MATH4406 - Control Theory Course Summary

Luke Marshall (41521209)

November 13, 2012

How do you get a satellite to stay in orbit, or land a rocket on the moon? and how on earth do those segway machines stay upright? The answer is Control Theory, a powerful field of mathematics that has many practical applications in our world today.

The concept of control has been around throughout history, but until recently it has predominantly involved human interaction (e.g. steering / maintaining speed of a horse, car or bike), in particular it wasn't until around 1920 when there was a rapid development in automatic control devices [1, p.vii]. Control theory uses mathematical notation to analyse a problem; it does not particularly care how solutions are actually implemented (e.g. hardware, software or even human), as long as the characteristics of the system are adequately described.

The majority of control problems can be generalized by the following question. Based on an input and control, how does the state of our system change over time (with continuous or discrete steps) in order to achieve a desired output? Thus our typical model is of the form [2, p5]:

$$\begin{array}{rcl|rcl} \dot{x}(t) & = & f(t, x(t), u(t)) & x(k+1) & = & f(k, x(k), u(k)) \\ y(t) & = & g(t, x(t), u(t)) & y(k) & = & g(k, x(k), u(k)) \end{array} \tag{1}$$

Where u is the *input* vector $(m \times 1)$, x the *state* vector $(n \times 1)$, y the *output* vector $(p \times 1)$, with f and g vector valued functions for continuous-time systems (above left) and discrete-time systems (above right).

From our understanding of ordinary differential equations (ODE), we know that many complex real-world problems can be written as a system of first-order ODE's, and if these equations are continuous differentiable (C^1) then we can linearize this system around a particular solution. It is clear then, that we can model (or approximate) many problems with the simplified equations below, and for the purposes of our discussion we will only consider these linear systems; i.e. with matrices $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{p\times n}$, and $D(t) \in \mathbb{R}^{p\times m}$:

$$\begin{array}{rcl|rcl} \dot{x}(t) & = & A(t)x(t) + B(t)u(t) & x(k+1) & = & A(k)x(k) + B(k)u(k) \\ y(t) & = & C(t)x(t) + D(t)u(t) & y(k) & = & C(k)x(k) + D(k)u(k) \end{array} \tag{2}$$

NOTE: since most fundamental concepts are applicable to both continuous / discrete systems, for simplicity we shall focus only on the continuous case.

If our input is dependent on our output (i.e. $u(t) = g(y(t), t)$) we say our system is a closed loop (otherwise open loop). An intuitive notation for modelling these problems is via block diagrams, where figure 1 shows an example of a controlled system using a closed loop.

In these diagrams we can easily combine our inputs (via addition / multiplication etc) to describe complex systems; but mathematically this requires us to operate on signals, that is, using convolutions $(f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau)$. Since these can be difficult - how can we possibly solve our problems?

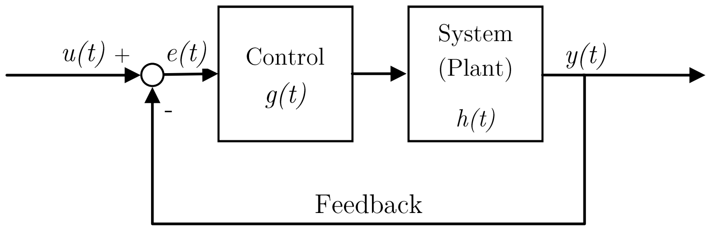

Figure 1: An example block diagram for a closed loop system

Laplace transforms ($\mathcal{L}_{\{f(t)\}}(s) = \int_0^\infty f(t)\, e^{-st}\, dt$) are a powerful tool that maps (i.e. isomorphism - a bijective structure-preserving map) difficult time-based differential equations into easier frequency-based algebraic equations [3, p63]. In this way our system of equations are much simpler to solve, with the main challenge in transforming to/from our system.

Conveniently our transform is a linear operator; recall that an operator $\mathcal{O}$ is called linear if given $y_1(t) = \mathcal{O}(u_1(t))$, and $y_2(t) = \mathcal{O}(u_2(t))$, we have $y(t) = \mathcal{O}(c_1 u_1(t) + c_2 u_2(t)) = c_1\mathcal{O}(u_1(t)) + c_2\mathcal{O}(u_2(t)) = c_1 y_1(t) + c_2 y_2(t)$, for all c_1,c_2. Since this transform and our system (by assumption) are linear, many complex problems can be practically solved using this method.

To see this in action we consider the popular Proportional-Integral-Derivative (PID) controller (figure 2), which is so effective that it is the standard controller in industrial systems [4, p185].

