Linear State Feedback

\( \overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C}, \overrightarrow{D} \)

\( y = Ax + Bu \)
\( y = cx + 0u \)

\( u(t) = Fx(t) + r(t) \)

Thm:
There exists \( F \in \mathbb{R}^{m \times n} \)
such that \( A + BF \) has
any eigenvalues at our choice.

\[ x(t) = (A + BF)x(t) + Bu(t) \]
\[ y(t) = (C + DF)x(t) + Du(t) \]
\[ (A + BF, B, (C + DF, D)) \]

Stability:
\[ \dot{x}(t) = f(x) \quad x(0) = 0 \]

Definition: The system is stable if
\[ \exists \theta > 0, \exists \delta > 0, \text{ such that } \]
\[ \| x(t) \| < \delta \quad \text{if} \quad \| x_0 \| < \theta \]
\[ \forall t \geq T \]
Stability and modes

\[ x(t) = f(x) \quad x(0) = 0 \]

Definition: The system is stable if \( \forall \epsilon > 0, \exists \delta > 0 \) such that

\[ \|x(t)\| < \epsilon \quad \text{if} \quad \|x_0\| < \delta \]

For \( x = Ax \) where

\[ A, B, C + DF, D \]

Definiton:
The system is asymptotically stable if it is stable and also \( \lim_{t \to \infty} \|x(t)\| = 0 \) whenever \( \|x_0\| \leq \delta \).
\[ x(t) = 0 \]
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**Definition:**

The system is asymptotically stable if it is stable and also \( \lim_{t \to \infty} \| x(t) \| = 0 \) whenever \( \| x(0) \| \leq \rho \).

For \( x = Ax \) when \( t \to \infty \)

stable / asymptotically stable / unstable.
Stability of \( x = Ax \) is determined as follows:

1. Asymptotically stable \( \rightarrow \Re(\lambda_i) < 0 \) for all \( i \).

2. Stable \( \rightarrow \Re(\lambda_i) \leq 0 \) for all \( i \), and for \( i \) with \( \Re(\lambda_i) = 0 \),

\[ \lambda_i = \alpha_i + \beta_i j \]

and multiplicity \( n_i \).

\[ \lim_{s \to \lambda_i} \frac{(s-\lambda_i)^n_i}{(s-\lambda_i)^n_i} = 0 \]

\[ x(t) + B(t) \]

Ordinary: \[\begin{pmatrix} x(t) + D(t) \\ y(t) \end{pmatrix} \]

Ref.: A state is controllable if \( \exists u \) such that the system goes from \( x(0) = 0 \) to \( x(t) = 0 \) for some \( t \).

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Notes (Discrete, Continuous)
Result: For CT systems
reachable = controllable
For DT systems with det(A) ≠ 0
reachable = controllable

Then: \((A, B)\) is controllable
\[
\text{rank} \left( \text{con}(A, B) \right) = n.
\]
\[
\text{con}(A, B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times m}.
\]
\[ X(n+1) = AX(n) + BU(n) \]
\[ X(0) = AX(0) + BU(0) \]
\[ X(t) = A \left( \sum_{i=0}^{k-1} A^{k-i-1} BU(i) \right) \]

Lemma:
(1) For $k \geq n$:
\[ \text{range}(\text{con}_k(AB)) = \text{range}(\text{con}(A,B)) \]
(2) For $k < n$:
\[ \text{range}(\text{con}_k(A,B)) = \text{range}(\text{con}(A,B)) \]

Proof:

If $k < n$,
\[ \text{con}(A,B) \]

This is a property of controllability.

\[ X(0) \]

\[ X(k) \]

\[ A^k \]

\[ B, AB, A^2B, \ldots, A^{k-1}B \]

\[ \text{con}_k(AB) \]

\[ U(k) \]

\[ V(k) \]

\[ MK \]
Lemma:

1) For \( k \in \mathbb{N} \)
\[
\text{range} (\text{con}(A, B))
\]
2) For \( k \in \mathbb{N} \)
\[
\text{range} (\text{con}(A, B))
\]
   \[
   = \text{range} (\text{con}(A, B))
\]

Proof:

(i) Trivial.

(ii) MK (1) ....
Result: Cayley-Hamilton

\[ A_{n \times n} P_A(s) \det(sI - A) = \alpha_0 + \alpha_1 s + \ldots + \alpha_s s^s \]

\[ O_{n \times n} = \alpha_0 + \alpha_1 A + \alpha_2 A^2 + \ldots + \alpha_{n-1} A^{n-1} + \alpha_n A^n \]

\[ \text{Thm.} \ (A, B) \text{ is controllable} \]

\[ \text{rank} (\text{con}(A, B)) = n. \]

\[ \text{con}(A, B) = \begin{bmatrix} B, AB, A^2B, \ldots, A^{n-1}B \end{bmatrix}_{n \times n} \in \mathbb{R}^{n \times n} \]
\[ A^k = \frac{\alpha_{n-1}}{\alpha_n} A^{n-1} - \cdots - \frac{\alpha_1}{\alpha_n} A - \frac{\alpha_0}{\alpha_n} I \]

\[ A^k B = \frac{\alpha_{n-1}}{\alpha_n} B - \frac{\alpha_{n-2}}{\alpha_1} AB - \cdots - \frac{\alpha_1}{\alpha_n} A^N B \]

\[ A^k B \text{ is the new term in } \text{con}(A, B) \]

\[ \text{Range}(\text{con}_m(A, B)) = \text{Range}(\text{con}_m(A, B)) \]

\[ \text{Proof:} \]

\[ X(k) = A^k + [B, AB, A^2B, \ldots, A^{k-1}B] \]

\[ \text{con}_k(A, B) \]

If \( k = n \) this is controllability rank.
Proof: \[ [\text{conv}(A, B) = n \iff (A, B) \text{ reachable}] \]

\[ \text{conv}_n(A, B) \bar{X} = X1 - A^T x \]

We can reach in \( k \) steps into \( Xd \) by \( \text{range}(\text{conv}_n(A, B)) \)

\[ \Rightarrow \text{range}(\text{conv}_n(A, B)) = \mathbb{R}^n \]

If reachable, since \( Xd \) is arbitrary,

there is a \( K \) for which

\[ \text{range}(\text{conv}_n(A, B)) = \mathbb{R}^n \]

We must have (by Lemma) \( KN \)
\[
\dot{x}(t) = A x(t) + B u(t) + K (y(t) - \hat{y}(t))
\]
\[
\hat{y}(t) = C x(t) + D u(t)
\]
\[
\dot{x}(t) = (A - K C) x(t) + [B - K D, K] [u] \\
\hat{y}(t) = C x(t) + [D, 0] [u] [y]
\]
\[
\dot{e}(t) = \begin{pmatrix} f(t) \\ \hat{f}(t) \end{pmatrix}
\]
\[ \dot{e}(t) = \dot{x}(t) - \dot{x}(t) \\
= (A - KC)x(t) \]