MATH4406 (Control Theory)
Unit 3: Elements of Classic Engineering Control

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Unit Outline

Goal: Taste the mathematical activities of classical control engineering

Unit Highlights:

- Some Plant Examples
- Control Design Goals
- Routh-Hurwitz stability criterion
- System type
- The PID Controller
- Profiling the Step Response
- Examples in MATLAB and Mathematica
- Frequency Response
- Nyquist Stability Criterion
- Pade Approximations
- Nonminimum phase systems (HW)

Some Plant Examples

Car Driving Straight (Newton's law: $F = ma$)

- F - Force
- m - Mass
- a - Acceleration

Assume:

- Rotational inertia of the wheels is negligible.
- Friction retarding the motion of the car is proportional to the car's speed with constant β (in practice it may be proportional to speed squared). If $x(t)$ is location:

$$u(t) - \beta \dot{x}(t) = m\ddot{x}(t)$$

Set $y(t) = \dot{x}(t)$,

$$\dot{y}(t) + \frac{\beta}{m} y(t) = \frac{1}{m} u(t).$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{m^{-1}}{s + \beta m^{-1}}.$$

Rotating Satellite (Newton's law: $M = I\alpha$)

- M - The sum of all external moments
- I - Mass moment of inertia
- α - Angular acceleration

Assume:

- Thrust working at force $u(t)$

$$u(t) = I\ddot{y}(t)$$
$$H(s) = \frac{Y(s)}{U(s)} = I^{-1}\frac{1}{s^2}.$$

Pendulum (Newton's law: $M = I\alpha$)

Assume:

- $\theta(t)$ is angle relative to hanging down position.
- Mass m, Length ℓ, Gravity g
- Torque with direction of θ, $u(t)$

$$u(t) - mg\ell \sin\theta(t) = m\ell^2\ddot{\theta}(t)$$

This is non-linear, yet for $\theta = 0$, $\sin\theta \approx \theta$ So we get,

$$\ddot{y}(t) + \frac{g}{\ell}y(t) = \frac{1}{m\ell^2}u(t)$$

set $\omega_n = \sqrt{g/\ell}$,

$$H(s) = \frac{Y(s)}{U(s)} = \frac{m^{-1}\ell^{-2}}{s^2 + \omega_n^2}.$$

DC Motor

From a torque ODE, an electrical ODE and some simplifying assumptions, one can get the following ODE:

$$J_m\ddot{\theta}(t) + (b + \frac{K_t K_e}{R_a})\dot{\theta}(t) = \frac{K_t}{R_a}v(t)$$

Here, J_m, b, K_t, K_e and R_a are physical constants of the motor and related electrical circuit. $\theta(t)$ is the motor angle. $v(t)$ is voltage applied to the motor. Thus for input: voltage and output: angle, we get this transfer function:

$$H(s) = \frac{K}{s(\tau s + 1)}, \quad K = \frac{K_t}{bR_a + K_t K_e}, \tau = \frac{R_a J_m}{bR_a + K_t K_e}.$$

And for input: voltage and output: speed ($\dot{\theta}$), we get this:

$$H(s) = \frac{K}{\tau s + 1}.$$

Basic model and overview of control design goals

The Basic Model

Signals: $R(s)$ is input/reference. $W(s)$ and $V(s)$ are disturbances. Systems: $H(s)$ is the plant, $G_1(s)$ and $G_2(s)$ are control components (compensator / sensor).

$$Y = H\Big(W + G_1\big(R - G_2(Y + V)\big)\Big)$$

we get,

$$Y = H_c R + \frac{H_c}{G_1} W - H_c G_2 V, \quad H_c := \frac{HG_1}{1 + HG_1 G_2}$$

$$E = R - Y = \frac{1}{1 + HG_1 G_2} R - \frac{H}{1 + HG_1 G_2} W + \frac{HG_1 G_2}{1 + HG_1 G_2} V$$

