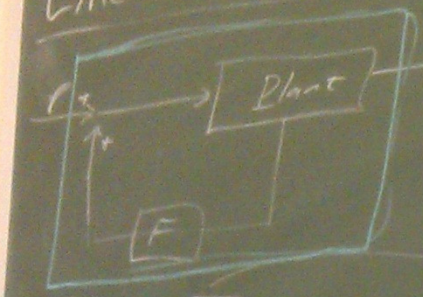
## Linear State Feedback




$(A, B, C, D)$

$y = Ax + Bu$

$y = Cx + Du$

$u(t) = \underset{m \times n}{F}\, x(t) + r(t)$

$\dot{x}(t) = (A + BF)\, x(t) + Br(t)$

$y(t) = (C + DF)\, x(t) + Dr(t)$

$(A + BF,\ B,\ C + DF,\ D)$

**Thm:** There exists $F \in \mathbb{R}^{m \times n}$ such that $A + BF$ has eigenvalues of our choice. (any)

$\Updownarrow$

$(A, B)$ is controllable

## Stability

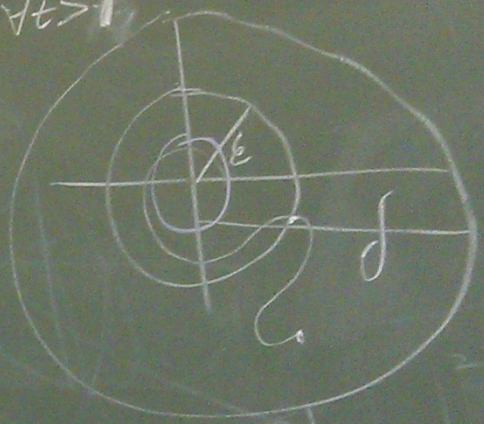
$\dot{x} = f(x) \quad x(0) = 0$

$f(x_e) = 0$

**Definition:** The system is Stable* if $\forall \varepsilon > 0,\ \exists \delta > 0,\ \exists T > 0$ such that

$\|x(t)\| < \varepsilon$ if $\|x_0\| < \delta$

$\forall t > T$



LK

(A, B, C, D)

= Ax + Bu

= Cx + Du

$\dot{x}(t) = (A+BF)x(t) + Br(t)$

$y(t) = (C+DF)x(t) + Dr(t)$

$(A+BF, B, C+DF, D)$

## Stability and modes

$\dot{x} = f(x) \quad x(0) = 0$

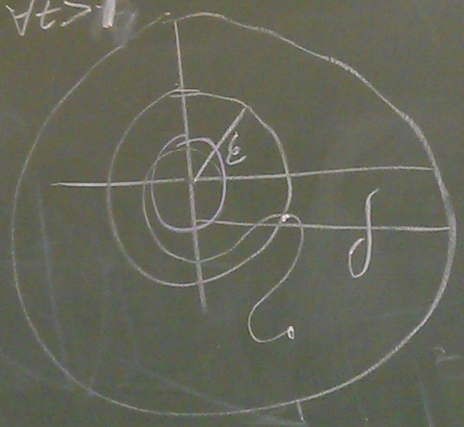
$x_e$ :
$f(x_e) = 0$

**Definition:** The system is **Stable**(**) if $\exists T > 0$

if $\forall \varepsilon > 0, \exists \delta > 0$, such that

$\|x(t)\| < \varepsilon$ if $\|x_0\| < \delta$.

$\forall t > T$




## Definition:

The system is asym

Stable if it is St

and also $\lim_{t \to \infty} \|x($

whenever $\|x_0\|$

For $\dot{x} = Ax$ whe

Stable / asymp- Sta

modes

$x_e$ :
$f(x_e) = 0$

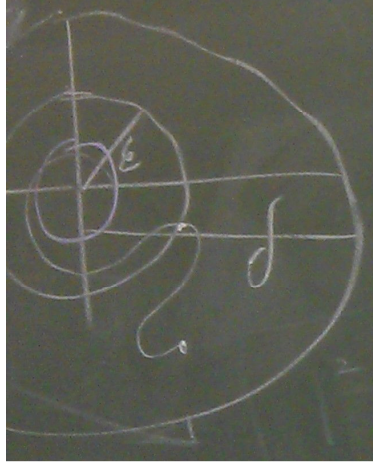
$f(x_e) = 0$

e system is Stable (**)

$\exists T > 0$

$\exists \delta > 0$, such that

$< \varepsilon$ if $\|x_0\| < \delta$.




