MATH4406 (Control Theory)
Unit 1: Introduction

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Unit Outline

- Introduction to the course: Course goals, assessment, etc...
- What is "Control Theory"
- A bit of jargon, history and applications
- A survey of the mathematical problems handled in this course: Units 2 – 11

Course Goal

- A “mathematically mature” student will get a taste of several of the key paradigms and methods appearing in Control Theory. As well as in Systems Theory.
- Following this course the student will have capacity to independently “dive deeper” into enginerring control pardigms and/or optimal control theory.

Assessment

- 7 HW
- 4 Quizzes
- Final course summary

Mode of Presentation

- Board
- Lecture Notes
- Slides (presentation)
- or… mixes of the above

What is "Control Theory"

Hard to find a precise definition. So here are some terms very strongly related:

- Dynamical systems and systems theory
- Linear vs. non-linear systems
- Open Loop vs. Closed Loop (Feedback)
- Optimal trajectories
- Trajectory tracking
- Disturbance rejection
- Stabilization vs. fine tuning
- Chemical systems, Electrical systems, Mechanical systems, Biological systems, Systems in logistics...
- Adaptive control and learning

A system (plant)

- *state*: $x \in \mathbb{R}^n$
- *input*: $u \in \mathbb{R}^m$
- *output*: $y \in \mathbb{R}^k$

The basic system model:

$$\begin{aligned} \dot{x}(t) &= f\big(x(t), u(t), t\big), \\ y(t) &= h\big(x(t), u(t), t\big). \end{aligned}$$

Or in *discrete time*,

$$\begin{aligned} x(n+1) &= f\big(x(n), u(n), n\big), \\ y(n) &= h\big(x(n), u(n), n\big). \end{aligned}$$

Open loop control

$$\begin{aligned}\dot{x}(t) &= f\big(x(t), u(t), t\big),\\ y(t) &= h\big(x(t), u(t), t\big).\end{aligned}$$

Choose a "good" input $u(t)$ from onset (say based on $x(0)$).
$y(t)$ is not relevant.

Open loop *optimal control* deals with choosing $u(t)$ as to,

$$\min \quad J\Big(\{x(t), u(t),\ 0 \le t \le T_{\max}\}\Big),$$

subject to constraints on x and u.

E.g. Make a "thrust plan" for a spaceship.

Closed loop (feedback) control

$$\begin{aligned}\dot{x}(t) &= f\big(x(t), u(t), t\big),\\ y(t) &= h\big(x(t), u(t), t\big).\end{aligned}$$

Choose (design) a controller (control law), $g(\cdot)$:

$$u(t) = g\big(y(t), t\big).$$

The design may be "optimal" w.r.t. some performance measure, or may be "sensible".

Combination of open loop and closed loop

Consider a jet flight from Brisbane to Los Angeles (approx $13:45$).

$$\dot{x}(t) = f\big(x(t), u(t), t\big).$$

Dynamical system modeling: What are possibilities for x? For u? Why is the problem time dependent?

Open loop methods can be used to set the best $u(\cdot)$. Call this a *reference* control. This also yields a *reference state*. We now have, $x_r(t)$ and $u_r(t)$ and we know that, if the reference is achieved then the output is,

$$y_r(t) = h\big(x_r(t), u_r(t), t\big).$$

We can now try to find a controller $g(\cdot)$,

$$u(t) = g\big(y(t), t, x_r, u_r\big),$$

such that $||x(t) - x_r(t)||$ "behaves well".

Combination of open loop and closed loop, cont.

The *trajectory tracking* problem of having $||x(t) - x_r(t)||$ "behave well" is essentially called the *regularization problem*. "Behaving well" stands for:

- Ideally 0
- Very important: $\lim_{t\to\infty} ||x(t) - x_r(t)|| = 0$ (stability)
- Also important: How $||x(t) - x_r(t)||$ is regulated to 0. E.g. fast/slow. E.g. with many osculations or not

Many (perhaps most) controllers are "regulators" and are not even associated with a trajectory, instead they try to regulate the system at a set point. E.g. cruise control in cars.

The Basic Feedback Loop

- plant
- controller
- actuators
- sensors.

(Historic) Applications of Control Theory

- Watt's fly-ball governor
- The feedback amplifier
- Aviation
- Space programs 50's, 60's
- Everywhere today: Head position control of hard disk drives (typically PID controllers)

Key Developers of (Highly Used) Ideas and Methods

Up to 60's:

- E. J. Routh
- A. M. Lyapunov
- H. Nyquist
- W. R. Evans
- R. Bellman
- L. S. Pontryagin
- R. E. Kalman

- The developers of Model Predictive Control (MPC)

Implementations

- Mechanical
- Analog
- Digital

A survey of the mathematical problems handled in this course

Unit 2: Signals, Systems and Math Background

- Essentials of “Signals and Systems” (no state-space yet)
- Convolutions, integral transforms (Laplace, Fourier, Z...) and generalized functions (δ)
- The unit ends with negative feedback:

Consider a system with plant transfer function, $H(s)$, and negative feedback controller, $G(s)$. The transfer function of the whole system is:

$$Q(s) = \frac{H(s)}{1 + G(s)H(s)}.$$

Unit 3: Elements of Classic Engineering Control

The main object is,

$$Q(s) = \frac{H(s)}{1 + G(s)H(s)}.$$

- The "objectives" in controller (regulator) design
- PID Controllers (Proportional, Integral, Derivative)
- The Nyquist Stability Criterion

Unit 4: Linear State-Space Systems and Control

The main object of study is,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t).\end{aligned}$$

Together with the discrete time analog.