Goals in Designing G_1 and G_2

- **Stability**:$H_c(s)$ should be stable system
- **Regulation** (for $R(s) = 0$): $E(s)$ small. Properties of the disturbances $W(s)$ and $V(s)$ can be taken into consideration
- **Tracking** (for $R(s) \neq 0$): For "desired" references, $R(s)$, the error $E(s)$ should be "small"
- **Robustness**: Model error of the plant, e.g. the plant $G'(s) = G(s)\big(1 + \delta(s)\big)$ should still be controlled well.
- **Simplicity**: Often a three parameter PID controller (or even simpler) "does the job"
- **Practicality**: Staying within dynamic limits, not using too many components, etc...

Stability

Types of Stability

Two basic types:

1. The ability of the system to produce a bounded output for any bounded input (BIBO).
2. The ability of the system to return to equilibrium after an arbitrary displacement away from equilibrium (internal stability)

For non-linear and/or time-varying systems these categories are generally distinct. For LTI (SISO and MIMO) systems, both categories are essentially equivalent.

In Unit 2, we dealt with BIBO stability and in Units 4 and 5 we deal with internal stability.

For now: A system with a given transfer function is stable if the pole locations are with negative real part.

Interlude: Routh-Hurwitz stability criterion

Given, $H_c(s)$ or $H(s) = \frac{N(s)}{D(s)}$ the standard way to check for stability is to solve,

$$D(s) = 0,$$

and see all solutions have are in the LHP.

Routh-Hurwitz is an alternative (today still good for analytic purposes).

See for example Sec 7.3 of [PolWil98].

Later on in this Unit we will look at Nyquist's Stability Criterion as another approach.

A simple necessary condition for stability ($d_k > 0$ for all k)

If a polynomial, $D(s) = s^n + d_{n-1}s^{n-1} + \ldots + d_1 s + d_0$ has $d_k \leq 0$ then it can **not** be stable. I.e. then there exists $p :, D(p) = 0$ with $Re(p) \geq 0$.

Proof.

Denote the real roots of the polynomial by $p_k = \lambda_k$ and the strictly complex roots by $p_k = \lambda_k \pm i\omega_k$. Now represent the polynomial as:

$$D(s) = \Big(\prod_{k:\text{real}} (s - \lambda_k) \Big) \Big(\prod_{k:\text{complex}} (s - \lambda_k)^2 + \omega_k^2 \Big)$$

$\square$

Controlling for Stability

In case $H(s)$ is not stable. A first goal in designing $G_1(s)$ and $G_2(s)$ is to achieve stability of

$$H_c(s) := \frac{H(s)G_1(s)}{1 + H(s)G_1(s)G_2(s)}.$$

It is further important from the view point of disturbances and robustness to have good *stability margins*. Common are:

- *gain margin* - The amount that the loop gain can be changed at the frequency at which the phase shift is 180° without reducing the return difference to zero.
- *phase margin* - The amount of phase lag that can be added to the open-loop transfer function, at the frequency at which its magnitude is unity, without making the return difference zero.

These concepts will become clear as we look at Nyquist and Bode plots.

The Pit-Fall of Open Loop Cancellation

Given that $H(s)$ has unwanted (e.g. unstable) poles, p_i. One may set $G_1(s)$ to have such zeros, hence cancelling the poles in the combined $G_1(s)H(s)$. In a non-ideal world, this is a bad idea - it simply won't work.

System Type

System Type - Definition

The system is of *"type m"* if it can track a polynomial input of degree m with finite but nonzero steady state error in the basic proportional loop: $G_2(s) = 1$, $G_1(s) = K_1$.

Interlude: The final value theorem.

Assume the Laplace transform of $f(t)$ exists for $s = 0$ and that $\lim_{t\to\infty} f(t) < \infty$:

$$\lim_{s\to 0}\int_0^\infty f'(t)e^{-st}dt = \int_0^\infty f'(t)dt = \lim_{t\to\infty} f(t) - f(0),$$

But also,

$$\int_0^\infty f'(t)e^{-st}dt = s\hat{f}(s) - f(0)$$

So,

$$\lim_{s\to 0} s\hat{f}(s) = \lim_{t\to\infty} f(t).$$

System Type

Recall, $E(s) = \frac{1}{1+K_1 H(s)} R(s)$.

