MATH4406 (Control Theory)
Unit 2: SISO Linear Systems and Background

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1 About

The first sections of this unit are about some key properties of convolutions, integral transforms and related concepts. Conditions for existence and finiteness are not the focus here. We use the term continuous time and discrete time functions/signals to mean that their domains are subsets of $\mathbb{R}$ or $\mathbb{Z}$ respectively.

The remaining sections (section 8 and onwards) about linear time invariant (LTI) systems with a single input and a single output (SISO). The basics of some of the aspects appearing in an undergraduate "Signals and Systems" course are covered. The treatment of indefinite integrals, generalized functions and other limiting objects is not rigourous yet suffices for the understanding of this material needed in engineering applications of control.

Warning: Don't confuse "continuous time" (in these notes) with "continuous".

2 Signals

The term *signal* is essentially synonymous with a function, yet a possible difference is that a signal can be described by various different representations, each of which is a different function.

Signals may be discrete time or continuous time. Although some signals are "digitized", their values are typically taken as real (or complex). Signals may be either scalar or vector.

When talking about SISO linear systems, a signal, $u(t)$ may be viewed as: $u : \mathbb{R} \to \mathbb{R}$ if the setting is that of continuous time or $u : \mathbb{Z} \to \mathbb{R}$ in the discrete time setting.

It is typical and often convenient to consider a signal through an integral transform (e.g. the Laplace transform) when the transform exists.

Example 1 *Consider the signal,*

$$u(t) = \begin{cases} 0 & t < 0, \\ e^{-t} & 0 \le t. \end{cases}$$

The Laplace transform is,

$$\hat{u}(s) = \int_0^\infty e^{-st} e^{-t} dt = \frac{1}{s+1}, \quad \textit{for } s \ge -1.$$

In this case, both $u(t)$ and $\hat{u}(s)$ represent the same signal. We often say that $u(t)$ is the time-domain *representation of the signal where as $\hat{u}(s)$ is the* frequency-domain *representation.*

It is common to do operations on signals. Here are a few very common examples:

- $\tilde{u}(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$: Add, subtract, scale or more generally take linear combinations.
- $\tilde{u}(t) = u(t-\tau)$: Translation. Shift forward in case $\tau > 0$ (delay) by τ.
- $\tilde{u}(t) = u(-t)$: Reverse time.
- $\tilde{u}(t) = u(\alpha t)$: Time scaling. Stretch (for $0 < \alpha < 1$). Compress (for $1 < \alpha$).
- $\tilde{u}(n) = u(nT)$: Sample to create a discrete time signal from a continuous one signal.
- $\tilde{u}(t) = \sum_n u(nT) K\big(\frac{t-nT}{T}\big)$, where $K(\cdot)$ is an *interpolation function*. I.e. it has the properties $K(0) = 1, K(n) = 0$ for other integers $n \neq 0$.

Exercise 1 *Find the $K(\cdot)$ that will do* linear interpolation, *i.e. connect the dots. Illustrate how this works on a small example.*

3 Convolutions

Definitions and Applications

Let $f(\cdot)$, $g(\cdot)$ be two functions. The convolution of f and g is the function $\big(f * g\big)(\cdot)$:

$$\big(f * g\big)(t) := \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau.$$

If the functions are of positive support ($= 0$ for $t < 0$) the range of integration in the convolution integral reduces to $\tau \in [0, t]$.

For a probabilist, the convolution is the basic tool of finding the distribution of the sum of two independent random variables X and Y, say with densities $f_X(\cdot)$ and $f_Y(\cdot)$:

$$F_{X+Y}(t) := P(X + Y \leq t) = \int_{-\infty}^{\infty} \int_{-\infty}^{t-x} \mathbb{P}\big((X,Y) \in [x, x+dx) \times [y, y+dy)\big) dy\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{t-x} f_X(x) f_Y(y) dy\, dx = \int_{-\infty}^{\infty} f_X(x) \Big(\int_{-\infty}^{t-x} f_Y(y) dy \Big)\, dx.$$

So for the density, $f_{X+Y}(t) := \frac{d}{dt}F_{X+Y}(t)$, we have

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(x)\Big(\frac{d}{dt}\int_{-\infty}^{t-x} f_Y(x)dy\Big)\, dx = \int_{-\infty}^{\infty} f_X(x)f_Y(t-x)dx = \big(f_X * f_Y\big)(t).$$

Convolution is also defined for discrete time functions (in probability theory this often corresponds to the probability mass function of the sum of two independent discrete random variables):

$$P_{X+Y}(n) = \mathbb{P}\big(X+Y=n\big) = \sum_{k=-\infty}^{\infty} \mathbb{P}\big(X=k\big)\mathbb{P}\big(Y=n-k\big) = \big(P_X * P_Y\big)(n).$$

Note again that if P_X and P_Y are of positive support ($= 0$ for $t < 0$) then the range of summation in the convolution sum reduces to $k \in \{0, \dots, n\}$.

Another way to view discrete convolutions is as a representation of the coefficients of polynomial products. Denote,

$$A(x) = \sum_{j=0}^{n-1} a_j x^j, \quad B(x) = \sum_{j=0}^{n-1} b_j x^j, \quad C(x) = A(x)B(x) = \sum_{j=0}^{2n-2} c_j x^j.$$

Exercise 2 *Show that* $c_j = \sum_{k=0}^{j} a_k b_{j-k}$. *Note: Assume* $a_i, b_i = 0$ *for* $i \notin \{0, \dots, n-1\}$.

Our use of convolutions will be neither for probability nor for polynomial products - but rather for the "natural" application of determining the action of a linear system on an input signal.

Algebraic Properties

- Commutativity:

$$\big(f*g\big)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau = \big(g*f\big)(t)$$

- Associativity:

$$\big(f*g\big)*h = f*\big(g*h\big)$$

- Distributivity:

$$f*(g+h) = f*g + f*h.$$

- Scalar multiplication:

$$\alpha(g*h) = (\alpha g)*h = g*(\alpha h).$$

- Shift/Differentiation:

$$D(g * h) = (Dg) * h = g * (Dh),$$

where D is either the "delay by one" operator for discrete time or the differentiation operator for continuous time.

