MATH4406 (Control Theory) Quiz 1 (Units 2 and 3) - August 23, 2012. Prepared by Yoni Nazarathy

Quiz duration: 40 minutes.

An input output system (plant) follows this differential equation:

$$\dot{y}(t) + c_1 y(t) = c_2 u(t),$$

with $c_1, c_2 \in \mathbb{R}$.

1) Treating $u(\cdot)$ as the input and $y(\cdot)$ as the output, write the transfer function of the plant.

Solution (10 points): By setting y(0) = 0 and taking the Laplace transform, we get,

$$H_1(s) = \frac{Y(s)}{U(s)} = \frac{c_2}{s+c_1}.$$

2) For what values of c_1 , c_2 is the plant strictly stable?

Solution (10 points):

$$(c_1, c_2) \in \mathbb{R}_+ \times \mathbb{R}.$$

i.e. $c_1 > 0$ and c_2 is not restricted.

3) What is the step response of the plant (i.e. the output resulting from input $\mathbf{1}(t)$)?

Solution (10 points):

Since the impulse response is $h_1(t) = \mathcal{L}^{-1}(H_1(\cdot))(t) = c_2 e^{-c_1 t} \mathbf{1}(t)$, the step response is,

$$\int_0^t h(s)ds = \frac{c_2}{c_1}(1 - e^{-c_1t})\mathbf{1}(t).$$

<< There was no question #4 >>.

Consider now a second plant with transfer function:

$$H(s) = \frac{1}{\frac{c_1}{2} + s}.$$

5) The output signal of the first plant is fed into the second plant to form a *combined* plant. Assume that $c_1, c_2 > 0$. Describe the combined plant as a second order system: 5a) What are the parameters, ζ and ω_n ?

Solution (20 points):

The combined plant is $H(s) = \frac{c_2}{(s+c_1)(s+\frac{c_1}{2})}$. I.e. it has two real poles, one at $s = -c_1$ and one at $s = -c_1/2$. One of the forms of second order systems is,

$$\frac{K\omega_n^2}{(s-p_1)(s-p_2)},$$

where $p_1, p_2 = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$. Since the poles are real, it must mean that on $-\zeta \omega_n$ lies on the half way point, i.e.

$$-\zeta\omega_n = -\frac{3}{4}c_1,$$

and further,

$$\omega_n \sqrt{\zeta^2 - 1} = \frac{c_1}{4}.$$

I.e,

$$\omega_n = \frac{c_1}{\sqrt{2}}, \qquad \zeta = \frac{3}{2\sqrt{2}}.$$

 $K = 2\frac{c_2}{c_1}.$

Further,

5b) Does it have a resonant peak in any frequencies?

Solution (10 points):

No, there is no resonant peak since the poles are pure real ($\zeta > 1$) and there are no zeros.

6) Assume that the plant is now controlled in closed loop with a PD compensator with parameters $K_P = \frac{1}{2}$ and $K_D = 2$. Consider now the controlled system: 6a) Is there a steady state error to the step response or is it zero? If there is a steady state error, explain briefly why? Further, suggest a modification to the controller that will remove the error.

Solution (20 points):

$$H_c(s) = \frac{(\frac{1}{2} + 2s)H(s)}{1 + (\frac{1}{2} + 2s)H(s)} = \frac{(\frac{1}{2} + 2s)c_2}{(s + c_1)(s + \frac{c_1}{2}) + (\frac{1}{2} + 2s)c_2} = \frac{2c_2s + \frac{1}{2}c_2}{s^2 + (\frac{3}{2}c_1 + 2c_2)s + \frac{1}{2}(c_2 + c_1^2)}$$

It is now clear that the system type is zero because $H_c(\cdot)$ is of the form $\frac{N(s)}{s^0 D(s)}$ where $N(\cdot)$ and $D(\cdot)$ don't have zeros at s = 0. Hence there is a strictly $\neq 0$ steady state error. One can also calculate this error using the final value theorem (although this was not requested for the question):

$$e_{\infty} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\left(\frac{1}{s} - \frac{1}{s}H_c(s)\right) = \lim_{s \to 0} \left(1 - H_c(s)\right) = \frac{c_1^2}{c_1^2 + c_2}.$$

The standard modification to the controller that will remove the steady state error (by increasing the effective system type to 1) is to add an integrator term to the controller.

6b) For values of $c_1, c_2 = 1$ is there a resonant peak? If so, explain briefly (1-3 lines) what this means.

Solution (10 points) :

In this case,

$$H_c(s) = \frac{2s + \frac{1}{2}}{s^2 + \frac{7}{2}s + 1}.$$

The two poles are at $-\frac{7}{4} \pm \frac{\sqrt{33}}{4}$ and there is a zero at $-\frac{1}{4}$. If there was no zero, it would be clear that there is NO resonant peak, but since there is a zero it is not clear. There is no easy way to see (to the best of my knowledge) if there is a resonant peak other than substituting $i\omega$ for s and seeing if there is a local maximum in $|H_c(i\omega)|$. It was not the intention of the question - and it is thus all answers that make sense (and don't say wrong things - other than guessing) got credit. Note that in fact THERE IS a resonant peak at $\omega = 0.73$.

6c) Assume that in the same controlled system, there is a pure delay of 3 time units on the feedback loop. I.e. the signal value that is subtracted from the reference at time t_0 is the output of the system at time t_0-3 . Write the closed loop transfer function.

Solution (10 points):

$$H_c(s) = \frac{(\frac{1}{2} + 2s)H(s)}{1 + (\frac{1}{2} + 2s)H(s)e^{-3s}}.$$

(Bonus) Return back to the single original plant. Set $c_1, c_2 = 1$. Let now y[k] be a discrete time sequence of samples at times k = 0, 1, 2, ..., of the output resulting from the input $u(t) = \mathbf{1}(t-1)$ (and y(0) = y[0] = 0). Write an expression for y[k].

Solution (15 points bonus):

Note: This is actually a stupid question. Use the answer of question 3:

$$y(t) = \mathcal{O}(\mathbf{1}(t))(t-1) = (1 - e^{-(t-1)})\mathbf{1}(t-1).$$

In the way the question is worded there is no distinction between y[k] and y(t).

Good Luck.