MATH4406 Assignment 1

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Unless it's otherwise stated, each exercise uses the definitions and variables currently in use at the point of the exercise in the notes (e.g. when Yoni has defined a random variable X and asked us to show something about X, the same definition of X is implicitly assumed).

Exercise 1. *Part I.* As $\emptyset \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementations, we have $\emptyset^c = \Omega \in \mathcal{F}$.
Part II. Observe firstly that $\cap_i A_i = (\cup_i (A_i^c))^c$ Now, we know that $A_i^c \in \mathcal{F}$ because $\mathcal{F}$ is closed under complementation. Hence $\cup_i (A_i^c) \in \mathcal{F}$, because $\mathcal{F}$ is closed under countable unions. Finally, closure under complementation (again) gives us $\cap_i A_i = (\cup_i (A_i^c))^c \in \mathcal{F}$.

Exercise 2. *Part I.* Observe $A \cap A^c = \emptyset$ and $A \cup A^c = \Omega$, so $1 = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c)$. Rearranging, $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
Part II. Observe $\emptyset = \Omega^c$, so by previous part $\mathbb{P}(\emptyset) = 0$.
Part III. $A_1 \cup A_2 = A_1 \cup (A_2 \setminus (A_2 \cap A_1))$ (and these two sets are now disjoint). Hence $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2 \setminus (A_2 \cap A_1))$. Now $A_2 = (A_2 \cap A_1) \cup (A_2 \setminus (A_2 \cap A_1))$, so $\mathbb{P}(A_2) = \mathbb{P}(A_2 \cap A_1) + \mathbb{P}(A_2 \setminus (A_2 \cap A_1))$. Combining the two, $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_2 \cap A_1)$.

Exercise 4. A and B are independent iff $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. Now by (1.1), $\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\mathbb{P}(B)$. Combining the two, A and B are independent iff $\mathbb{P}(A \mid B)\mathbb{P}(B) = \mathbb{P}(A)\mathbb{P}(B)$, that is $\mathbb{P}(A \mid B) = \mathbb{P}(A)$.

Exercise 5. We are calculating $\mathbb{P}(\text{roll } 6 \mid \text{even}) = \mathbb{P}(\text{roll } 6 \cap \text{even}) / \mathbb{P}(\text{even}) = \frac{1/6 \times 1/2}{1/2}$. Evaluating this, the required probability is $1/3$.

Exercise 8. *Part I.* Recall $F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \le x\})$. Then $\lim_{x \to -\infty} F_X(x) = \lim_{x \to -\infty} \mathbb{P}(\{\omega \in \Omega : X(\omega) \le x\}) = \mathbb{P}(\emptyset)$. Now $\mathbb{P}(\emptyset) = 0$, so $\lim_{x \to -\infty} F_X(x) = 0$
Part II. Similarly $\lim_{x \to \infty} F_X(x) = \lim_{x \to \infty} \mathbb{P}(\{\omega \in \Omega : X(\omega) \le x\}) = \mathbb{P}(\Omega)$. Now $\mathbb{P}(\Omega) = 1$, so $\lim_{x \to \infty} F_X(x) = 1$.
Part III. Consider $x > y$. Then $F_X(x) - F_X(y) = \mathbb{P}(X \le x) - \mathbb{P}(X \le y)$. Reverting to the longhand, we have

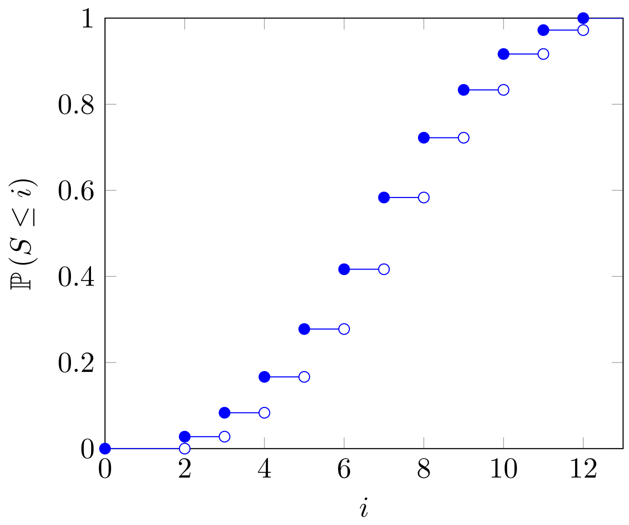
$$\begin{aligned} F_X(x) - F_X(y) &= \mathbb{P}(\{\omega \in \Omega : X(\omega) \le x\} \cap \{\omega \in \Omega : X(\omega) \le y\}) \\ &= \mathbb{P}(\{\omega \in \Omega : y < X(\omega) \le x\}) \ge 0, \end{aligned}$$

because probabilities are positive. Hence $F_X(x) - F_X(y) \ge 0$, and so $F_X(\cdot)$ is non-decreasing.