Exercise 3 *Show the shift/differentiation property. Do both shift (discrete time) and differentiation (continuous time).*

Sometimes the notation f^{*m} is used for $f * f * \ldots * f$, m times.

If f is a probability density with mean μ and finite variance σ^2, the central limit theorem (CLT) in probability says that as $m \to \infty$, $\frac{f^{*m}(t)-m\mu}{\sqrt{m}\sigma}$ converges to the normal (Gaussian) density:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

Exercise 4 *Let $f(t) = \mathbf{1}(t)\mathbf{1}(1-t)$. Find f^{*k}, $k = 2, 3, 4$. Do the same for $f_2(t) = e^{-t}\mathbf{1}(t)$. Plot the resulting functions and relate this to the CLT.*

Sufficient conditions for existence of the convolution

The *support* of a function f is the (closure of the) set of values for which $f(t) \neq 0$. We often talk about *positive support* if the the support does not contain negative values, and also about *bounded support* if the support is a bounded set..

A continuous time function, f is *locally integrable* if $\int_a^b |f(t)|dt$ exists and is finite for every a, b.

Theorem 1 *The convolution $f_1 * f_2$ in continuous time exists if both signals are locally integrable and if one of the following holds*

1. *Both signals have bounded support.*
2. *Both signals have positive support.*
3. *$||f_1||_2$ and $||f_2||_2$ are both finite.*

Theorem 2 *The theorem above holds for discrete time signals without the locally integrable requirement. In that case the L_2 norms above are taken as ℓ_2 norms.*

4 Laplace Transforms

Let s be a complex number, the Laplace transform of a continuous time function $f(t)$ at the "*frequency*" $f(\cdot)$ is,

$$\mathcal{L}\{f(\cdot)\}(s) = \int_{0^-}^{\infty} e^{-st} f(t) dt. \tag{1}$$

We shall often denote $\mathcal{L}\{f(\cdot)\}$ by $\hat{f}$. Observe the lower limit to be 0^- and read that as,

$$\lim_{\epsilon \to 0^-} \int_{\epsilon}^{\infty} e^{-st} f(t).$$

This is typical "engineering notation" as the function $f(\cdot)$ may sometimes have "peculiarities" at 0. For example may have a generalized function component. In applied probability and other more rigorous mathematical contexts, the Laplace-Stiltijes Transform is often used,

$$\int_0^{\infty} e^{-st} dF(t),$$

where the above is a Stiltijes integral. We shall not be concerned with this here. Our Laplace transform, (1) is sometimes referred to as the *one-sided Laplace transform*. Whereas,

$$\mathcal{L}_B\{f(\cdot)\}(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt,$$

is the *bilateral Laplace transform*. The latter is not as useful and important as the former for control purposes. An exception is the case of $s = i\omega$ (pure imaginary) in which case,

$$\hat{f}(\omega) = \mathcal{L}_B\{f(\cdot)\}(i\omega),$$

is the (up to a constant) Fourier transform of f (here we slightly abuse notation by using the "hat" for both Laplace and Fourier transforms). Note that in most engineering text the symbol $i = \sqrt{-1}$ is actually denoted by j.

In probability, the Laplace transform of a density, f_X of a continuous random variable has the meaning, $\mathbb{E}\left[e^{-sX}\right]$. This has many implications and applications which we shall not discuss.

Existence, Convergence and ROC

A function $f(t)$ is said to be of *exponential order* as $t \to \infty$ if there is a real σ and positive real M, T such that for all $t > T$,

$$|f(t)| < M e^{\sigma t}. \tag{2}$$

The function e^{t^2} is not of exponential order but most signals used in control theory are.

Exercise 5 *Show that the following is an alternative definition to* exponential order*: There exists a real $\tilde{\sigma}$ such that,*

$$\lim_{t\to\infty} \left| f(t)e^{-\tilde{\sigma}t} \right| = 0.$$

Exercise 6 *Show that any rational function is of exponential order.*

For a function of exponential oder, the *abscissa of convergence*, σ_c, is the greatest lower bound (infimum) of all possible values σ in (2). Hence for polynomials, $\sigma_c = 0$ while for functions of the form $e^{t\alpha}$ with $\alpha > 0$, $\sigma_c = \alpha$.

Exercise 7 *What is the abscissa of convergence of a rational function $f(t) = \frac{a(t)}{b(t)}$ (here $a(t)$ and $b(t)$ are polynomials and $a(t)$ is of lower degree)?*

Theorem 3 *Functions $f(t)$ that are locally integrable and are of exponential order with σ_c have a Laplace transform that is finite for all $Re(s) > \sigma_c$.*

The regieon in the complex *s-plane*: $\{s : Re(s) > \sigma_c\}$ is denoted the *regieon of convergence* (ROC) of the Laplace transform.

Proof

$$|\hat{f}(s)| = \Big| \int_{0^-}^{\infty} e^{-st} f(t)dt \Big| \leq \int_{0^-}^{\infty} \big|e^{-st}\big|\big|f(t)\big|dt.$$

Writing $s = \sigma + i\omega$ we have $|e^{-st}| = e^{-\sigma t}$, so for all $\sigma' > \sigma_c$

$$|\hat{f}(s)| \leq M \int_{0^-}^{\infty} e^{-\sigma t} e^{\sigma' t} dt = M \int_{0^-}^{\infty} e^{-(\sigma - \sigma')t} dt.$$

This integral is finite whenever $\sigma = Re(s) > \sigma'$. Now since σ' can be chosen arbitrarily close such that $\sigma' > \sigma_c$ we conclude that the transform exists whenever $\sigma > \sigma_c$. □

Uniqueness

Laplace transforms uniquely map to their original "time-functions". In fact, this is the inversion formula:

$$f(t) = \lim_{M\to\infty} \frac{1}{2\pi i} \int_{\sigma - iM}^{\sigma + iM} e^{st} \hat{f}(s) ds,$$

for any $\sigma > \sigma_c$. The integration is in the complex plane and is typically not the default method.