- When can such system be well controlled (e.g. stabilized)?
- Linear state feedback controllers are of the form $u(t) = Ky(t)$ for some matrix K
- Is there a way to estimate x based on y? The Luenberger observer
- The Kalman decomposition

This unit is allocated plenty of time (approx. 9 lecture hours).

Unit 5: Lyapunov Stability Theory

Here we "leave" control for a minute and look at the *autonomous system*:

$$\dot{x}(t) = f\big(x(t)\big).$$

Note that this system may be the system after a regulating control law is applied.

An equilibrium point is a point x_0, such that $f(x_0) = 0$.

Lyapunov's stability results provide methods for verifying if equilibrium points are stable (in one of several senses).

One way is to linearize $f(\cdot)$ and see if the point is locally stable.

Another (stronger) way is to find a so called *Lyapunov function* a.k.a. *summarizing function* or *energy function*.

Unit 6: LQR and MPC

- Getting into optimal control for the first time in the course.
- The Linear Quadratic Regulator (LQR) problem deals with fully observed linear dynamics: $\dot{x}(t) = Ax(t) + Bu(t)$, and can be posed as follows:

$$\min_{u(\cdot)} \; x(T)'Q_f x(T) + \int_0^T x(t)'Qx(t) + u(t)'Ru(t)'dt$$

- It is beautiful that the solution is of the form $u(t) = Kx(t)$ (linear state feedback) where the matrix K can be found by what is called a Ricatti equation
- The unit only summarizes results that are to be proved down the road after both dynamic programming and calculus of variations are studied.
- The second part of the unit introduces Model Predictive Controller variations of LQR in which u is constrained to be in a set depending on x. In this case quadratic programs arise.

Unit 7: Dynamic Programming

Bellman's principle of optimality can be briefly described as follows:

If x-y-w-z is the "optimal" path from x to z then y-w-z is the optimal path from y to z.

The unit generally deals with systems of the form $\dot{x}(t) = f\big(x(t), u(t)\big)$ and a cost function:

$$J\big(x(t_0), t_0\big) = \min_u \int_{t_0}^{T} C\big(x(s), u(s)\big)\, ds + D\big(x(T)\big).$$

Bellman's optimality principle implies that the optimal cost satisfies:

$$J^*\big(x(t), t\big) = \min_u C\big(x(t+dt), u(t+dt)\big)\, dt + J^*\big(x(t+dt), t+dt\big).$$

This leads to the Hamilton-Jacobi-Bellman PDE which will be studied in the unit.

Unit 8: Calculus of Variations and Pontryagin's Minimum Principle

Calculus of variations is a classic subject dealing with finding optimal functions. E.g., find a function, $u(\cdot)$ such that, $u(a) = a_u$, $u(b) = b_u$ and the following integral is kept minimal:

$$\int_a^b \sqrt{\frac{1 + u'(t)^2}{u'(t)}} dt.$$

Question: What is the best $u(\cdot)$ if we remove the denominator?

Pontryagin's Minimum Principle is a result allowing to find $u(\cdot)$ in the presence of constraints, generalising some results of the calculus of variations, yet suited to optimal control.

Unit 9: Back to LQR

Now that the essentials of dynamic programming and Pontryagin's minimum principle are known - we present two proofs (using both methods) for the celebrated LQR result (of Unit 6).

Unit 10: Systems with Stochastic Noise

The Kalman filter deals with the following type of model:

$$
\begin{aligned}
x(n+1) &= Ax(n) + Bu(n) + \xi_1(n), \\
y(n) &= Cx(n) + Du(n) + \xi_2(n),
\end{aligned}
$$

where ξ_i are *noise processes*, typically taken to be Gaussian.

The Kalman filter is a way to optimally reconstruct x based on y. It is a key ingredient in many tracking, navigation and signal processing applications.

Related is the Linear Quadratic Gaussian Regulator (LQG) which generalises the LQR by incorporating noise processes as in the Kalman filter.

This is the only "stochastic" unit in the course.

Unit 11: Closure

The course is just the tip of the ice-berg. There is much-much more to control theory practice and research. Some further developments will be surveyed:

- Control of Non-linear (smooth) systems (e.g. in Robotics)
- Adaptive Control
- Robust Control
- Applications of linear matrix inequalities
- Supervisory control
- Control of inherently stochastic systems
- Control of stochastic queueing networks (***)

Last Slide

Suggestion: *Wiki* the terms appearing in this presentation to get a feel for the scope of the course.