Exercise 9. We need to calculate the probability of getting each value in $\{2, \dots, 12\}$. We do so by counting the number of ways to get that sum (including order) and that there are 36 possible 2-tuples. For example there is 3 ways to get a sum of 4 ($(1,3)$, $(2,2)$ and $(3,1)$), so the probability of a sum of 4 from to dice rolls is $3/36 = 1/12$. The probabilities are (here S is the sum):

i	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{P}(S = i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36
$\mathbb{P}(S \leq i)$	1/36	3/36	6/36	10/36	15/36	21/36	26/36	30/36	33/36	35/36	36/36
(i.e.)	1/36	1/18	1/6	5/18	5/12	7/12	13/18	5/6	11/12	35/36	1

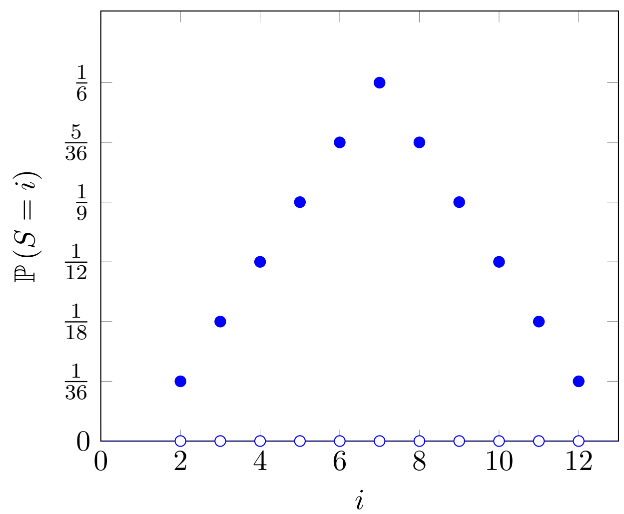
Hence the plot is:



Exercise 10. We have the same probabilities as in Exercise 9:

i	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{P}(S = i)$	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36

Hence the plot is:



Although it is hard to see, there is a blue line running along the x-axis (other than where it is clearly interrupted by open circles).
The CDF increases at the points where the PMF is non-zero. Indeed it increases by exactly the value of the PMF. Equivalently, the CDF is the running "height" of the PMF so far.

Exercise 11. Using the table from Exercise 10, we have $\mathbb{E}[S] = \sum_{i=2}^{12} i\mathbb{P}(S=i) = 7$. (Makes intuitive sense as we have symmetry about 7).

Exercise 12. X is non-negative, so

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=1}^{\infty} kp_X(k) \\ &= \sum_{k=1}^{\infty}\sum_{i=1}^{k} p_X(k) \\ &= \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} p_X(k) \qquad (1) \\ &= \sum_{i=1}^{\infty} \overline{F}_X(x).\end{aligned}$$

By Tonelli's theorem the interchange of order of summation in (1) is justified.

Exercise 13. I'm not quite sure what Yoni meant by illustrate this through the meaning of a random variable, so you can skip marking this question (I've included 41 exercises in total).

Part I. We have $\mathbb{E}[cX] = \sum_{k=-\infty}^{\infty} ckp_X(k) = c\sum_{k=-\infty}^{\infty} kp_X(k)$, via the law of the unconscious statistician. But $\sum_{k=-\infty}^{\infty} kp_X(k) = \mathbb{E}[X]$, so $\mathbb{E}[cX] = c\mathbb{E}[X]$.

