# MATH4406 (Control Theory) <br> Solution to: Quiz 2 (Unit 4) - September 6, 2012. <br> Prepared by Yoni Nazarathy 

Quiz duration: 40 minutes.
Consider the system,

$$
\begin{aligned}
\dot{x}(t) & =A x(t)+B u(t), \\
n y(t) & =C x(t)+D u(t),
\end{aligned}
$$

with,

$$
A=\left[\begin{array}{cc}
\alpha & 0 \\
0 & \alpha
\end{array}\right], \quad B=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right], \quad C=\left[\begin{array}{ll}
1 & 0 \\
1 & 2
\end{array}\right], \quad D=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] .
$$

1) Write the impulse response matrix of the system, i.e. the output $y(t)$ resulting from input, $u(t)=(\delta(t), \delta(t))^{\prime}$ with $x(0)=(0,0)$.

Solution (20pts):

$$
H(t)=\mathbf{1}(t)\left(C e^{A t} B+D \boldsymbol{\delta}(t)\right)=\left[\begin{array}{cc}
e^{\alpha t}+\delta(t) & 0 \\
e^{\alpha t} & 2 e^{\alpha t}+\delta(t)
\end{array}\right]
$$

2) For what values of $\alpha$ is the system controllable?

Solution (20pts):

$$
\operatorname{con}(A, B)=\left[\begin{array}{ll}
I & A
\end{array}\right]=\left[\begin{array}{llll}
1 & 0 & \alpha & 0 \\
0 & 1 & 0 & \alpha
\end{array}\right]
$$

This matrix has full row rank $(=2)$ for any $\alpha$ hence the system is controllable for any value of $\alpha$.
3) For what values of $\alpha$ is the system observable?

## Solution (20pts):

$$
\operatorname{obs}(A, C)=\left[\begin{array}{c}
C \\
C A
\end{array}\right]=\left[\begin{array}{cc}
1 & 0 \\
1 & 2 \\
\alpha & 0 \\
\alpha & 2 \alpha
\end{array}\right]
$$

This matrix has full column rank $(=2)$ for any $\alpha$ hence the system is observable for any value of $\alpha$.
4) Set now $\alpha=0$. Determine a feedback control law,

$$
u(t)=F x(t)+r(t)
$$

so that the eigenvalues of the resulting system are -1 and -2 and the resulting system is not observable.

Note: If you are not able to satisfy both of the above criteria, partial points will be given for satisfying one of the above. Further, if you can formulate the problem of satisfying both of the above criteria clearly, but are not able to solve it, you will also get partial points.

Solution (40pts): The system controlled by state feedback is,

$$
(A+B F, B, C+D F, D)=(F, I, C+F, I)
$$

For this system, obs $(\mathrm{F}, \mathrm{C}+\mathrm{F})=\left[\begin{array}{c}C+F \\ (C+F) F\end{array}\right]$.

It is now evident that if $F$ is set to be $-C$ then the observability matrix is all 0 's and is thus of rank $0<2$ and thus the system is not observable.

Further, observe that the eigenvalues of $F=-C$ are -1 and -2 exactly as required.

Good Luck.