Exercise 8 *Optional (only for those that have taken a complex analysis course). Apply the inversion formula to show that,*

$$\mathcal{L}^{-1}\Big(\frac{1}{(s+a)^2}\Big) = te^{-at}.$$

Basic Examples

Example 2 *The Laplace transform of $f(t) = c$:*

$$\mathcal{L}(c) = \int_0^\infty e^{-st} c\,dt = \lim_{T\to\infty} \int_0^T e^{-st} c\,dt = \lim_{T\to\infty} \Big[-\frac{c}{s} e^{-st} \Big]_0^T = \frac{c}{s}\big(1 - \lim_{T\to\infty} \ e^{-sT}\big).$$

When does the limit converge to a finite value? Take $s = \sigma + i\omega$,

$$\lim_{T\to\infty} e^{-sT} = \lim_{T\to\infty} \ e^{-\sigma T}(\cos\omega T + i\sin\omega T).$$

So we need $\sigma > 0$ to get $\lim_{T\to\infty} \ e^{-sT} = 0$, hence,

$$\hat{f}(s) = \frac{c}{s}, \quad Re(s) > 0.$$

Exercise 9 *Show that the transform of $f(t) = e^{\alpha t}$ is,*

$$\hat{f}(s) = \frac{1}{s-\alpha}, \quad Re(s) > Re(\alpha).$$

Exercise 10 *Derive the Laplace transform (and find ROC) of*

$$f(t) = e^{-at}\cos(bt).$$

For other examples see a *Laplace transform table*.

Basic Properties

You should derive these.

- Linearity:
$$\mathcal{L}\big(\alpha_1 f_1(t) + \alpha_2 f_2(t)\big) = \alpha_1 \hat{f}_1(t) + \alpha_2 \hat{f}_2(t).$$

- Time shift:
$$\mathcal{L}\big(f(t-\theta)\big) = \int_{0^-}^\infty f(t-\theta)e^{-st}dt = \int_{0^-}^\infty f(t)e^{-s(t+\theta)}dt = e^{-s\theta}\hat{f}(t).$$

- Frequency shift:
$$\mathcal{L}\big(e^{-at}f(t)\big) = \hat{f}(s+a).$$

- Time Scaling:
$$\mathcal{L}\big(f(at)\big) = \frac{1}{|a|}\hat{f}\big(\frac{s}{a}\big).$$

- Differentiation:

$$\mathcal{L}\big(f'(t)\big) = \int_{0^-}^{\infty} f'(t)e^{-st}dt = f(t)e^{-st}\big|_0^{\infty} + s\int_{0^-}^{\infty} f(t)e^{-st}dt = -f(0^-) + s\hat{f}(s).$$

- Integration:

$$\mathcal{L}\Big(\int_0^t f(x)dx\Big) = \frac{1}{s}\hat{f}(s).$$

More basic properties are in one of tens of hundreds of tables available in books or on the web.

Relation To Differential Equations

The differentiation formula allows to transform differential equations into algebraic equations for s. Then the equations may be solved in the *s-plane* and transformed back to obtain the solutions.

Exercise 11 *Solve using the Laplace transform:*

$$\ddot{x}(t) + 6x(t) = \cos\big(\frac{t}{2}\big),$$

with $x(0) = 0$, $\dot{x}(0) = 0$.

Relation To Convolution

This property is very important:

$$\mathcal{L}\big(f_1(t) * f_2(t)\big) = \hat{f}_1(s)\hat{f}_2(s).$$

Exercise 12 *Prove it.*

5 Rational Laplace Transforms and Partial Fraction Expansion

Often Laplace (as well as Fourier and Z) transforms are of the rational form,

$$\hat{f}(s) = \frac{p(s)}{q(s)} = \frac{p_m s^m + \ldots + p_1 s + p_0}{q_n s^n_+ \ldots\ldots\ldots + q_1 s + a_0},$$

with p_i, q_i either real or complex coefficients (we mostly care about real coefficients) such that, $p_m, q_n \neq 0$. The function $\hat{f}(\cdot)$ is called *proper* if $m \leq n$, *strictly proper* if $m < n$ and *improper* if $m > n$.

If $\hat{f}(s)$ is not strictly proper, then by performing *long division* it may be expressed in the form,

$$r(s) + \frac{v(s)}{q(s)},$$

where $r(s)$ is a polynomial of degree $m - n$ and $v(s)$ is a polynomial of degree $< n$.

Exercise 13 *Carry long division out for,*

$$\hat{f}(s) = \frac{s^4 + 2s^3 + s + 2}{s^2 + 1},$$

to express it in the form above.

The action of performing *partial fraction expansion* is the action of finding the coefficients A_{ik} such that a strictly proper $\hat{f}(\cdot)$ in the form,

$$\hat{f}(s) = \sum_{i=1}^{K} \Big(\sum_{k=1}^{m_i} \frac{A_{ik}}{(s - s_i)^k} \Big),$$

where $s_1, \ldots, s_K$ are the distinct real or complex roots of $q(s)$, and the multiplicity of root s_i is m_i.

After carrying out long division (if needed) and partial fraction expansion, $\hat{f}(s)$ may be easily inverted, term by term.

Example 3 *Consider,*

$$\hat{f}(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}.$$

We want the form,

$$\hat{f}(s) = \frac{A_{11}}{s+1} + \frac{A_{21}}{s+2}.$$

This to equation,

$$1 = A_{11}(s+2) + A_{21}(s+1). \tag{3}$$

or,

$$1 = (A_{11} + A_{21})s + (2A_{11} + A_{21}). \tag{4}$$

Now "identity coefficients of terms with like powers of s" to get a set of linear equations:

$$\begin{aligned} A_{11} + A_{21} &= 0 \\ 2A_{11} + A_{21} &= 1 \end{aligned}$$

to get $A_{11} = 1$ and $A_{21} = -1$.