Part II. $\mathbb{E}[X+Y] = \sum_{k=-\infty}^{\infty} k\mathbb{P}(X+Y=k) = \sum_{k=-\infty}^{\infty} k\sum_{i=-\infty}^{\infty}\mathbb{P}(X=i, Y=k-i)$, where the second equality is due to the law of total probability. Now, letting $j = k-i$, we have

$$\begin{aligned}\mathbb{E}[X+Y] &= \sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}(i+j)\mathbb{P}(X=i, Y=j) \\ &= \sum_{i=-\infty}^{\infty} i\sum_{j=-\infty}^{\infty}\mathbb{P}(X=i,Y=j) + \sum_{j=-\infty}^{\infty} j\sum_{i=-\infty}^{\infty}\mathbb{P}(X=i,Y=j) \\ &= \sum_{i=-\infty}^{\infty} i\mathbb{P}(X=i) + \sum_{j=-\infty}^{\infty} j\mathbb{P}(Y=j),\end{aligned}$$

where the interchange of summation order is by the Fubini–Tonelli theorem and the last equality is by the law of total probability. Recognising the expectations of X and Y, we have $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

Exercise 14. We have $\mathrm{Var}(c_1X+c_2) = \mathbb{E}\left[(c_1X+c_2)^2\right] - (\mathbb{E}[c_1X+c_2])^2$, but from Exercise 13 we know $\mathbb{E}[c_1X+c_2] = c_1\mathbb{E}[X]+c_2$. Now expanding the squares,

$$\begin{aligned}\mathrm{Var}(c_1X+c_2) &= \mathbb{E}\left[c_1^2X^2 + 2c_1c_2X + c_2^2\right] - \left(c_1^2\mathbb{E}[X]^2 + 2c_1c_2\mathbb{E}[X] + c_2^2\right) \\ &= c_1^2\mathbb{E}\left[X^2\right] + 2c_1c_2\mathbb{E}[X] + c_2^2 - c_1^2\mathbb{E}[X]^2 - 2c_1c_2\mathbb{E}[X] - c_2^2 \\ &= c_1^2\left(\mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2\right) \\ &= c_1^2\,\mathrm{Var}(X)\end{aligned}$$

Hence, $\mathrm{Var}(c_1X+c_2) = c_1^2\,\mathrm{Var}(X)$.

Exercise 17. We will show the sum is 1 in the reverse direction using the Binomial Theorem. We have,

$$\begin{aligned} 1 &= 1^n \\ &= (p + (1-p))^n \\ &= \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \qquad (2) \\ &= \sum_{i=0}^{n} \mathbb{P}(X = i), \end{aligned}$$

where (2) is by the Binomial Theorem. Hence, $\sum_{i=0}^{n} \mathbb{P}(X = i) = 1$.

Exercise 18. *Part I.* Let $(X_i, i = 1, \dots n)$ be the value of the ith trial (1 for success, 0 for failure), so that $X = \sum_{i=1}^{n} X_i$. Then $\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n}(1p + 0(1-p))$, because the X_i are iid Bernoulli trials. Hence $\mathbb{E}[X] = \sum_{i=1}^{n} p = np$.
Part II. The distribution of Z is binomial with parameter $(1-p)$ (and n trials). To see this observe that if we let $Z_i = 1 - X_i$ (where X_i are as above), then $Z = n - X = n - \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Z_i$. Now Z_i are iid Bernoulli random variables with probability $(1-p)$ of success, so $Z \sim \mathsf{Bin}(n, 1-p)$.

Exercise 19. Assuming that all answers are equally likely, the probability of success is $\frac{1}{4}$. Let X be the number of correct questions. Then $X \sim \mathsf{Bin}(20, 1/4)$, so the probability of getting 10 or more correct answers is given by $\mathbb{P}(X \geq 10) \sum_{i=10}^{20} \binom{20}{i} (1/4)^i (3/4)^{20-i}$. Evaluating this, we have $\mathbb{P}(X \geq 10) \approx 0.01386$.

Exercise 20. To get our first success at exactly the kth trial, we must have all of the first $k-1$ trials fail (which happens with probability $(1-p)^{k-1} = \prod_{i=1}^{k-1}(1-p)$, recalling each trial is independent), *and* the kth trial must be a success (which happens with probability p). Combining these, the probability our first success is the kth trial (i.e. $\mathbb{P}(X = k)$) is $(1-p)^{k-1}p$. So $\mathbb{P}(X = k) = (1-p)^{k-1}p$.

Exercise 21. We need *all* twenty rides to not have a flat tire. If the probability of a flat tire is 0.01, then the probability of not getting a flat tire is 0.99. Hence the probability of not getting a flat tire in 20 consecutive rides (assuming all bike rides are independent) is $0.99^{20} \approx 0.8179$.

