Solving constrained LQR using MPC

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Unconstrained LQR

Problem:

$$J = \min_{u(k)} \sum_{k=0}^{\infty} x(k)^T Q x(k) + u(k)^T R u(k)$$

subject to

$$x(k+1) = Ax(k) + Bu(k) \qquad x(0) = x_0$$

Assumptions:

- R positive definite
- Q positive semi-definite
- (A, B) stabilizable
- (C, A) detectable ($Q = C^T C$ with rank C=rank Q)

Unconstrained LQR

Solution:

$$u(k) = -Fx(k)$$

where

$$F = (R + B^T PB)^{-1} B^T PA$$

and P the unique positive definite solution to the discrete time algebraic Riccati equation (DARE)

$$P = Q + A^T \big(P - PB(R + B^T PB)^{-1} B^T P\big) A$$

The resulting costs, starting in $x(0) = x_0$ is given by

$$J(x_0) = x_0^T P x_0$$

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Constrained LQR

Problem:

$$J = \min_{u(k)} \sum_{k=0}^{\infty} x(k)^T Q x(k) + u(k)^T R u(k)$$

subject to

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \qquad\qquad x(0) = x_0 \\ u(k) &\in \mathbb{U} = \{u \mid K_u u \leq k_u\} \\ x(k) &\in \mathbb{X} = \{x \mid K_x x \leq k_x\} \end{aligned}$$

with $k_u > 0$, $k_x > 0$, i.e. $0 \in \mathbb{U}$ and $0 \in \mathbb{X}$.

Constrained LQR

Let $F = (R + B^T P B)^{-1} B^T P A$ with $P = P^T > 0$ solution to the DARE.

Define a terminal set $\mathbb{X}_T \subset \mathbb{X}$ satisfying

$$(A - BF)\mathbb{X}_T \subset \mathbb{X}_T \qquad\qquad -F\mathbb{X}_T \subset \mathbb{U}$$

Constrained LQR

Consider the following MPC-problem:

$$J = \min_{u(0),u(1),\ldots,u(N-1)} x(N)^T P x(N) + \sum_{k=0}^{N-1} x(k)^T Q x(k) + u(k)^T R u(k)$$

subject to

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \qquad\qquad x(0) = x_0 \\ u(k) &\in \mathbb{U} \\ x(k) &\in \mathbb{X} \\ x(N) &\in \mathbb{X}_T \end{aligned}$$

Let $u^*(0), u^*(1), \ldots, u^*(N-1)$ denote the solution to this QP (assuming it is feasible for $x(0) = x_0$).
Then $u^*(0), u^*(1), \ldots, u^*(N-1)$, $u^*(k) = -Fx(k)$ for $k \geq N$ is a feasible solution to the constrained LQR problem.

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Constrained LQR

Typical choice for $\mathbb{X}_T$: maximal output admissible set, see [GT91].

Typical choice for $N(x_0)$: such that $x * (N(x_0)) \in \mathbb{X}_T$ for MPC-problem without the constraint $x(N) \in \mathbb{X}_T$.

Also possible to determine upperbound for $N^*(x_0)$ for given x_0, or initial state set, see e.g. [ZL08]

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