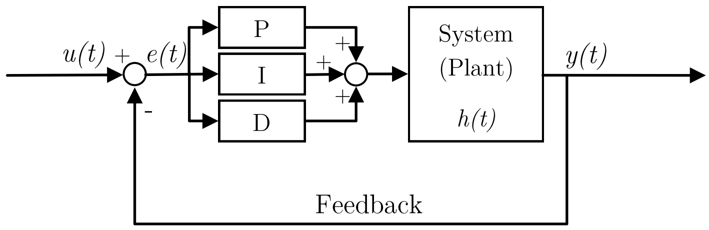

Figure 2: A PID controller

PID minimizes the error of a system by using a linear combination of three controllers:

$$P(t) = K_p e(t), \quad I(t) = K_i \int_0^t e(\tau)\, d\tau, \quad D(t) = K_d \frac{d}{dt} e(t)$$

which gives (via Laplace) the combined controller: $G(s) = K_p + \frac{K_i}{s} + K_d s$ Transforming our entire problem, gives $Y(s) = U(s)\,G(s)\,H(s) - Y(s)\,G(s)\,H(s) = \frac{U(s)G(s)H(s)}{1+G(s)H(s)}$. Since the inverse is relatively straight forward to find, the real challenge is choosing appropriate parameters (K_p, K_i, K_d) to control the system.

But in general, what do we want to control? Are we trying to stabilize a system, or guide it to another state? Perhaps we'd like to find the optimal path or simply keep overshoot within a tolerance? Or more fundamentally, do solutions even exist and if so are they unique? and what about asymptotic behaviour? All these questions and more drive at the heart of control theory.

Often our systems share common characteristics and so we can generalize our answers to classes of problems; for example when the output $y(t)$ only depends on the inputs up to time t (i.e. it doesn't depend on future input values) we call the system causal, or nonanticipative; when A, B, C, D does not depend on t we call this linear time-invariant (LTI) - many physical processes are modelled by causal LTI systems, which have nice analytic properties (see equation (3)), and we can visualize these responses using Bode plots. [5, p46-51,74,436]

Taking the Laplace transform (with initial value equal to zero) on our LTI system gives:

$$Y(s) = \underbrace{\left(C\,(sI - A)^{-1}\,B + D\right)}_{\text{transfer function}} U(s) \tag{3}$$

If for equation (2) we have $m = p = 1$ then our system is said to have single-input-single-output (SISO), otherwise we have a multiple-input-multiple-output (MIMO) or various combinations, and if $\|u(t)\| < \infty$ and $\|y(t)\| < \infty \; \forall t$ we have bounded-input-bounded-output (BIBO).

Boundedness is a concept that's linked with stability, which is a very important consideration since often we need to know how/if the system can explode. Informally, stability means that small inputs do not cause our output to diverge; mathematically we often define stability in terms of an equilibrium point, this is known as Lyapunov stability [2, p144].

Many methods exist that test for stability, for example a causal LTI SISO system is BIBO stable if all the poles (i.e. roots of the denominator) of the transfer function (see equation (3)) have negative complex parts [6, p370]. Numerically this is simple to check with computers, even with high order polynomials; historically however this was a major problem, thus the Routh test, Hurwitz test and the Nyquist plot were developed to check for stability without actually calculating the roots of the polynomial.

A popular approach for testing internal stability of LTI systems (i.e. ignoring inputs) is via Lyapunov's Direct Method [2]: assuming an equilibrium at the origin, the system is stable if we can find a *Lyapunov function* $v(x)$, i.e. that satisfies the following characteristics: locally positive definite, continuous and has continuous partial derivatives on our domain (D), has a unique minimum at the equilibrium point, and $\dot{v}(x) \leq 0 \; \forall x \in D$ (for asymptotic stability we instead require a strict inequality). A common function choice is $v(x) = x^T P x$, with positive definite matrix P.

Of course stability is not the only important system property, for instance other useful concepts are *controllability* and *observability.* A state $x_0 \in D$ is called *controllable* if there exists an input that transfers the system from x_0 to the zero state in some finite time [2, p196] and a system is *observable* if is possible to determine the present state $x(t_0)$ from the knowledge of current and future outputs $y(t)$, and inputs $u(t)$, $t \geq t_0$ [2, p219].

This is easily verified if we have an LTI system, by checking that the controllability matrix $\mathcal{C}$ has full rank, and similarly the observability matrix $\mathcal{O}$ has full rank. An alternative approach uses *Kalman Decomposition* to convert the system to a special form that allows us to easily identify the observable and controllable parts [2, p245].

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C & CA & CA^2 & \dots & CA^{n-1} \end{bmatrix}^T$$

Controllability and observability have important implications on state feedback and state estimation:

State feedback: Consider an unstable LTI system, if it is fully controllable then we can add a feedback control F such that the system $\dot{x} = (A + BF)x$ is asymptotically stable, by arbitrarily assigning the eigenvalues of $A+BF$. If we have some uncontrollable eigenvalues, but all of them are stable, then the system is called stabilizable [2, p354-356].