Example 4 *Consider,*

$$\hat{f}(s) = \frac{s-1}{s^3 - 3s - 2} = \frac{s-1}{(s+1)^2(s-2)}.$$

We want the form,

$$\hat{f}(s) = \frac{A_{11}}{s+1} + \frac{A_{12}}{(s+1)^2} + \frac{A_{21}}{s-2}.$$

Similar to before, we may get a system of equations for the A_{ik}.

Exercise 14 *Complete the partial fraction expansion of the above example.*

When the coefficients of $q(\cdot)$ are real, the roots are complex conjugate pairs (say with multiplicity m_i). In this case we may write for any pair of roots, s_i and $\overline{s_i}$,

$$(s - s_i)(s - \overline{s_i}) = s^2 + a_i s + b_i,$$

where a_i and b_i are real coefficients. In this case, the partial fraction expansion is of the form,

$$\hat{f}(s) = \ldots\ldots + \frac{B_{i1}s + A_{i1}}{s^2 + a_i s + b_i} + \frac{B_{i2}s + A_{i2}}{(s^2 + a_i s + b_i)^2} + \ldots + \frac{B_{im_i}s + A_{im_i}}{(s^2 + a_i s + b_i)^{m_i}} + \ldots\ldots.$$

A similar technique may be used to find the B's and A's.

Exercise 15 *Carry out a partial fraction expansion for,*

$$\hat{f}(s) = \frac{s+3}{(s^2 + 2s + 5)(s+1)}.$$

6 The Fourier Transform in Brief

The Fourier transform of $f(t)$ is:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.$$

The inverse fourier transform is,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} dw.$$

Exercise 16 *Find the Fourier transform of $f(t) = \frac{\sin(t)}{t}$.*

Conditions for convergence:

Theorem 4 *A sufficient condition for convergence of the Fourier integral is that $f(\cdot)$ satisfies the following:*

- $\int_{-\infty}^{\infty} \big|f(t)\big| dt < \infty$.
- $f(\cdot)$ *has a finite number of maxima and minima in any finite interval.*
- $f(\cdot)$ *has a finite number of discontinuities within any finite interval. Furthermore each of these discontinuities must be finite.*

By means of generalized functions, the Fourier transform may also be defined (and convergences) for periodic functions that are not absolutely integrable.

Basic Properties

Many properties are very similar to the Laplace transform (the Fourier transform is a special case of the bilateral Laplace transform).

Some further important properties are:

- The transform of the product $f_1(t)f_2(t)$ is $\big(\hat{f}_1 * \hat{f}_2\big)(\cdot)$. This has far reaching implications in signal processing and communications.
- Parseval's Relation (energy over time = energy over spectrum):

$$\int_{-\infty}^{\infty} \big|f(t)\big|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \big|\hat{f}(\omega)\big|^2 d\omega.$$

Graphical Representaions

Plots of $\big|\hat{f}(\omega)\big|$ and $\angle \hat{f}(\omega)$ are referred to by engineers as *Bode plots*. It is typical to stretch the axis of the plots so that the horizontal axis is $\log_{10}(\omega)$ and the vertical axis are $20 \log_{10} \big|\hat{f}(\omega)\big|$ and $\angle \hat{f}(\omega)$. There is a big tradition in engineering to generate approximate bode plots by hand based on first and second order system approximations.

An alternative plot is the *Nyquist plot*

Exercise 17 *Generate a Bode and a Nyquist plot of a system with transfer function,*

$$H(s) = \frac{1}{s^2 + s + 2}.$$

7 The Z Transform in Brief

This is the analog of the Laplace transform for discrete time functions, $f(n)$. The *Z-transform* is defined as follows,

$$\hat{f}(n) = \sum_{k=-\infty}^{\infty} f(n) z^{-n}.$$

Many of the things we do for continuous time using the Laplace transform may be done for discrete time using the Z-transform. We will not add further details in this unit, but rather touch discrete time systems when we talk about general (MIMO) linear systems.

8 The Delta Function and Generalized (Singular) Signals

Engineering (and mathematics) practice of continuous time signals is often greatly simplified by use of *generalized signals*. The archetypal such signal is the *delta*-function denoted by $\delta(t)$, also called *impulse*. This "weird" mathematical object has the following two basic properties:

1. $\delta(t) = 0$ for $t \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(t) dt = 1$.

Now obviously there is no such function $\delta : \mathbb{R} \to \mathbb{R}$, that obeys these two properties if the integral is taken in the normal sense (e.g. Reiman integral). The rigorous description of delta functions is part of the theory of distributions (not to be confused with probability distributions). We shall overview it below informally and then survey a few useful properties of the delta function. First, one should be motivated by the fact that in practice the delta function can model the following:

1. The signal representing the energy transfer from a hammer to a nail.
2. The "derivative" of the unit step function,
3. A Gaussian (normal) density of variance 0.

A more formal (yet not fully rigorous) way to define delta functions is "under the integral sign". It can be thought of as an "entity" that obeys,

$$\int_{-\infty}^{\infty} \delta(t)\phi(t) dt = \phi(0), \tag{5}$$

for every (regular) function ϕ that is continuous at 0 and has bounded support (equals 0 outside of a set containing the origin). Entities such as $\delta(t)$ are not regular functions - we will never talk about the "value" of $\delta(t)$ for some t, but rather always consider values of integrals involving $\delta(t)$. Yet from a practical perspective they may often be treated as such.