**Definition:**

The system is asymptotically Stable if it is Stable and also $\lim_{t \to \infty} \|x(t)\| = 0$ whenever $\|x_0\| \leq R$

For $\dot{x} = Ax$ when is it Stable / asymp-Stable / Unstable

Stability of $\dot{x} = Ax$ is determined as follows:

1) Asymp stable $\iff$ $\mathrm{Re}(\lambda_i) < 0$ $i = 1 \ldots n$

2) Stable $\iff$ $\mathrm{Re}(\lambda_i) \leq 0$ $i = 1 \ldots n$
and for $\lambda_i$ with $\mathrm{Re}(\lambda_i) = 0$
and multiplicity $n_i$

$$\lim_{s \to \lambda_i} \left[ (s - \lambda_i)^{n_i} (sI - A)^{-1} \right] \quad (n_i - 1 - k)$$
$$= 0$$

$x(t) + Bu(t)$

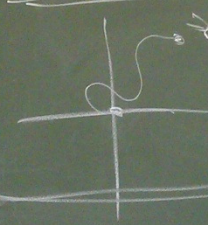
$x(t) + Du(t)$

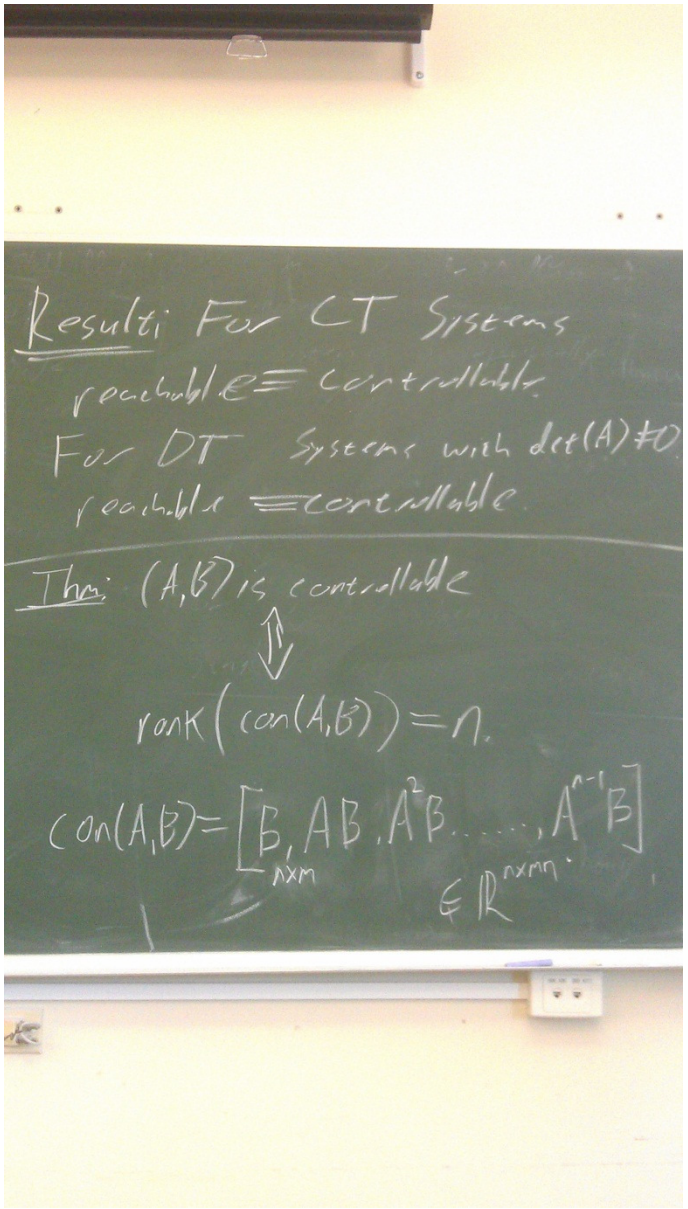
$(+DF, D)$

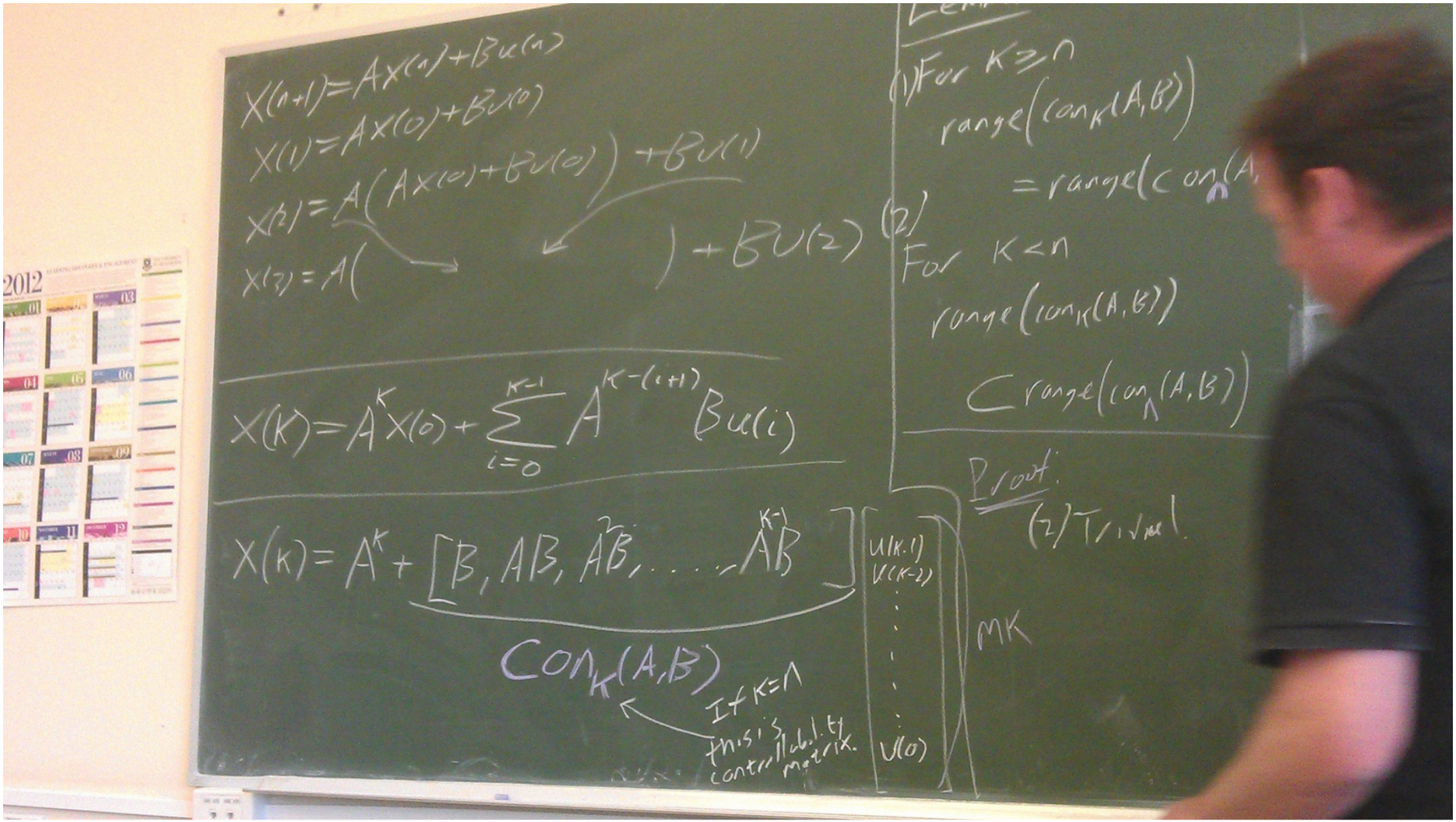
(3) Otherwise
unstable

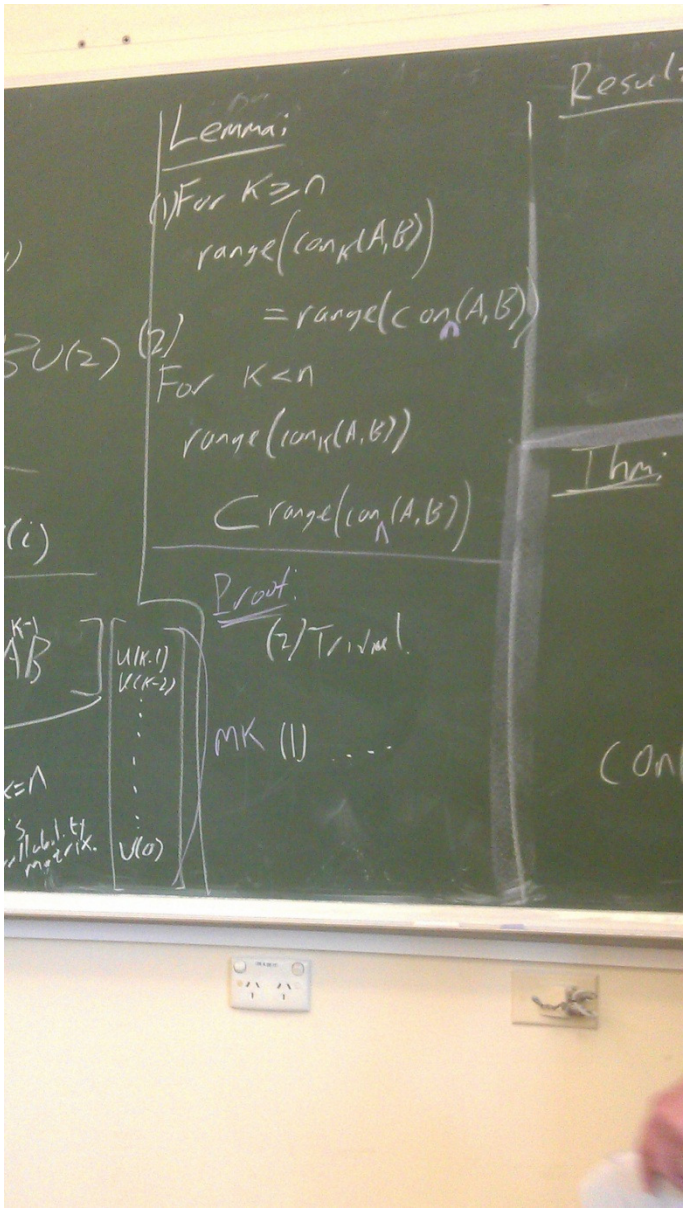
Def: A state $x_d \in \mathbb{R}^n$ is reachable if $\exists\, u$ such that system goes from $x(0) = 0$ to $x(T) = x_d$ for some finite $T$. (Discrete, Continuous)