Consider reference: $r(t) = c_0 + c_1 t + \frac{c_2}{2} t + \ldots + \frac{c_m}{m!} t^m$. So,

$$R(s) = \frac{c_0}{s} + \frac{c_1}{s^2} + \frac{c_2}{s^3} + \ldots + \frac{c_m}{s^{m+1}}.$$

Denote plant by the form, $H(s) = \frac{N(s)}{s^p D(s)}$, where $N(\cdot)$ and $D(\cdot)$ don't have zeros at $s = 0$. Then,

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{s^{p-m} D(s)}{s^p D(s) + K_1 N(s)} (c_0 s^{m+1} + c_1 s^m + \ldots + c_m)$$

$$= \begin{cases} 0 & \text{if} \quad p > m, \\ e_\infty < \infty & \text{if} \quad p = m, \\ \infty & \text{if} \quad p < m. \end{cases}$$

Hence the system type of $H(s) = \frac{N(s)}{s^p D(s)}$ is p.

Truxal's formula for e_∞ in Type-1 systems

Suppose for type 1 system,

$$H_c(s) = K\frac{(s-z_1)\cdot \ldots \cdot (s-z_m)}{(s-p_1)\cdot \ldots\ldots \cdot (s-z_n)}$$

$$E(s) = R(s) - Y(s) = R(s)\Big(1 - \frac{Y(s)}{R(s)}\Big) = R(s)\Big(1 - H_c(s)\Big).$$

With "ramp" input, we get $E(s) = \frac{1-H_c(s)}{s^2}$.

$$e_\infty = \lim_{s\to 0}\frac{1-H_c(s)}{s} = -\lim_{s\to 0}\frac{dH_c(s)}{ds} = -\lim_{s\to 0}\frac{dH_c(s)}{ds}\frac{1}{H_c(0)},$$

since, $H_c(0) = 1$ because the system is type-1. Thus,

$$e_\infty = -\lim_{s\to 0}\frac{dH_c(s)}{ds}\frac{1}{H_c(s)} = -\lim_{s\to 0}\frac{d}{ds}\log H_c(s) = \sum_{i=1}^{n} z_i^{-1} - \sum_{i=1}^{m} p_i^{-1}.$$

Software Interlude

Basic MATLAB commands for systems

- Define a system through **sys = tf(num,den)**
- Visualize in s-plane: **pzmap**, **bode**, **margin**, **nyquist**,
- Visualize in time: **step**, **impulse**
- Conversion: **tf2ss**, **ss2zp**, **zp2tf** etc...
- Misc: **roots**, **pade**

Mathematica

It is new in Mathematica.

Look for: guide/ControlSystems .

The PID (Proportional – Integral – Derivative) Controller

Parameterized by k_P, k_I and k_D

Controller Transfer Function:

$$G_1(s) = k_P + k_I \frac{1}{s} + k_D s$$

Response of controller to error:

$$u(t) = k_P e(t) + k_I \int_0^t e(\tau) d\tau + k_D \frac{d}{dt} e(t)$$

Closed Loop System Transfer Function:

$$H_c(s) = \frac{H(s)(k_P + k_I \frac{1}{s} + k_D s)}{1 + H(s)(k_P + k_I \frac{1}{s} + k_D s)}$$

Example: P-Controller for second order system

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

The closed loop (controlled) system:

$$H_c(s) = \frac{k_P\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2 + k_P\omega_n^2}$$

Since the original system is type-0, the resulting system will have a constant error term. If k_P is made large there will be small steady state error, but the damping may be much too low.

Numeric illustration on MATLAB/Mathematica.

Incorporating an integrator term.

Now second order systems may now reject constant disturbances since it “upgrades” the system type to 1.

Incorporating a derivative term.