An important operation in the study of linear systems is the *convolution*. We use '$*$' to denote the convolution operator. For two continuous time signals, $f(t)$ and $g(t)$. The signal $h = f * g$ resulting from their convolution is,

$$h(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau.$$

The delta function gives a way to represent any signal $u(t)$. Consider the convolution, $\delta * u$:

$$\int_{-\infty}^{\infty} \delta(\tau)u(t-\tau)d\tau = u(t-0) = u(t). \tag{6}$$

Thus we see that the δ function is the identity "function" with respect to convolutions:

$$\delta * u = u.$$

The discrete time version of the convolution $h = f * g$ is,

$$h(n) = \sum_{k=-\infty}^{\infty} f(k)g(n-k).$$

In this case a discrete parallel of (6) is,

$$\big(\delta * u\big)(n) = \sum_{k=-\infty}^{\infty} \delta[k]u(n-k) = u(n). \tag{7}$$

Here $\delta[n]$ is the *discrete delta function* (observe the square brackets), a much simpler object than $\delta(t)$ since it is defined as,

$$\delta[n] = \begin{cases} 1 & n = 0, \\ 0 & n \neq 0. \end{cases}$$

Thus we have again that $\delta * u = u$. Note that part of the motivation for introducing for the continuous time delta function is to be able to mimic the representation (7).

We shall soon present other generalized signals related to the delta function. Since such functions are "defined under the integral" sign, two signals $\eta_1(t)$ and $\eta_2(t)$ are equal if,

$$\int_{-\infty}^{\infty} \eta_1(t)\phi(t)dt = \int_{-\infty}^{\infty} \eta_2(t)\phi(t)dt,$$

for a "rich enough class" of functions, $\phi(\cdot)$.

For generalized signals $\eta_1(\cdot)$ and $\eta_2(\cdot)$ and for scalars α_1, α_2, we define the signal of the linear combination as,

$$\int_{-\infty}^{\infty} \big(\alpha_1\eta_1 + \alpha_2\eta_2\big)(t)\phi(t)dt = \alpha_1 \int_{-\infty}^{\infty} \eta_1(t)\phi(t)dt + \alpha_2 \int_{-\infty}^{\infty} \eta_2(t)\phi(t)dt.$$

Exercise 18 *Prove that:* $\alpha_1\delta + \alpha_2\delta = (\alpha_1 + \alpha_2)\delta$.

For regular functions $f(\cdot)$ and $\alpha \neq 0$ we have (by a simple change of variables) that,

$$\int_{-\infty}^{\infty} f(\alpha t)\phi(t)dt = \frac{1}{|\alpha|}\int_{-\infty}^{\infty} f(\tau)\phi(\frac{\tau}{\alpha})d\tau.$$

For generalised signals this is taken as the definition of time scaling:

$$\int_{-\infty}^{\infty} \delta(\alpha t)\phi(t)dt = \frac{1}{|\alpha|}\int_{-\infty}^{\infty} \delta(\tau)\phi(\frac{\tau}{\alpha})d\tau = \frac{1}{|\alpha|}\phi(0) = \frac{1}{|\alpha|}\int_{-\infty}^{\infty} \delta(t)\phi(t)dt.$$

Here the first equality is a definition. and the second and third equalities come from the defining equation (5). This then implies that

$$\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t).$$

Consider now translation. Take some time shift θ:

$$\int_{-\infty}^{\infty} \delta(t-\theta)\phi(t)dt = \int_{-\infty}^{\infty} \delta(\tau)\phi(\tau+\theta)d\tau = \phi(0+\theta) = \phi(\theta).$$

Hence delta functions translated by θ, denoted $\delta(t-\theta)$ are defined by

$$\int_{-\infty}^{\infty} \delta(t-\theta)\phi(t)dt = \phi(\theta).$$

Consider now what happens when $\delta(t)$ is multiplied by a function $f(t)$ continuous at 0. If $\delta(t)$ was a regular function then,

$$\int_{-\infty}^{\infty} \Big(f(t)\delta(t)\Big)\phi(t)dt = \int_{-\infty}^{\infty} \delta(t)\Big(f(t)\phi(t)\Big)dt = f(0)\phi(0)$$

It is then sensible to define the generalized function, $f(t)\delta(t)$ (for any regular function $f(\cdot)$) as satisfying:

$$\int_{-\infty}^{\infty} \Big(f(t)\delta(t)\Big)\phi(t) = f(0)\phi(0)$$

Hence we have that,

$$f(t)\delta(t) = f(0)\delta(t).$$

This again follows from (5).

Exercise 19 *Take τ as fixed and $t \in R$. Show that,*

$$f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau).$$

Example 5 *A useful generalized function is the so-called "Dirac Comb", also known as "impulse train":*

$$\Delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

Here of course one needs to justify the existence of the series (of generalized functions !!!) etc, but this is not our interest.

Impulse trains are very useful for representing the operation of sampling a continuous time (analog) signal. This is done by taking the signal $u(t)$ and multiplying by $\Delta_T(t)$. The resulting signal has values $u(t)$ for $t = kT$, $k \in N$ and 0 elsewhere.

The derivation of the famous Nyquist-Shannon sampling theorem is greatly aided by the impulse train. That theorem says that a "band limited" analog signal $u(t)$ can be perfectly reconstructed if sampled at a rate that is equal or greater than twice its highest frequency.

Related to the delta function is the *unit step function*,

$$\mathbf{1}(t) = \begin{cases} 0 & t < 0, \\ 1 & 0 \leq t. \end{cases} \tag{8}$$

This is sometimes called the "Heaviside unit function". Other standard notation for it is $u(t)$, but in control theory we typically reserve $u(t)$ for other purpuses (i.e. the input to a system). While it is a function in the regular sense, it can also be defined as a generalized function:

$$\int_{-\infty}^{\infty} \mathbf{1}(t)\phi(t)dt = \int_{0}^{\infty} \phi(t)dt, \tag{9}$$

where $\phi(t)$ is any integrable function.

