1}{a}$$
 and $1 - \frac{d_1}{a} < u_2 < \frac{d_2}{a}$.

If u_1 and u_1 are in these ranges, then this system will be stable.

Consider $V(x) = \frac{1}{2}x_1^2 + x_2^2$, choose $\epsilon = 1$, and define

$$K = \{ x : d_1 x_1 + d_2 x_2 - a(x_1 I_{x_1 \le x_2} + x_2 I_{x_2 < x_1}) < 2 \}.$$

If we assume $a < d_1 + d_2$, then K is nonempty, as $x = 0 \in K$. As $x_1, x_2 \ge 0$ and $a, d_1, d_2 > 0$,

$$K \subset \{x : x_1 \le \max(\frac{2}{d_1}, \frac{2}{d_1 + d_2 - a}), x_2 \le \max(\frac{2}{d_2}, \frac{2}{d_1 + d_2 - a})\} = J$$

To see this, consider $x \notin J$, so $x_1 > \frac{2}{d_1+d_2-a}, x_2 > \frac{2}{d_1+d_2-a}$. Then if $x_1 \leq x_2$,

$$d_1x_1 + d_2x_2 - a(x_1I_{x_1 \le x_2} + x_2I_{x_2 < x_1}) = d_1x_1 + d_2x_2 - ax_1 > x_1(d_1 + d_2 - a) > \frac{2}{(d_1 + d_2 - a)}(d_1 + d_2 - a) = 2.$$

so
$$x \notin K$$
. Also, if $x_2 < x_1$,

$$d_1x_1 + d_2x_2 - a(x_1I_{x_1 \le x_2} + x_2I_{x_2 < x_1}) = d_1x_1 + d_2x_2 - ax_2 > x_2(d_1 + d_2 - a) > \frac{2}{(d_1 + d_2 - a)}(d_1 + d_2 - a) = 2,$$

so $x \notin K$. As a, d_1, d_2 are fixed and finite, so J is finite, which means K is a finite set.

Now, recall that $X_i(k+1) = X_i(k) + A_i(k) - D_i(k) + L_i(k)$. By definition of $L_i(k)$, $X_i(k+1)$ is either equal to $X_i(k) + A_i(k) - D_i(k)$ or 0, so $X_i(k+1)^2 \le (X_i(k) + A_i(k) - D_i(k))^2$.

$$\mathbb{E}(V(X(1))|X(0)) = \frac{1}{2}\mathbb{E}\left(\left((X_1(0) + A_1(0) - D_1(0) + L_1(0)\right)^2 + (X_2(0) + A_2(0) - D_2(0) + L_2(0))^2\right)|X(0)\right)$$

$$\leq \frac{1}{2}\mathbb{E}\left((X_1(0) + A_1(0) - D_1(0))^2 + (X_2(0) + A_2(0) - D_2(0))^2|X(0)\right)$$

$$= \frac{1}{2}\mathbb{E}\left(X_1(0)^2 + 2X_1(0)(A_1(0) - D_1(0)) + (A_1(0) - D_1(0))^2 + X_2(0)^2 + 2X_2(0)(A_2(0) - D_2(0)) + (A_2(0) - D_2(0))^2|X(0))\right)$$

$$= \frac{1}{2}\mathbb{E}\left(X_1(0)^2 + 2X_1(0)(A_1(0) - D_1(0)) + X_2(0)^2 + 2X_2(0)(A_2(0) - D_2(0))|X(0)\right)$$

$$+ \mathbb{E}\left((A_1(0) - D_1(0))^2 + (A_2(0) - D_2(0))^2\right)$$

as $A_i(0)$ and $D_i(0)$ are independent of $X_i(0)$. For each *i*, at most, only one of $A_i(0), D_i(0)$ can be nonzero, so $A_i(0) - D_i(0)$ can only be either 1, 0 or -1. Thus $(A_i(0) - D_i(0))^2$ can only be 0 or 1, which means $\mathbb{E}((A_i(0) - D_i(0))^2) \leq 1$. So we have

$$\mathbb{E}\left(V(X(1))|X(0)\right) \\ \leq \frac{1}{2} \mathbb{E}\left(X_1(0)^2 + 2X_1(0)(A_1(0) - D_1(0)) + X_2(0)^2 + 2X_2(0)(A_2(0) - D_2(0))|X(0)\right) + 1 \\ = \frac{1}{2}(x_1^2 + x_2^2) + a(x_1I_{x_1 \le x_2} + x_2I_{x_2 < x_1}) - d_1x_1 - d_2x_2 + 1.$$

This is finite if on any finite subset of the sample space \mathbb{N}^2 , in particular, K. Similarly to above we have

$$\begin{split} \mathbb{E} \left(V(X(1)) - V(X(0)) | X(0) \right) \\ &= \mathbb{E} \left(\frac{1}{2} (X_1(1)^2 + X_2(1)^2) - \frac{1}{2} (X_1(0)^2 + X_2(0)^2) | X(0) \right) \\ &\leq \frac{1}{2} \mathbb{E} \left((X_1(0) + A_1(0) - D_1(0))^2 + (X_2(0) + A_2(0) - D_2(0))^2 - X_1(0)^2 - X_2(0)^2 | X(0) \right) \\ &= \frac{1}{2} \mathbb{E} \left(2X_1(0) (A_1(0) - D_1(0)) + (A_1(0) - D_1(0))^2 + 2X_2(0) (A_2(0) - D_2(0)) + (A_2(0) - D_2(0))^2 | X(0) \right) \\ &\leq \mathbb{E} \left(X_1(0) (A_1(0) - D_1(0)) + X_2(0) (A_2(0) - D_2(0)) | X(0) \right) + 1 \\ &= a(x_1 I_{x_1 \le x_2} + x_2 I_{x_2 \le x_1}) - d_1 x_1 - d_2 x_2 + 1 \end{split}$$

If $x \notin K$,

$$\mathbb{E}\left(V(X(1)) - V(X(0))|X(0)\right) \le 1 - (d_1x_1 + d_2x_2 - a(x_1I_{x_1 \le x_2} + x_2I_{x_2 < x_1})) \le 1 - 2 = -1 = -\epsilon.$$

Thus $\mathbb{E}\left(V(X(1)) - V(X(0))|X(0)\right) \le -\epsilon$, where $\epsilon > 0$ and independent of state.

Thus, all the conditions in the Foster-Liapounov Theorem hold, we have that this system is positive redcurrant, i.e. this system is stable. As the only assumption we made was that $a < d_1 + d_2$, then this condition is a sufficient condition for stability for the "join the shortest queue policy". However, this condition is a necessary condition. This means that every other policy must also satisfy this condition for stability. So for any other policy, the conditions for stability must be at least as strict as the conditions on the "join the shortest queue policy". Thus, the "join the shortest queue policy" is as good as can be in terms of stability.