# MATH(4/7)406 (Control Theory)
# Quiz 1 (Unit 0) - August 19, 2014.
**Prepared by Yoni Nazarathy**

Quiz duration: 40 minutes. Total points: 100.

**Name:**
**Student ID:**

**Problem 1 (35pts):** You are throwing a ball into a hoop, one throw after another. The chance of success in each throw is $p \in (0, 1]$. Assume throws are independent. You throw until you get 2 successes. Write an expression (as simple as you can make it) for the mean of the number of throws.

**Answer (with working if needed):**
The number of throws, $X$, has a negative binomial distribution with parameters $m = 2$ and $p$ (as in Exercise 23 in the HW). This means,

$$\begin{aligned}\mathbb{P}(X = k) &= \binom{k-1}{m-1} p^m (1-p)^{k-m}, \quad k = m,\, m+1,\, m+2, \dots \\ &= (k-1)p^2(1-p)^{k-2}, \quad k = 2, 3, \dots\end{aligned}$$

So in principle you can calculate,

$$\mathbb{E}\,[X] = \sum_{k=m}^{\infty} k \binom{k-1}{m-1} p^m (1-p)^{k-m} = \ldots = \frac{m}{p} = \frac{2}{p},$$

although this calculation isn't so simple (especially in a short quiz). For $m = 2$ it is somewhat manageable:

$$\mathbb{E}\,[X] = \sum_{k=2}^{\infty} k(k-1)p^2(1-p)^{k-2} = \ldots = \frac{2}{p}.$$

But here again, this isn't the quickest calculation in the world (effort comparable to computing both the mean and variance of the geometric distribution).

Expressions such as the one above (not evaluating the summation) will yield 30pts. Not a bad deal (except for the frustration during the quiz).

Now here is the real slick way of getting the answer: $X = X_1 + X_2$ where $X_1$ and $X_2$ are independent geometric random variables with success probability $p$. After all, the time till 2 successes can be split up into the time till the first plus the time till the second.

So $\mathbb{E}\,[X] = \mathbb{E}\,[X_1] + \mathbb{E}\,[X_2] = 2\mathbb{E}\,[X_1]$. And the expectation of a geometric random variable (with support $k = 1, 2, \ldots$) is $1/p$, so the result follows. This is the neat way to get the result $2/p$ (and 35pts).

**Problem 2 (35pts):** Let $X$ be an exponential random variable with mean 2. Let $Y = \lceil X \rceil$. The notation $\lceil z \rceil$ implies the smallest integer not less than $z$; e.g. $\lceil 7.4 \rceil = 8$. What is the mean of the random variable $Y$?

**Answer (with working):**

This exercise is very much like Exercise 46. Set $\lambda = 1/2$ (you need to figure out for this that if $X \sim \exp(\lambda)$ then $\mathbb{E}[X] = 1/\lambda$).

$$\begin{aligned}
\mathbb{E}[Y] = \mathbb{E}[\lceil X \rceil] &= \int_0^\infty \lceil x \rceil \lambda e^{-\lambda x}\, dx \\
&= \sum_{k=1}^\infty \int_{k-1}^k k\lambda e^{-\lambda x}\, dx \\
&= \sum_{k=1}^\infty k \Big[ -e^{-\lambda x} \Big]_{x=k-1}^{x=k} \\
&= \sum_{k=1}^\infty k \left( e^{-\lambda(k-1)} - e^{-\lambda k} \right) \\
&= (e^\lambda - 1) \sum_{k=1}^\infty k e^{-\lambda k} \\
&= \frac{e^\lambda}{e^\lambda - 1}
\end{aligned}$$

The summation in the last step is the same summation you need to do when evaluating the mean of a geometric random variable. If last step wasn't carried out, only 5 points are deducted (i.e. a correct answer up to the final summation yields 30 points).

**Problem 3:** Consider a Markov chain, $\{X(t),\, t = 0, 1, 2 \ldots\}$ on state space $\{0, 1\}$. Denote the transition probability matrix by,

$$P = \begin{bmatrix} (1-p) & p \\ q & (1-q) \end{bmatrix},$$

where the parameters $p$ and $q$ are values in the interval $[0, 1]$.

**(5 pts)** For what parameters is $X(\cdot)$ non-periodic?

**Answer:**
As long as $p \neq 1$ or $q \neq 1$ the chain is non-periodic.

**(5 pts)** For what parameters is $X(\cdot)$ irreducible?

**Answer:**
As long as $p \neq 0$ and $q \neq 0$ the chain is irreducible.

**(10pts)** Assume the chain is irreducible and non-periodic.
Write an expression for $\lim_{t\to\infty} \mathbb{P}\big(X(t) = 1\big)$.

**Answer (with working if needed):**

The stationary distribution is,

$$\pi = \Big[\frac{q}{p+q} \quad \frac{p}{p+q}\Big].$$

This is easy to obtain and even easier to check. This was effectively also handled in Exercise 57. So,

$$\lim_{t\to\infty} \mathbb{P}\big(X(t) = 1\big) = \frac{p}{p+q}.$$

**(10pts)** Assume the chain is irreducible and non-periodic.
Denote, $S_n = X(n) + X(2n) + X(3n) + \ldots + X(10n)$.
Write an expression for $\lim_{n\to\infty} \mathbb{P}(S_n = 4)$.

**Answer (with intuitive explanation or working):**

The answer is,

$$\lim_{n\to\infty} \mathbb{P}(S_n = 4) = \binom{10}{4}\Big(\frac{p}{p+q}\Big)^4\Big(\frac{q}{p+q}\Big)^6.$$

What happens is that for $n$ large each $X(kn)$ is approximately Bernoulli with probability $p/(p+q)$ and the time till $(k+1)n$ is large $(n)$ so it is independent (approx.) of $X(kn)$.... Thus we basically have a binomial random variable.

This can be proved rigorously without doing anything fancy, since we have the explicit values of $P_{i,j}^{(n)}$ as in exercise 57. But this is a bit involved for such a short quiz.