Exercise 20 *Derive (8) from (9).*

Given a generalized function $\eta(t)$, we define it's *generalized derivate*, $\eta'(t)$ (again a generalized function) by:

$$\int_{-\infty}^{\infty} \eta'(t)\phi(t)dt = -\int_{-\infty}^{\infty} \eta(t)\phi'(t)dt.$$

The above definition applied to $\mathbf{1}(t)$ yields,

$$\int_{-\infty}^{\infty} \mathbf{1}'(t)\phi(t)dt = -\int_{-\infty}^{\infty} \mathbf{1}(t)\phi'(t)dt = -\int_{0}^{\infty} \phi'(t)dt = -\big(\phi(\infty) - \phi(0)\big) = \phi(0).$$

We have just shown that $\mathbf{1}' = \delta$.

Exercise 21 *Show that* $\mathbf{1}'(t-\theta) = \delta(t-\theta)$.

We can also look at the derivative of the delta function:

$$\int_{-\infty}^{\infty} \delta'(t)\phi(t)dt = -\int_{-\infty}^{\infty} \delta(t)\phi'(t)dt = -\phi'(0).$$

This generalized function is sometimes called a *doublet*. Higher order derivatives of a generalized function η are defined by,

$$\int_{-\infty}^{\infty} \eta^{(n)}(t)\phi(t)dt = (-1)^n \int_{-\infty}^{\infty} \eta(t)\phi^{(n)}(t)dt,$$

here $\phi(t)$ needs to be any function from a "suitable" set of test functions. We will not discuss generelized function in any more depth than covered here. Students interested in functional analysis and related fields can study more about Schwartz's theory of distributions indepdently.

9 Systems - Basic Definitions

A system is a mapping of an input signal to an output signal. When the signals are scalars the system is called SISO. When inputs are vectors and outputs are vectors the system is called MIMO (Multi Input Multi Output). Other combinations are MISO and SIMO. We concentrate on SISO in this unit. As in the figure below, we typically denote the output of the system by $y(t)$.

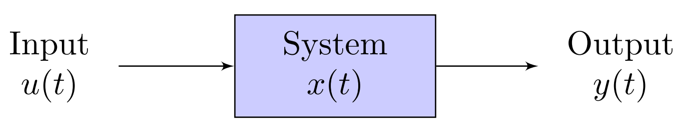

Figure 1: A system operates on an input signal $u(\cdot)$ to generate an output signal $y(\cdot) = \mathcal{O}\big(u(\cdot)\big)$. The system may have a *state*, $x(t)$. This unit does not focus on state representations and thus, $x(t)$ is often ignored here.

A system is *memoryless* if the output at time t depends only on the input at time t. I.e. $y(t) = g\big(u(t)\big)$ for some scalar function $g(\cdot)$. These systems are typically quite boring.

A system is *non-anticipating* (or *causal*) if the output at time t depends only on the inputs during times up to time t. This is defined formally by requiring that for all t_0, whenever the inputs u_1 and u_2 obey $u_1(t) = u_2(t)$ for all $t \le t_0$, the corresponding outputs y_1 and $y_2(t)$ obey $y_1(t) = y_2(t)$ for all $t \le t_0$.

A system is *time - invariant* if its behaviour does not depend on the actual current time. To formally define this, let $y(t)$ be the output corresponding to $u(t)$. The system is *time-invariant* if the output corresponding to $u(t-\tau)$ is $y(t-\tau)$, for any time shift τ.

A system is *linear* if the output corresponding to the input $\alpha_1 u_1(t) + \alpha_2 u_2(t)$ is $\alpha_1 y_1(t)+\alpha_2 y_2(t)$, where y_i is the corresponding input to u_i and α_i are arbitrary constants.

Exercise 22 *Prove that the linearity property generalises to inputs of the form* $\sum_{i=1}^{N} \alpha_i u_i(t)$.

Systems that are both linear and time-invariant are abbreviated with the acronym LTI. Such systems are extremely useful in both control and signal processing. The LTI systems of control are typically casual while those of signal processing are sometimes not.

Exercise 23 *For discrete time input $u(n)$ define,*

$$y(n) = \frac{1}{N+M+1} \sum_{m=-M}^{N} \big(u(n+m)\big)^{\alpha+\beta\cos(n)}.$$

When $\alpha = 1$ and $\beta = 0$ this system is called a sliding window averager. *It is very useful and abundant in time-series analysis and related fields. Otherwise, there is not much practical meaning for the system other than the current exercise.*

Determine when the system is memory-less, casual, linear, time-invariant based on the parameters N, M, α, β.

A final general notion of systems that we shall consider is *BIBO stability*. BIBO stands for bounded-input-bounded-output. A system is defined to be BIBO stable if whenever the input u satisfies $||u||_\infty < \infty$ then the output satisfies $||y||_\infty < \infty$. We will see in the sections below that this property is well characterised for LTI systems.

10 LTI Systems - Overview of Representations

Many useful systems used practice (signal processing and control) are both linear and time-invariant. For this we have the acronym LTI. The remainder of this unit deals with LTI SISO systems.

We now overview several ways of representing linear systems:

1. IO Mapping representation.
2. Representation using the impulse response.
3. Representation using the transfer function.

4. Representation as a difference (discrete time) or differential (continuous time) equation.

5. State space representation.

11 The Impulse Response and Convolutions

We begin the discussion with equation (7). This is merely a representation of a discrete time signal $u(n)$ using the shifted (by k) discrete delta function,

$$\delta[n-k] = \begin{cases} 1 & n = k, \\ 0 & n \neq k. \end{cases}$$

Treat now $u(n)$ as input to an LTI system with output $y(n)$. In this case since the input as a function of the time n, is represented as in (7), the output may be represented as follows:

$$y(n) = \mathcal{O}\Big(u(n)\Big) = \mathcal{O}\Big(\sum_{k=-\infty}^{\infty} \delta[k]u(n-k)\Big) = \mathcal{O}\Big(\sum_{k=-\infty}^{\infty} u(k)\delta[n-k]\Big) = \sum_{k=-\infty}^{\infty} u(k)\mathcal{O}\big(\delta[n-k]\big)$$

Now if denote $h(n) = \mathcal{O}\big(\delta[n]\big)$ and since the system is time invariant we have that $h(n-k) = \mathcal{O}\big(\delta[n-k]\big)$. So we have:

$$y(n) = \sum_{k=-\infty}^{\infty} u(k)h(n-k) = \big(u * h\big)(n).$$

This very nice fact shows that the output of LTI systems can in fact be described by the convolution of the the input with the function $h(n)$. This function deserves a special name: *impulse response*.

