# MATH4406 (Control Theory) <br> Quiz 1 (Units 2 and 3) - August 23, 2012. <br> Prepared by Yoni Nazarathy 

Quiz duration: 40 minutes.
An input output system (plant) follows this differential equation:

$$
\dot{y}(t)+c_{1} y(t)=c_{2} u(t)
$$

with $c_{1}, c_{2} \in \mathbb{R}$.

1) Treating $u(\cdot)$ as the input and $y(\cdot)$ as the output, write the transfer function of the plant.

Solution (10 points): By setting $y(0)=0$ and taking the Laplace transform, we get,

$$
H_{1}(s)=\frac{Y(s)}{U(s)}=\frac{c_{2}}{s+c_{1}}
$$

2) For what values of $c_{1}, c_{2}$ is the plant strictly stable?

Solution (10 points):

$$
\left(c_{1}, c_{2}\right) \in \mathbb{R}_{+} \times \mathbb{R}
$$

i.e. $c_{1}>0$ and $c_{2}$ is not restricted.
3) What is the step response of the plant (i.e. the output resulting from input $\mathbf{1}(t))$ ?

Solution (10 points):
Since the impulse response is $h_{1}(t)=\mathcal{L}^{-1}\left(H_{1}(\cdot)\right)(t)=c_{2} e^{-c_{1} t} \mathbf{1}(t)$, the step response is,

$$
\int_{0}^{t} h(s) d s=\frac{c_{2}}{c_{1}}\left(1-e^{-c_{1} t}\right) \mathbf{1}(t)
$$

$\ll$ There was no question $\# 4 \gg$.

Consider now a second plant with transfer function:

$$
H(s)=\frac{1}{\frac{c_{1}}{2}+s}
$$

5) The output signal of the first plant is fed into the second plant to form a combined plant. Assume that $c_{1}, c_{2}>0$. Describe the combined plant as a second order system: 5a) What are the parameters, $\zeta$ and $\omega_{n}$ ?

## Solution (20 points):

The combined plant is $H(s)=\frac{c_{2}}{\left(s+c_{1}\right)\left(s+\frac{c_{1}}{2}\right)}$. I.e. it has two real poles, one at $s=-c_{1}$ and one at $s=-c_{1} / 2$. One of the forms of second order systems is,

$$
\frac{K \omega_{n}^{2}}{\left(s-p_{1}\right)\left(s-p_{2}\right)},
$$

where $p_{1}, p_{2}=-\zeta \omega_{n} \pm \omega_{n} \sqrt{\zeta^{2}-1}$. Since the poles are real, it must mean that on $-\zeta \omega_{n}$ lies on the half way point, i.e.

$$
-\zeta \omega_{n}=-\frac{3}{4} c_{1},
$$

and further,

$$
\omega_{n} \sqrt{\zeta^{2}-1}=\frac{c_{1}}{4}
$$

I.e,

$$
\omega_{n}=\frac{c_{1}}{\sqrt{2}}, \quad \zeta=\frac{3}{2 \sqrt{2}} .
$$

Further,

$$
K=2 \frac{c_{2}}{c_{1}} .
$$

$5 b)$ Does it have a resonant peak in any frequencies?

## Solution (10 points):

No, there is no resonant peak since the poles are pure real $(\zeta>1)$ and there are no zeros.
6) Assume that the plant is now controlled in closed loop with a PD compensator with parameters $K_{P}=\frac{1}{2}$ and $K_{D}=2$. Consider now the controlled system:
6a) Is there a steady state error to the step response or is it zero? If there is a steady state error, explain briefly why? Further, suggest a modification to the controller that
will remove the error.

## Solution (20 points):

$$
H_{c}(s)=\frac{\left(\frac{1}{2}+2 s\right) H(s)}{1+\left(\frac{1}{2}+2 s\right) H(s)}=\frac{\left(\frac{1}{2}+2 s\right) c_{2}}{\left(s+c_{1}\right)\left(s+\frac{c_{1}}{2}\right)+\left(\frac{1}{2}+2 s\right) c_{2}}=\frac{2 c_{2} s+\frac{1}{2} c_{2}}{s^{2}+\left(\frac{3}{2} c_{1}+2 c_{2}\right) s+\frac{1}{2}\left(c_{2}+c_{1}^{2}\right)}
$$

It is now clear that the system type is zero because $H_{c}(\cdot)$ is of the form $\frac{N(s)}{s^{0} D(s)}$ where $N(\cdot)$ and $D(\cdot)$ don't have zeros at $s=0$. Hence there is a strictly $\neq 0$ steady state error. One can also calculate this error using the final value theorem (although this was not requested for the question):

$$
e_{\infty}=\lim _{s \rightarrow 0} s E(s)=\lim _{s \rightarrow 0} s\left(\frac{1}{s}-\frac{1}{s} H_{c}(s)\right)=\lim _{s \rightarrow 0}\left(1-H_{c}(s)\right)=\frac{c_{1}^{2}}{c_{1}^{2}+c_{2}} .
$$

The standard modification to the controller that will remove the steady state error (by increasing the effective system type to 1 ) is to add an integrator term to the controller.
$6 \mathrm{~b})$ For values of $c_{1}, c_{2}=1$ is there a resonant peak? If so, explain briefly (1-3 lines) what this means.

## Solution (10 points) :

In this case,

$$
H_{c}(s)=\frac{2 s+\frac{1}{2}}{s^{2}+\frac{7}{2} s+1}
$$

The two poles are at $-\frac{7}{4} \pm \frac{\sqrt{33}}{4}$ and there is a zero at $-\frac{1}{4}$. If there was no zero, it would be clear that there is NO resonant peak, but since there is a zero it is not clear. There is no easy way to see (to the best of my knowledge) if there is a resonant peak other than substituting $i \omega$ for $s$ and seeing if there is a local maximum in $\left|H_{c}(i \omega)\right|$. It was not the intention of the question - and it is thus all answers that make sense (and don't say wrong things - other than guessing) got credit. Note that in fact THERE IS a resonant peak at $\omega=0.73$.
$6 \mathrm{c})$ Assume that in the same controlled system, there is a pure delay of 3 time units on the feedback loop. I.e. the signal value that is subtracted from the reference at time $t_{0}$ is the output of the system at time $t_{0}-3$. Write the closed loop transfer function.

## Solution (10 points):

$$
H_{c}(s)=\frac{\left(\frac{1}{2}+2 s\right) H(s)}{1+\left(\frac{1}{2}+2 s\right) H(s) e^{-3 s}} .
$$

(Bonus) Return back to the single original plant. Set $c_{1}, c_{2}=1$. Let now $y[k]$ be a discrete time sequence of samples at times $k=0,1,2, \ldots$, of the output resulting from the input $u(t)=\mathbf{1}(t-1)$ (and $y(0)=y[0]=0)$. Write an expression for $y[k]$.

## Solution (15 points bonus):

Note: This is actually a stupid question. Use the answer of question 3:

$$
y(t)=\mathcal{O}(\mathbf{1}(t))(t-1)=\left(1-e^{-(t-1)}\right) \mathbf{1}(t-1)
$$

In the way the question is worded there is no distinction between $y[k]$ and $y(t)$.

Good Luck.