Exercise 22. Let Y be the number of failures until success. Then $Y = X - 1$ (where X is the number of trial until success). Hence Y has support over $\{0, 1, \dots, \infty\}$. Further the distribution of Y is given by $\mathbb{P}(Y = k) = \mathbb{P}(X = k + 1) = (1-p)^k p$.

Exercise 26. We have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
&= \lambda e^{-\lambda} e^{\lambda} \qquad (3) \\
&= \lambda,
\end{aligned}$$

where we have used the Taylor series definition of the exponential in (3). Hence $\mathbb{E}[X] = \lambda$. Now,

$$\begin{aligned}
\mathbb{E}\left[X^2\right] &= \sum_{k=0}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} \\
&= \lambda e^{-\lambda} \left(\sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} \right) \\
&= \lambda e^{-\lambda} \left(\sum_{k=2}^{\infty} (k-1) \frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right) \\
&= \lambda e^{-\lambda} \left(\lambda \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right) \\
&= \lambda e^{-\lambda} \left(\lambda \sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!} + \sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!} \right) \\
&= \lambda(\lambda + 1) \qquad (4) \\
&= \lambda^2 + \lambda. \qquad (5)
\end{aligned}$$

Recalling that $\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$, we have $\mathrm{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$.

Exercise 27. We have

$$\lim_{n\to\infty} \binom{n}{k} (\frac{\lambda}{n})^k (1 - \frac{\lambda}{n})^{n-k} = \left(\lim_{n\to\infty} \lambda^k \right) \left(\lim_{n\to\infty} (1 - \frac{\lambda}{n})^{-k} \right) \left(\lim_{n\to\infty} (1 - \frac{\lambda}{n})^{n} \right) \left(\lim_{n\to\infty} \binom{n}{k} \frac{1}{n^k} \right).$$

Now we must simply evaluate each factor. Clearly $\lim_{n\to\infty} \lambda^k = \lambda^k$ and $\lim_{n\to\infty}(1 - \frac{\lambda}{n})^{-k} = 1$. It is well known that $\lim_{n\to\infty}(1 + \frac{-\lambda}{n})^n = e^{-\lambda}$ (the exponential function is sometimes defined this way). Finally,

$$\begin{aligned}
\lim_{n\to\infty}\binom{n}{k}\frac{1}{n^k} &= \lim_{n\to\infty}\frac{\prod_{i=0}^{k+1}(n-i)}{k!n^k} \\
&= \lim_{n\to\infty}\frac{\prod_{i=0}^{k+1}(n-i)}{k!n^k} \\
&= \lim_{n\to\infty}\frac{n^k+O(n^{k-1})}{k!n^k} \\
&= \lim_{n\to\infty}\frac{n^k}{k!n^k}+\lim_{n\to\infty}\frac{O(n^{k-1})}{k!n^k} \\
&= \frac{1}{k!}+0.
\end{aligned}$$

Now combining each of these factors, we have

$$\lim_{n\to\infty}\binom{n}{k}(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k} = \lambda^k \times 1 \times e^{-\lambda} \times \frac{1}{k!} = e^{-\lambda}\frac{\lambda^k}{k!},$$

as required.

Exercise 28. We have $\mathbb{E}[X] = \sum_{k=1}^{\infty}\frac{k}{k(k+1)} = \sum_{k=1}^{\infty}\frac{1}{k+1}$. Now letting $i = k+1$, $\mathbb{E}[X] = \sum_{i=2}^{\infty} 1/i = \sum_{i=0}^{\infty} 1/i - 3/2$. It is well known that $\sum_{i=0}^{\infty} 1/i$, i.e. the harmonic series, diverges to infinity, and so it is clear that $\mathbb{E}[X]$ also diverges to infinity.

Exercise 29. We have $\mathbb{P}(X=k) = \sum_{l=-\infty}^{\infty}\mathbb{P}(X=k, Y=l)$ (via the law of total probability, provided X and Y are discrete random variables). Matching with the form of the law of total probability given in the notes, we've used $A = \{\omega\in\Omega : X(\omega) = k\}$, and $B_l = \{\omega\in\Omega : X(\omega) = l\}$, for $l \in \{0, 1, \ldots\}$. Now substituting $p_X(k) = \mathbb{P}(X=k)$ and $p_{X,Y}(k,l) = \mathbb{P}(X=k, Y=l)$, we have $p_X(k) = \sum_{l=-\infty}^{\infty} p_{X,Y}(k,l)$, as required.