For continuous time systems the same argument essentially follows, this time using (6):

$$y(t) = \mathcal{O}\Big(u(t)\Big) = \mathcal{O}\Big(\int_{-\infty}^{\infty} \delta(t)u(t-\tau)d\tau\Big) = \mathcal{O}\Big(\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau\Big)$$
$$= \int_{-\infty}^{\infty} u(\tau)\mathcal{O}\big(\delta(t-\tau)\big)d\tau = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = \big(u * h\big)(t).$$

Observe that in the above we had a "leap of faith" in taking our system to be linear in sense that,

$$\mathcal{O}\Big(\int_{-\infty}^{\infty} \alpha_s u_s ds\Big) = \int_{-\infty}^{\infty} \alpha_s \mathcal{O}\big(u_s\big)ds.$$

We have thus seen that a basic description of LTI systems is the impulse response. Based on the impulse response we may thus be able to see when the system is memory-less, causal and BIBO-stable.

Exercise 24 *Show that an LTI system is memory less if and only if the impulse response has the form* $h(t) = K\delta(t)$*.*

Exercise 25 *Show that an LTI system is causal if and only if* $h(t) = 0$ *for all* $t < 0$*.*

Example 6 *Consider the sliding window averager of exercise 23 with* $\alpha = 1$ *and* $\beta = 0$*. Find it's impulse response and verify when it is casual.*

The following characterises LTI systems that are BIBO stable:

Theorem 5 *A SISO LTI system with impulse response* $h(\cdot)$ *is BIBO stable if and only if,*

$$||h||_1 < \infty.$$

Further if this holds then,

$$||y||_\infty \leq ||h||_1 \; ||u||_\infty, \tag{10}$$

for every bounded input.

Proof The proof is for discrete-time (the continuous time case is analogous). Assume first that $||h||_1 < \infty$. Then,

$$|y(n)| = \Big|\sum_{k=-\infty}^{\infty} h(n-k)u(k)\Big| \leq \sum_{k=-\infty}^{\infty} |h(n-k)|\,|u(k)| \leq \Big(\sum_{k=-\infty}^{\infty} |h(n-k)|\Big)||u||_\infty$$

So,

$$||y||_\infty \leq ||h||_1 \, ||u||_\infty.$$

Now to prove that $||h||_1 < \infty$ is also a necessary condition. We assume the input is real, the complex case is left as an exercise, choose the input,

$$u(n) = \text{sign}\big(h(-n)\big).$$

So,

$$y(0) = \sum_{k=-\infty}^{\infty} h(0-k)u(k) = \sum_{k=-\infty}^{\infty} |h(-k)| = ||h||_1.$$

Thus if $||h||_1 = \infty$ the output for input $u(\cdot)$ is unbounded, so $||h||_1 < \infty$ is a necessary condition. $\square$

Exercise 26 *What input signal achieves equality in* (10)*?*

Exercise 27 *Prove the continuous time version of the above.*

Exercise 28 *Prove the above for signals that are in general complex valued.*

12 Integral Transforms and the Transfer Function

It is now useful to consider our LTI SISO systems as operating on complex valued signals. Consider now an input of the form $u(t) = e^{-st}$ where $s \in \mathbb{C}$. We shall denote $s = \sigma + i\omega$, i.e. $\sigma = Re(s)$ and $\omega = Im(s)$. We now have,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = \Big(\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\Big)e^{st}.$$

Denoting $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$ we found that for exponential input, e^{st}, the output is simply a multiplication by the complex constant (with respect to t), $H(s)$:

$$y(t) = H(s)e^{st}.$$

This is nice as it shows that inputs of the form e^{st} are the "eigensignals" of LTI systems where as for each s, $H(s)$ are "eigenvalues".

Observe that $H(s)$ is nothing more than the Laplace transform of the impulse response. It is central to control and system theory and deserves a name: the *transfer function*. When the input signal under consideration has real part $\sigma = 0$, i.e. $u(t) = e^{i\omega t}$ then the output can still be represented interns of the transfer function:

$$y(t) = H(i\omega)e^{i\omega t}$$

In this case $y(t)$ is referred to as the *frequency response* of the *harmonic* input $e^{i\omega t}$ at frequency ω. In this case, $F(\omega) = H(i\omega)$ is the Fourier transform of the impulse response. Note that both the Fourier and Laplace transform are referred to in practice as the transfer function. Further, analogies exist in discrete time systems (e.g. the *Z-transform*).

In general since we have seen that $y(t) = \big(u * h\big)(t)$, we have that,

$$Y(s) = U(s)H(s).$$

So the transfer function, $H(s)$ can also be viewed as,

$$H(s) = \frac{Y(s)}{U(s)}.$$

This takes practical meaning for pure imaginary $s = i\omega$ as it allows to *measure* $H(i\omega)$ based on the ratios of $Y(i\omega)$ and $U(i\omega)$. The *frequency response* of a system is the Fourier transform of the impulse response.