State estimation: Consider a LTI system where we do not know some (or all) values of our state. This is often the case in practice, since it may not be possible to measure some or all of our values using appropriately placed sensors, for these cases we can estimate the missing states only if the system is observable. With this in mind, the *Luenberger observer* models the estimated state $\hat{x}$ using the system: $\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y})$ with $\hat{y} = C\hat{x} + Du$. Now if the eigenvalues of $[A - KC]$ have negative complex parts, then it can be shown that the estimated state asymptotically converges to the actual state, regardless of initial values [2, p378]. Alternatively, the *Kalman filter* is another well known estimator, which is quite effective when our state is a stochastic process.

Realistic models are often faced with the problem of how to recover from disturbances to the system, especially since it is practically impossible to predict these future values. One method is to incorporate stochastic noise into the model by using random variables:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + \xi_x(t) \\ y(t) &= C(t)x(t) + D(t)u(t) + \xi_y(t) \end{aligned}$$

Here $\xi_x(t), \xi_y(t)$ are iid Normal random variables with zero mean and Σ_x, Σ_y covariance matrices. Solving these systems can be difficult, and numerical solutions (via computers) have profoundly influenced theory development. The *Kalman filter* was a particularly significant advancement which recursively calculates the estimates (weighted by covariance) over time, with very good results. The separation theorem shows that this can be used to solve *optimal* stochastic control problems, like the Linear Quadratic Gaussian (LQG) which tries to minimize the expected value of a quadratic cost function. By using Kalman filters to estimate the state, we can then use deterministic dynamic programming to find the optimal solution. [7, p5-8]

For many problems we often desire to find the "best" solution, which is ranked by a cost function of our choosing. Optimal control is a large field of research that attempts to cover this class of problems via various techniques. For example, lets consider the Linear Quadratic Regulator (LQR) problem which is the deterministic equivalent to LQG (above). Our system is simply $\dot{x} = Ax + Bu$, $x(0) = x_0$, and our cost function is defined as

$$J(u) = \int_0^T \left[x^T Q x + u^T R u\right] dt + x(T)^T Q_f x(T)$$

with $Q = Q^T \geq 0$, $Q_f = Q_f^T \geq 0$, $R \succ 0$, and we are trying to optimize (minimize) for $u(t)$. By using classic techniques like calculus of variations (Pontryagin's minimum principle) and dynamic programming (Hamilton-Jacobi-Bellman equation) we can find the optimal solution to be

$$u^*(t) = \left[-RB^T P(t)\right] x(t)$$

where $P(t)$ is found by solving the Riccati equation $-\dot{P}(t) = A^T P + PA - PBR^{-1}B^T P + Q$ (with $P(T) = Q_f$) which is straightforward via numerical methods. [8] [9]

Linear models can solve many problems, however in some applications the nonlinear terms dominate (eg. involving large rotations, like robotics etc) and so we must consider non-linear control. Many of the concepts mentioned above (stability, controllability, optimal control, etc) still apply to our non-linear system, but extra care is required to avoid issues (eg. chaos).[10]

We hope this summary has given a tiny glimpse into the wonderful world of control theory, it is an often silent but major influence on our everyday lives.

References

[1] S. Bennett, *A History of Control Engineering: 1930-1955*. IET, 1993.

[2] P. J. Antsaklis and A. N. Michel, "Feedback control systems," in *A Linear Systems Primer*. Boston, MA: Birkhauser Boston, pp. 411–453.

[3] W. Arendt, C. J. K. Batty, M. Hieber, and F. Newbrander, *Vector-valued Laplace Transforms and Cauchy Problems*, 2nd ed., ser. Monographs in Mathematics. Springer, 2011, vol. 96.

[4] G. F. Franklin, J. D. Powell, and A. Emami-Naeini, *Feedback control of dynamic systems*. Pearson, 2010.

[5] A. V. A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, *Signals & Systems*. Prentice Hall (Sd), 1997.

[6] L. Chaparro, *Signals and Systems using MATLAB*. Burlington: Academic Press, 2010.

[7] K. J. Astrom, *Introdution to Stochastic Control Theory*. New York: Academic Press, 1970.

[8] F. L. Lewis, D. L. Vrabie, and V. L. Syrmos, *Optimal Control, Third Edition*, Jan. 2012.

[9] S. Boyd, "Continuous time linear quadratic regulator," Stanford Universtity, 2008. [Online]. Available: http://www.stanford.edu/class/ee363/

[10] A. Isidori, *Nonlinear Control Systems*. Springer, Aug. 1995.