Exercise 30. Now,

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] \\
&= \mathbb{E}[XY - \mathbb{E}[X]\,Y - \mathbb{E}[Y]\,X + \mathbb{E}[X]\,\mathbb{E}[Y]] \\
&= \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] - \mathbb{E}[Y]\,\mathbb{E}[X] + \mathbb{E}[X]\,\mathbb{E}[Y] \\
&= \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y],
\end{aligned}$$

as required. Here we've used that $\mathbb{E}[aZ] = a\mathbb{E}[Z]$ for any constant a, and that $\mathbb{E}[Z]$ is a constant for any random variable Z.

Exercise 31. $\mathbb{E}[XY] = \sum i = -\infty^{\infty}\sum j = -\infty^{\infty} ijp_{X,Y}(i,j)$. But for X and Y independent, $p_{X,Y}(i,j) = p_X(i)p_Y(j)$, so $\mathbb{E}[XY] = \sum i = -\infty^{\infty}\sum j = -\infty^{\infty} ijp_X(i)p_Y(j)$. Separating the sums, $\mathbb{E}[XY] = \sum i = -\infty^{\infty} ip_X(i)\sum j = -\infty^{\infty} jp_Y(j) = \mathbb{E}[X]\,\mathbb{E}[Y]$. Now substituting this into the previous result, $\mathrm{Cov}(X,Y) = \mathbb{E}[X]\,\mathbb{E}[Y] - \mathbb{E}[X]\,\mathbb{E}[Y] = 0$.

Exercise 32. *Part I.* Consider $p_{X,Y}$ as given by Table 1. Then $\mathbb{E}[XY] = 11/4$, $\mathbb{E}[X] = 7/4$ and $\mathbb{E}[Y] = 3/2$. Hence $\mathrm{Cov}(X,Y) = 11/4 - 21/8 = 1/8 \neq 0$.

	y		
	$p_{X,Y}(x,y)$	1	2
x	1	1/4	1/4
	2	1/4	0
	3	0	1/4

Table 1: $\mathrm{Cov}(X,Y) \neq 0$.

Part II. Consider $p_{X,Y}$ as given by Table 2. Then $\mathbb{E}[XY] = 13/3$, $\mathbb{E}[X] = 13/5$ and $\mathbb{E}[Y] = 5/3$. Hence $\mathrm{Cov}(X,Y) = 13/3 - 13/3 = 0$. However, $p_X(2)p_Y(1) = 2/15 \times 1/3 = 2/45 \neq 0 = p_{X,Y}(2,1)$, so X and Y are not independent.

	y		
	$p_{X,Y}(x,y)$	1	2
x	1	1/15	1/15
	2	0	2/15
	3	4/15	7/15

Table 2: $\mathrm{Cov}(X,Y) = 0$, but X and Y not independent.

Exercise 33. Firstly, $p_{X|Y=l}(k,l) \geq 0$, because $p_{X,Y}(k,l) \geq 0$ and $p_Y(l) > 0$. Further, we have

$$\sum_{k=-\infty}^{\infty} p_{X|Y=l}(k,l) = \frac{\sum_{k=-\infty}^{\infty} p_{X,Y}(k,l)}{p_Y(l)} = \frac{p_Y(l)}{p_Y(l)} = 1$$

(recalling that $p_Y(l) > 0$ so we are not dividing by 0 ever). Hence $p_{X|Y=l}(\cdot, l)$ is a valid PMF.

Exercise 34. We have $p_{X|Y=l}(k,l) = \frac{p_{X,Y}(k,l)}{p_Y(l)}$. But if X and Y are independent, then $p_{X,Y}(k,l) = p_X(k)p_Y(l)$, so $p_{X|Y=l}(k,l) = \frac{p_X(k)p_Y(l)}{p_Y(l)} = p_X(k)$.

Exercise 38.

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[h(X) \,|\, Y]] &= \sum_{l=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} h(k) p_{X|Y=l}(k,l) \right] p_Y(l) \\
&= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h(k) p_{X,Y}(k,l) \\
&= \sum_{k=-\infty}^{\infty} h(k) \sum_{l=-\infty}^{\infty} p_{X,Y}(k,l) \\
&= \sum_{k=-\infty}^{\infty} h(k) p_X(k) \\
&= \mathbb{E}[h(X)]\,,
\end{aligned}$$

where once again the swapped order of summation is valid by the Fubini–Tonelli theorem.