Gives sharp response to suddenly changing signals.

Second order systems may essentially be shaped in any way.

Transient Time Domain Behavior

Profiling the Step Response

Specification for type-1 regulators with respect to "change of reference point":

$$r(t) = \mathbf{1}(t).$$

If (controlled system) is BIBO then, $\lim_{t\to\infty} y(t) = 1$. But how does it get there?

- **Rise time** - The time it takes the system to reach the "vicinity" of the new point: $t_r = \inf\{t : y(t) = 0.9\}$.
- **settling time** - The time it takes the transients to "decay": $t_s = \inf\{t : |y(\tau) - 1| \leq 0.01,\ \forall \tau > t\}$.
- **overshoot** - The maximum amount the system overshoots its final value divided by its final value. If it exists: $M_p = \max\{y(t)\}$.
- **peak time** - The time it takes to reach the maximum overshoot, $t_p = \inf\{t : y(t) = M_p\}$.

There is often a tradeoff between low M_p and low t_r.

Analysis by means of Frequency Response

Bode Plots

See handout.

Resonance

The phenomenon of resonance.

The Nyquist Stability Criterion

Nyquist's Criterion: The Setup

We are interested in analyzed the closed loop transfer function:

$$H_c := \frac{HG_1}{1 + HG_1G_2} = \frac{\frac{N_H}{D_H}\frac{N_1}{D_1}}{1 + \frac{N_H}{D_H}\frac{N_1}{D_1}\frac{N_2}{D_2}} = \frac{N_H N_1 D_2}{D_H D_1 D_2 + N_H N_1 N_2}$$

For stability we need that the **roots** of $D_H D_1 D_2 + N_H N_1 N_2$ are in the LHP as they are the **poles** of the closed loop system.

Yet, Nyquist is best explained (and also holds for non-rational systems) if we take: $G_1 = K$ and $G_2 = 1$. In this case we need to make sure the **zeros** of $1 + KH(s)$ are in the LHP as they are the poles of the closed loop system.

Nyquist assumes that we can (by other means) determine the number of unstable (RHP) **poles** of $H(s)$, hence it should not be hard to also find the **zeros** of $1 + KH(s)$ and thus determine stability. Yet, the (historic) strength of the method is in being able to visualize how changes in the gain K will affect stability.

Nyquist's Criterion: The Setup - cont.

So we have a plant $H(s)$, we know the location (LHP or RHP) of the **poles** of $H(s)$ and we want to see:

- If the zeros of $1 + KH(s)$ are in the LHP and thus $H_c(s)$ is stable.
- How stability will be affected by changes in K.

Nyquist's idea is to graphically use the **principle of the argument**:

Let $R(s)$ be a complex function and $\mathcal{C}$ a closed contour such that there are no poles of $R(\cdot)$ on $\mathcal{C}$. As the complex number s traverses $\mathcal{C}$ once in the clockwise direction:

$$\text{number of clockwise encirclements of R(s) of the origin} = N - P,$$

where N and P are respectively the the number of zeros and number of poles of $R(\cdot)$ inside $\mathcal{C}$.

Interlude: The principle of the argument

Let $f(s)$ be an analytic function inside and on a closed contour $\mathcal{C}$ except for a finite number of poles inside $\mathcal{C}$ Then for $\mathcal{C}$ described in the clockwise direction,

$$E = \frac{1}{2\pi i} \oint \frac{f'(s)}{f(s)} ds = N - P,$$

where E is the number of encirclements of $f(s)$ of the origin, and N and P are respectively the number of zeros and number of poles inside $\mathcal{C}$.

For rational functions - it is easy to see intuitively why $E = N - P$... Start by envisioning a contour with one zero inside it and one outside, and look at the net change in phase of $f(s)$ as it traverses the contour. Then add more zeros and poles to the story.

Interlude: The principle of the argument - cont.