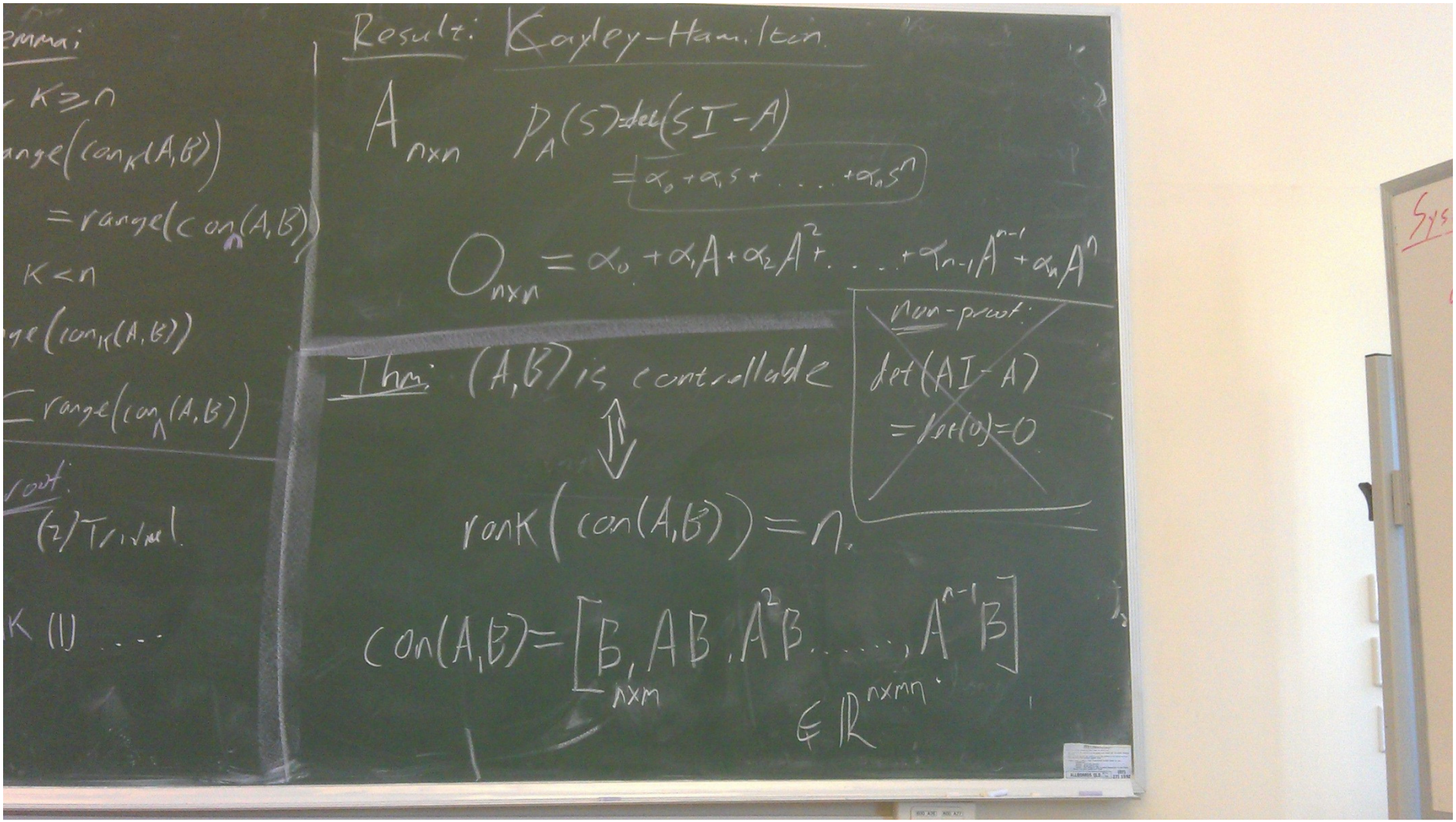
Def: A state $x_s$ is controllable if it can move from $x_s$ to 0.