Exercise 40. *Part I.* $\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = 1$.
Part II. The CDF is $\int_{-\infty}^{x} f_X(x)\,\mathrm{d}x$. This CDF is continuous provided the random variable X does not have positive probability of falling on any specific point. In other words, if X is a

purely continuous function with no discrete parts. The PDF does not have to be continuous (or even exist), see e.g. the Cantor distribution.
Part III. As well as being non-negative, there must exist some set $A \subset \mathbb{R}$ such that $\tilde{f}(x) > 0$ for all $x \in A$ (i.e. $\tilde{f} \not\equiv 0$). Then $f_X(x) = K\tilde{f}(x)$ will be a density for $K = \left(\int_{-\infty}^{\infty} \tilde{f}(x)\,\mathrm{d}x\right)^{-1}$. That is K is chosen as a normalising constant so that the integral of f_X over $\mathbb{R}$ is equal to 1.

Exercise 41. For X uniform on $[a,b]$, $\mathbb{E}[X] = \int_a^b \frac{x}{b-a}\,\mathrm{d}x = \frac{(a+b)(a-b)}{2(a-b)} = \frac{a+b}{2}$. Intuitively, $\mathbb{E}[X]$ is the midpoint of the interval. Now

$$\mathbb{E}\left[X^2\right] = \int_a^b \frac{x^2}{b-a}\,\mathrm{d}x = \frac{b^3-a^3}{3(b-a)}$$

, so

$$\begin{aligned}
\mathrm{Var}(X) &= \frac{b^3-a^3}{3(b-a)} - \frac{a^2+2ab+b^2}{4} \\
&= \frac{4b^3-4a^3-3(a^2b+2ab^2+b^3-a^3-2a^2b-b^2a)}{12(b-a)} \\
&= \frac{b^3-a^3-3ab^2+3a^2b}{12(b-a)} \\
&= \frac{(b-a)^3}{12(b-a)} \\
&= \frac{(b-a)^2}{12}.
\end{aligned}$$

Exercise 42. For $X \sim \mathsf{Unif}(a,b)$, the CDF of X is given by

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } x \in [a,b] \\ 1 & \text{if } x > b. \end{cases}$$

Exercise 46. $\mathbb{P}(Y = y) = \lambda \int_{\lfloor y \rfloor}^{\lfloor y \rfloor + 1} e^{-\lambda x}\,\mathrm{d}x = e^{-\lambda \lfloor y \rfloor} - e^{-\lambda(\lfloor y \rfloor + 1)}$, for all $y \in \{0,1,\ldots\}$ (and $\mathbb{P}(Y = y) = 0$ otherwise).

Exercise 51. The code used was:

```
n = 10000; X = rand(n,1) < 0.25;
xbar = sum(X)/n;
fprintf('sample mean: %f, sample var: %f\n', xbar, ...
    sum((X - xbar).^2)/n)
%not sure if we could use mean() and var(), so demonstrated
%my ability to do without here - will use them for the other
%questions.
```

The results are sample mean, 0.2546, and sample variance, 0.1898. The values are both within 0.01 of the theoretical values (0.25 and 0.1875 respectively).

Exercise 52. $F(x) = 1 - e^{-\lambda x}$, so $F^{-1}(u) = \frac{-1}{\lambda}\ln(1-u)$. Notice that since we will be drawing u from a uniform distribution on $[0,1]$, we can equivalently transform by $\frac{-1}{\lambda}\ln(u)$. The code used:

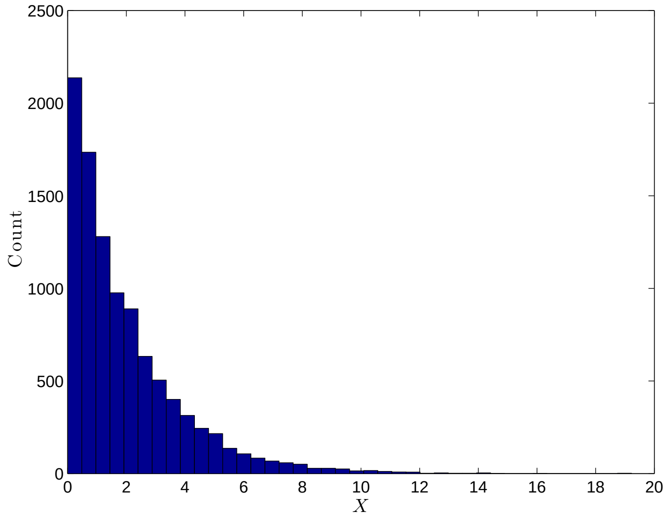


Figure 1: Histogram of simulated exponential random variables.

```
n = 10000;   X = -log(rand(n,1))/(1/2);
fprintf('sample mean: %f, sample var: %f\n', mean(X), var(X,1))
hist(X, 40);
xlabel('$X$', 'interpreter','latex', 'fontsize',14,'fontweight','bold')
ylabel('Count', 'interpreter','latex', 'fontsize',14,'fontweight','bold')
set(gca, 'fontsize',12)

filename = 'Ex52.pdf'; fig = gcf;
set(fig, 'Units', 'centimeters');
set(fig, 'PaperUnits','centimeters');
pos = get(fig, 'Position');
set(fig, 'PaperSize', pos(3:4));
set(fig, 'PaperPositionMode', 'auto');
print('-dpdf',filename)
```

The histogram is given in Figure 1. The sample mean was 1.9725, and the sample variance was 3.9078. This is once again quite close to the theoretical values of 2 and 4 respectively.

Exercise 53. The sample frequencies are (in the same order) $[0.3495, 0.2561, 0.1016, 0.2928]$. Once again these are quite close to the theoretical values (all within 0.01). The code use is:

```
%create an anonymous function which takes parameters p and n and
%returns n values drawn from the distribution p (or if p is not a
%proper distribution, then the distribution arising from
%snormalising p).
gen_rand = @(p, n)(sum(bsxfun(@gt, rand(n, 1),...
    reshape(cumsum(p)/sum(p), [1, length(p)])), 2) + 1);
n = 10000;
X = gen_rand([0.35, 0.25, 0.1, 0.3], n);
counts = sum(bsxfun(@eq, X, [1, 2, 3, 4]), 1)/n;
disp(counts);
```

Exercise 56. To ensure an iid sequence we need transitions to be independent of the state we are currently in (all values in each column are equal). We also need it to be equally likely to transition to each state (all values in each row are the same). Together these say that all entries must be the same. Now combining this with the knowledge rows must sum to 1, we know we should use the matrix P such that every entry is $1/N$.

Exercise 58. The transition probabilities are given by

$$p_{ij} = \begin{cases} 1 & \text{if } i \in \{0, L\} \wedge j = i \\ 0 & \text{if } i \in \{0, L\} \wedge j \neq i \\ p & \text{if } i \notin \{0, L\} \wedge j = i + 1 \\ 1 - p & \text{if } i \notin \{0, L\} \wedge j = i - 1. \\ 0 & \text{if } i \notin \{0, L\} \wedge j \notin \{i - 1, i + 1\} \end{cases}$$

Exercise 63. Let S be the set of states. Consider states i, j, k such that $i \leftrightarrow j$ and $j \leftrightarrow k$. Then we want to show $i \leftrightarrow k$ (i.e. transitivity). Now, $i \to j \implies \exists t_1 : p_{ij}^{(t_1)} > 0$, and likewise $j \to k \implies \exists t_2 : p_{jk}^{(t_2)} > 0$. Chapman-Kolmogorov say $p_{ik}^{(t_1+t_2)} = \sum_{l \in S} p_{il}^{(t_1)} p_{lk}^{(t_2)} \geq p_{ij}^{(t_1)} p_{jk}^{(t_2)} > 0$ (because all values of the summand are non-negative). Hence $\exists t : p_{ik}^{(t)} > 0$ (one such t is $t = t_1 + t_2$), and so $i \to k$.
The same argument applies in reverse to show $i \leftarrow k$. We have $i \leftarrow j \implies \exists t_3 : p_{ji}^{(t_3)} > 0$, and likewise $j \leftarrow k \implies \exists t_4 : p_{kj}^{(t_4)} > 0$. So $p_{ki}^{(t_4+t_3)} = \sum_{l \in S} p_{kl}^{(t_4)} p_{li}^{(t_3)} \geq p_{kj}^{(t_4)} p_{ji}^{(t_3)} > 0$. Hence $\exists t : p_{ki}^{(t)} > 0$ (one such t is $t = t_4 + t_3$), and so $i \leftarrow k$. So $i \to k$ and $i \leftarrow k$, i.e. $i \leftrightarrow k$, and we have proved transitivity.