Below is a proof of the first equality $E = \frac{1}{2\pi i}\oint \frac{f'(s)}{f(s)}ds$. The other equality, $\frac{1}{2\pi i}\oint \frac{f'(s)}{f(s)}ds = N - P$, can be obtained by using Cauchy's residue theorem:

$$\frac{1}{2\pi i}\oint g(s)ds = \sum_{i=1}^{n} \mathrm{Res}[g(s); s_i].$$

Let $s(t)$ be a function from $[a, b]$ to the contour $\mathcal{C}$ which parameterizes that curve:

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f'(s)}{f(s)}ds = \frac{1}{2\pi i}\int_a^b \frac{f'(s(t))}{f(s(t))}s'(t)dt = \frac{1}{2\pi i}\int_a^b \Big(\ln(f(s(t)))\Big)'dt$$

$$= \frac{1}{2\pi i}\int_a^b \Big(\ln R_{f(s(t))} + \ln e^{i\Phi(f(s(t)))}\Big)'dt = \frac{\Phi(f(s(b))) - \Phi(f(s(a)))}{2\pi} = E$$

Applying Nyquist

This is how you can apply the Nyquist criterion:

1. Plot $KH(s)$ for $s = i\omega$, $\omega \in [0, \omega_{\max})$.
2. Then revert the plot about the real axis.
3. Evaluate the number of clockwise encirclements of -1 and call that number N. Do this by drawing a straight line in any direction from -1 to ∞ and then counting the net number of left-to-right crossings of the straight line by $KH(s)$. For crossings in the counter-clockwise directions - count as -1.
4. Determine the number of unstable (RHP) poles of $H(s)$ and call that number P.
5. Calculate the number of unstable closed-loop roots, Z:

$$Z = N + P$$

For stability we want $Z = 0$.

Nyquist Example

Example of $H(s) = \frac{1}{s(s+1)^2}$.

Gain and Phase Margins - Revisited

- *gain margin* - The amount that the loop gain can be changed at the frequency at which the phase shift is 180° without reducing the return difference to zero.
- *phase margin* - The amount of phase lag that can be added to the open-loop transfer function, at the frequency at which its maginitude is unity, without making the return difference zero.

Pade Approximations

Pade Approximations

Given a function, $H(s)$ a Pade approximant of order m, n is a rational function,

$$\tilde{H}(s) = \frac{\sum_{i=0}^{m} b_i s^i}{1 + \sum_{i=1}^{n} a_i s^i}$$

such that,

$$H(0) = \tilde{H}(0), \quad H'(0) = \tilde{H}'(0), \quad , \ldots, H^{(m+n)}(0) = \tilde{H}^{(m+n)}(0).$$

This is basically obtained by equating the first $m + n$ Taylor series around $(s = 0)$ coefficients of $H(s)$ with those of $\tilde{H}(s)$ and solving for the sequences a_i and b_i.

Approximation of Pure Delay

In control the archetypal candidate for a Pade approximation is $H(s) = e^{-sT}$ - the pure delay.

For example: A Pade approximation of order $(1,1)$ compares the taylor expansion

$$e^{-sT} = 1 - sT + (sT)^2/2 - (sT)^3/3! + (sT)^4/4! - \dots$$

with,

$$\frac{b_0 s + b_1}{a_0 s + 1} = b_1 + (b_0 - a_0 b_1)s - a_0(b_0 - a_0 b_1)s^2 + a_0^2(b_0 - a_0 b_1)s^3 + \dots$$

And solves for the coefficients to get, $e^{-sT} \approx \frac{1 - Ts/2}{1 + Ts/2}$.

Nonminimum phase systems

Nonminimum phase systems

A system with zeros in the RHP is called **nonminimum phase**.

As an illustration consider the two systems,

$$H_1(s) = 10\frac{s+1}{s+10}, \quad H_2(s) = 10\frac{s-1}{s+10}.$$

Both systems have the same magnitude of frequency response:

$$|H_1(i\omega)| = |H_2(i\omega)|$$

yet as can bee seen through a Bode plot, the phases of the transfer function are drastically different.

These “exotic” systems are the subject of HW 2.