MATH4406 (Control Theory)
Solution to Quiz 3 (Unit 5) - September 20, 2012.

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Quiz duration: 40 minutes.

1) Consider the system,

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$

with state-feedback controller,

$$u(t) = [\alpha \quad 1]x(t).$$

Assume we want to show that the closed loop system (with 0 input is stable) using the Lyapounov function $V(x) = x'x$. For which values of α is this possible?

Solution: The closed loop system with feedback **(10 pts)** is:

$$\dot{x}(t) = \begin{bmatrix} \alpha - 1 & 1 \\ \alpha & 2 \end{bmatrix}.$$

The drift condition on the suggested Lyapounov function reduces to **(10 pts)**,

$$-\Big(\begin{bmatrix} \alpha - 1 & 1 \\ \alpha & 2 \end{bmatrix} + \begin{bmatrix} \alpha - 1 & \alpha \\ 1 & 2 \end{bmatrix} \Big) > 0,$$

or,

$$\begin{bmatrix} 2 - 2\alpha & -1 - \alpha \\ -1 - \alpha & -4 \end{bmatrix} > 0.$$

Now for which α does the above hold **(20 pts)**?

The short answer is that a necessary condition for a matrix to be positive definite is to have non-negative elements on the diagonal (easy to see and prove). Here this is violated with the entry -4. So there is not α that satisfies the requirement.

Alternatively, one can work (a bit) harder and check one of the several necessary and sufficient conditions, e.g. using eigenvalues or determinants.

2) Consider discrete time linear systems,

$$x(k + 1) = Ax(k),$$

with $A \in \mathbb{R}^{n \times n}$. Assume that we want to find a Lyapounov function of the form,

$$V(x) = Qx + x'Px.$$

Show that in that case the matrices Q and P need to satisfy,

$$(QA - Q)x + x'(A'PA - P)x < 0, \qquad \forall x \neq 0.$$

Solution: (20 pts) We need to have,

$$V(x(k+1)) - V(x(k)) < 0.$$

This can be written as,

$$V(Ax) - V(x) < 0,$$

or,

$$QAx + (Ax)'PAx - Qx - x'Px < 0,$$

or,

$$(QA - Q)x + x'A'PAx - x'Px = (QA - Q)x + x'(A'PA - P)x < 0,$$

as desired.

3) Consider the system:

$$\begin{aligned} \dot{x_1} &= x_2(1 - x_1) \\ \dot{x_2} &= -x_1(1 - x_2) \end{aligned}$$

a) Show that 0 is an equilibrium point.
b) Show that the system is stable. If you wish, try using the Lyapounov function,

$$V(x_1, x_2) = -x_1 - \log(1 - x_1) - x_2 - \log(1 - x_2).$$

Solution:
a) **(10 pts)** It is trivial to see this by setting $(x_1, x_2) = (0, 0)$ in the right hand side.

b)

$$\nabla V(x_1, x_2) = \left(\frac{1}{1 - x_1} - 1 \quad , \quad \frac{1}{1 - x_2} - 1 \right)'$$

(15 pts)

$$\begin{aligned} \frac{d}{dt}V\big(x(t)\big) = \nabla V(x_1, x_2)'\dot{x} &= \left(\frac{1}{1 - x_1} - 1\right)x_2(1 - x_1) - \left(\frac{1}{1 - x_2} - 1\right)x_1(1 - x_2) \\ &= (x_2 - x_2(1 - x_1) - x_1 + x_1(1 - x_2) \\ &= x_1x_2 - x_2x_1 = 0 \leq 0 \end{aligned}$$

(5 pts) Further, observe that $V(x_1, x_2)$ is continuous in a region containing 0.

(5 pts) Further, observe that $V(x_1, x_2)$ is uniquely minimized at 0 since $V(0, 0) = 0$ and $V(x_1, x_2) \geq 0$. To see the latter you can use the known inequality $e^{-x} > 1 - x$.

(5pts) Now since the three conditions above are satisfied 0 is a stable point.