Exercise 65. *Part I.* Assume the states are $\{1, 2, 3\}$. Then the equivalence classes are $\{1\}$ and $\{2, 3\}$.
Part II. States 2 and 3 are recurrent because once we enter the subchain defined by 2 and 3, we can't leave it, we simply move between 2 and 3 visiting each infinitely often. State 1 is transient - once we leave state 1 (which happens with probability 0.7 each jump) we can never return. The next part also shows that $f_{11} < 1$ and $f_{22} = f_{33} = 1$.
Part III. This uses the equation derived by first step analysis on page 26. Firstly observe that $i \not\to 1$ for $i = 2, 3$, so $f_{i1} = 0$ for $i = 2, 3$. Now,

$$\begin{aligned}
f_{11} &= (p_{12} f_{21} + p_{13} f_{31}) + p_{11} = 0.3, \\
f_{12} &= (p_{11} f_{12} + p_{13} f_{32}) + p_{12} = 0.3 f_{12} + 0 + 0.7 \implies 0.7 f_{12} = 0.7 \implies f_{12} = 1, \\
f_{13} &= (p_{11} f_{13} + p_{12} f_{23}) + p_{13} = 0.3 f_{13} + 0.7 f_{23} \implies f_{13} = f_{23}, && (6) \\
f_{22} &= (p_{21} f_{12} + p_{23} f_{32}) + p_{22} = 0 + 0.5 f_{32} + 0.5, && (7) \\
f_{23} &= (p_{22} f_{23} + p_{21} f_{13}) + p_{23} = 0.5 f_{23} + 0 + 0.5 \implies f_{23} = 1, && (8) \\
f_{32} &= (p_{33} f_{32} + p_{31} f_{12}) + p_{32} = 0.5 f_{32} + 0 + 0.5 \implies f_{32} = 1, && (9) \\
f_{33} &= (p_{32} f_{23} + p_{31} f_{13}) + p_{33} = 0.5 f_{23} + 0 + 0.5. && (10)
\end{aligned}$$

Equations (8) and (6) imply that $f_{13} = 1$, (8) and (10) imply that $f_{33} = 1$ and (9) and (7) imply that $f_{22} = 1$. Hence we have $f_{11} = 0.3$, $f_{21} = f_{31} = 0$, and $f_{12} = f_{13} = f_{22} = f_{23} = f_{32} = f_{33} = 1$.

Exercise 68. Consider a finite state DTMC run for an infinite number of steps. If the finite state DTMC has no recurrent state then the DTMC visits all states (of which there are finitely many) only finitely often. Hence it only makes a finite number of steps. We have a contradiction, and so there must be at least one recurrent state (this state(s) "soaks up" the infinite number of steps).

Exercise 73. We may (equivalently) represent our sets as a binary (ordered) tuple of length $|\mathcal{A}|$, where the ith element of $\mathcal{A}$ is in the set iff the ith entry of the tuple is 1. It is clear there is a bijective mapping between this representation and the sets. As such the number of sets in $2^{\mathcal{A}}$ is the number of possible tuples. There are 2 choices for each of the $|\mathcal{A}|$ entries in the tuple so there are $2^{|\mathcal{A}|}$ possible tuples (and hence $|2^{\mathcal{A}}| = 2^{|\mathcal{A}|}$).

Exercise 75. A one-to-one mapping $f : \mathbb{Z}_+ \mapsto \mathbb{N}$ is $f(i) = f(i) + 1$, hence $\mathbb{Z}_+$ is a countably infinite set. A one-to-one mapping $g : \mathbb{Z} \mapsto \mathbb{N}$ is

$$g(i) = \begin{cases} 2n & \text{if } n > 0 \\ -2n+1 & \text{if } n \leq 0, \end{cases}$$

and so $\mathbb{Z}_+$ is a countably infinite set. (To see this mapping is one-to-one notice that non-positive number go to unique odd natural numbers while positive numbers go to unique even natural numbers.)

Finally, an injective mapping from $\mathbb{Q}$ to $\mathbb{Z}$ is defined by

$$h!\left(\frac{p}{q}\right) = \operatorname{sgn}\left(\frac{p}{q}\right) 2^{|p|} 3^{|q|},$$

where p and q are coprimes (so that the rational is in simplest form). Then by the properties of mappings, $g \circ f$ is an injective function from $\mathbb{Q}$ to $\mathbb{N}$, so Q is countable. Now, $\mathbb{N} \subsetneq \mathbb{Q}$ so the cardinality of $\mathbb{Q}$ is at least that of $\mathbb{N}$. Hence $\mathbb{Q}$ is both countable and infinite, i.e. countably infinite.