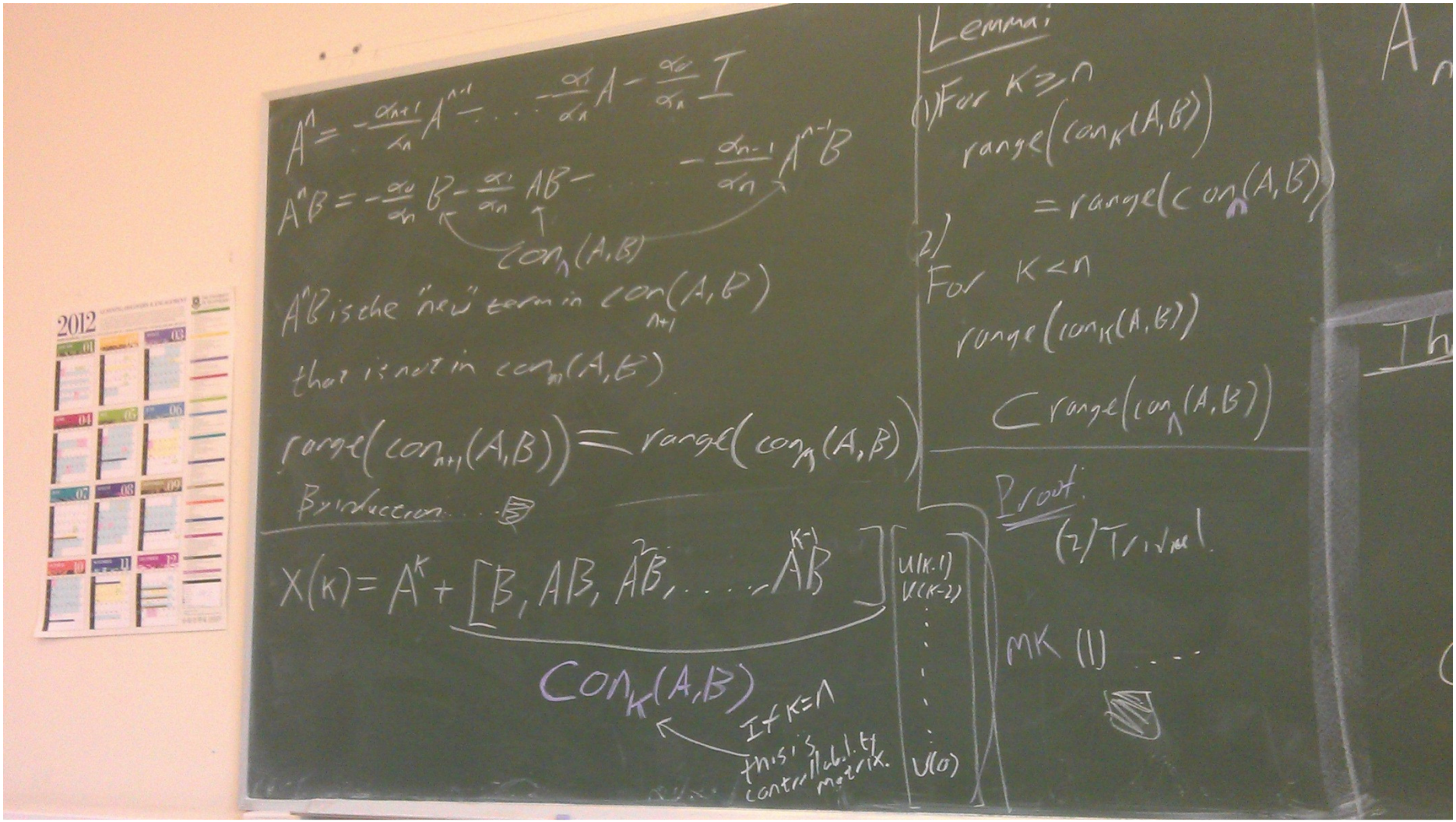



Result: For CT Systems
reachable ≡ controllable
For DT Systems with det(A) ≠ 0
reachable ≡ controllable
Thm: (A,B) is controllable
rank(con(A,B)) = n.
con(A,B) = [B, AB, A²B, ..., A^{n-1}B]
n×m
∈ R^{n×nm}


$x(t+1) = Ax(t) + Bu(t)$
$x(1) = Ax(0) + Bu(0)$
$x(2) = A(Ax(0) + Bu(0)) + Bu(1)$
$x(3) = A(\quad) + Bu(2)$
$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-(i+1)} Bu(i)$
$x(k) = A^k + [B, AB, A^2B, \ldots, A^{k-1}B]$
$Con_k(A,B)$
If $k=n$ this is controllability matrix.
$u(k-1)$
$u(k-2)$
$u(0)$
$mk$
(1) For $k \geq n$
range$(con_k(A,B))$
$=$ range$(con_n(A,$
(2) For $k<n$
range$(con_k(A,B))$
$\subset$ range$(con_n(A,B))$
Proof.
(2) Trivial