Response to Sinusoidal Inputs

As an illustration, take a system with rational transfer function, $H(s)$, and assume $H(s)$ has distinct poles, $p_1, \ldots, p_n$. Assume a sinusoidal input $u(t) = \sin(\omega_0 t)\mathbf{1}(t)$ is applied. The Laplace transform of the input is,

$$U(s) = \frac{\omega_0}{s^2 + \omega_0^2}.$$

So the Laplace transform of the output is,

$$Y(s) = H(s)\frac{\omega_0}{s^2 + \omega_0^2}.$$

Now applying a partial fraction expansion we will get the following form

$$Y(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s - p_i} + \frac{\alpha_0}{s + i\omega_0} + \frac{\overline{\alpha_0}}{s - i\omega_0}.$$

Exercise 29 *Carry out the above to find $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ for some explicit $H(s)$ of your choice having 3 distinct poles with negative real part.*

Inverting $Y(s)$ we get,

$$y(t) = \sum_{i=1}^{n} \alpha_i e^{p_i t} + 2|\alpha_0| cos(\omega_0 t + \phi), \quad t \geq 0,$$

with,

$$\phi = \tan^{-1}\Big(\frac{Im(\alpha_0)}{Re(\alpha_0)}\Big).$$

Now if all $p_i < 0$ the system will represent stable behavior and as t grows the output will be determined by the sinusoidal term.

Exercise 30 *Continuing the previous exercise, find ϕ and plot the system output as to illustrate convergence to the pure sinusoidal term.*

Exercise 31 *Now make/generate (probably using some software) a Bode plot of your system and relate the result of the previous exercise to the Bode plot. I.e. what is the frequency response for ω_0.*

13 Stability based on pole locations

When we study general linear systems we will relate BIBO stability to *internal stability*. The latter term is defined for systems with rational transfer functions as follows: A system is internally stable if the locations of all poles are in the left hand plane.

A classic criterion for this is the *Routh-Hurwitz Test*: We consider a polynomial $q(s) = q_n s^n_+ \dots\dots\dots + q_1 s + a_0$ and are interested to see if $Re(q_i) < 0$ for $i = 1, \dots, n$. In that case call the polynomial Hurwitz.

See section 7.3, pp 247 of [PolWil98] (handout).

Exercise 32 *Follow example 7.3.2 of [PolWil98]. Then choose a different polynomial of similar order and carry out the test again. Compare your results to the actual roots of the polynomial (which you can find using some software).*

14 First and Second Order Systems

A first order LTI system can be described by,

$$\frac{1}{\lambda}\dot{y}(t) + y(t) = u(t),$$

for some scalar λ. Thus the impulse response is the solution of,

$$\frac{1}{\lambda}\dot{h}(t) + h(t) = \delta(t), \quad h(0^-) = 0,$$

which is,

$$h(t) = \lambda e^{-\lambda t}\mathbf{1}(t).$$

Exercise 33 *Check that this is indeed the solution (use properties of $\delta(t)$).*

The transfer function of this system is,

$$H(s) = \frac{\lambda}{s + \lambda}.$$

It has a pole at $s = -\lambda$ and thus the ROC is $Re(s) > -\lambda$. So for $\lambda > 0$ the system is stable.

A second order LTI system (with complex conjugate roots) is generally a more interesting object. It can be described by,

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = \omega_n^2 u(t).$$

The two positive parameters have the following names:

- The parameter ω_n is called the *undamped natural frequency*.
- The parameter ζ is called the *damping ratio*.

Exercise 34 *Show that the transfer function of this system is (apply the Laplace transform to the ODE with input $\delta(t)$).*

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

The transfer function may be factored as follows:

$$H(s) = \frac{\omega_n^2}{(s - c_1)(s - c_2)},$$

where $c_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$.

Exercise 35 *Assume $\zeta \neq 1$. Carry out partial fraction expansion to get,*

$$H(s) = \frac{M}{s - c_1} - \frac{M}{s - c_2},$$

where $M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}$.

Exercise 36 *Use the above to show that if $\zeta \neq 1$,*

$$h(t) = M(e^{c_1 t} - e^{c_2 t})\mathbf{1}(t).$$

Exercise 37 *Assume $\zeta = 1$. Find $H(s)$ and $h(t)$.*

Exercise 38 *Investigate (numerically) $h(t)$ and the poles of $H(s)$ for a variety of parameter combinations. Explain now the names of ω_n and ζ.*

15 Feedback Configurations of LTI SISO Systems

The next unit dealing with classic control methods generally deals with designing LTI controllers G_1 and G_2 configured as in Figure 2. The whole system relating output y to input reference r is then also an LTI and may be analyzed in the frequency (or 's') domain easily.

The idea is as follows is to find the $\tilde{H}(s)$ that satisfies,

$$Y(s) = R(s)\tilde{H}(s).$$

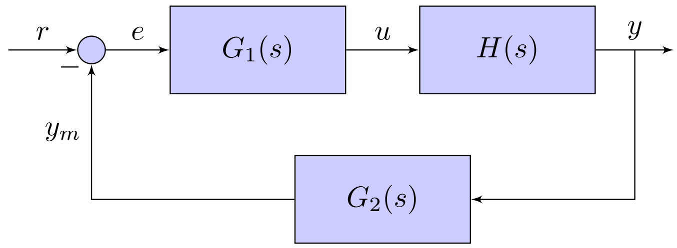

Figure 2: A *plant*, $H(s)$ is controlled by the blocks $G_1(s)$ and $G_2(s)$ they are both optional (i.e. may be set to be some constant K or even 1.

This can be done easily:

$$Y(s) = U(s)H(s) = E(s)G_1(s)H(s) = \big(R(s) - Y_m(s)\big)G_1(s)H(s) = \big(R(s) - G_2(s)Y(s)\big)G_1(s)H(s).$$

Solving for $Y(s)$ we have,

$$Y(s) = R(s)\frac{G_1(s)H(s)}{1 + G_2(s)G_1(s)H(s)}.$$

Hence the feedback system is:

$$\tilde{H}(s) = \frac{G_1(s)H(s)}{1 + G_2(s)G_1(s)H(s)}.$$

Exercise 39 *What would be the feedback system if there was* positive feedback *instead of negative. I.e. if the circle in the figure would have a '+' instead of '-'?*