Lemma:
(1) For $K \geq n$
$range(con_K(A,B)) = range(con_\wedge(A,B))$
(2) For $K < n$
$range(con_K(A,B)) \subset range(con_\wedge(A,B))$
Proof:
(2) Trivial
Result
Thm:
U(K-1)
V(K-2)
U(0)
MK (1) . . .


emma:
K ≥ n
range(con_K(A,B))
= range(con_n(A,B))
K < n
range(con_K(A,B))
⊂ range(con_n(A,B))
root.
(2) Trivial.
K (1) ...
Result: Kayley-Hamilton
A_{n×n}
P_A(s) = det(sI − A)
= α_0 + α_1 s + ... + α_n s^n
0_{n×n} = α_0 + α_1 A + α_2 A^2 + ... + α_{n-1} A^{n-1} + α_n A^n
Thm: (A,B) is controllable
rank(con(A,B)) = n.
con(A,B) = [B, AB, A^2 B, ..., A^{n-1} B]
n×m
∈ R^{n×mn}
non-proof:
det(AI − A)
= det(0) = 0


2012
$A^n = -\frac{\alpha_{n+1}}{\alpha_n} A^{n-1} - \cdots - \frac{\alpha_1}{\alpha_n} A - \frac{\alpha_0}{\alpha_n} I$
$A^n B = -\frac{\alpha_0}{\alpha_n} B - \frac{\alpha_1}{\alpha_n} AB - \cdots - \frac{\alpha_{n-1}}{\alpha_n} A^{n-1} B$
$con_n(A,B)$
$A^n B$ is the "new" term in $con_{n+1}(A,B)$
that is not in $con_n(A,B)$
$range(con_{n+1}(A,B)) = range(con_n(A,B))$
By induction...
$X(K) = A^K + [B, AB, A^2B, \ldots, A^{K-1}B]$
$con_K(A,B)$
If $K=n$ this is controllability matrix.
U(K-1)
U(K-2)
U(0)
Lemma:
(1) For $K \geq n$
$range(con_K(A,B)) = range(con_n(A,B))$
2) For $K < n$
$range(con_K(A,B)) \subset range(con_n(A,B))$
Proof:
(2) Trivial!
MK (1) ...

Proof: $\left[\text{range}\left(con_n(A,B)\right)=n \iff (A,B) \text{ reachable}\right]$

$con_K(A,B)\,\bar{u}_K = x_d - A^K x_0$

We can reach in $K$ steps if $x_d \in \text{range}\left(con_K(A,B)\right)$

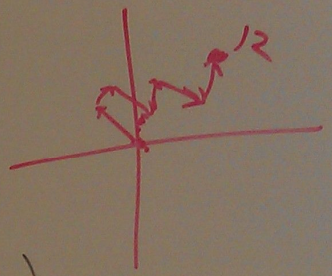
$\Rightarrow \text{rank}\left(con_n(A,B)\right)=n \iff \text{range}\left(con_n(A,B)\right)=\mathbb{R}^n$

If reachable, since $x_d$ is arbitrary,

there is a $K$ for which

$$\text{range}\left(con_K(A,B)\right)=\mathbb{R}^n.$$

We must have (by Lemma) $K \leq n$.



$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_{n\times p}\left(y(t) - \hat{y}(t)\right);$$

$$\hat{y}(t) = C\hat{x}(t) + Du(t)$$

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$$\dot{\hat{x}}(t) = (A - KC)\hat{x}(t) + \begin{bmatrix} B - KD & K \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

$$\hat{y}(t) = C\hat{x}(t) + \begin{bmatrix} D & 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

$$\dot{e}(t) =$$

$$=$$

$$\dot{e}(t) =$$

$\hat{y}(t)$:

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t)$$

$$= (A - KC)x \qquad \hat{x}(t)$$

$$\dot{e}(t) = (A - K$$

$$\begin{bmatrix} u \\ y \end{bmatrix}$$