Ex 5

Let X be the outcome of rolling the die. Denote $X=i$ by X_i

$$\mathbb{P}(X_6 \mid X_2 \cup X_4 \cup X_6)$$

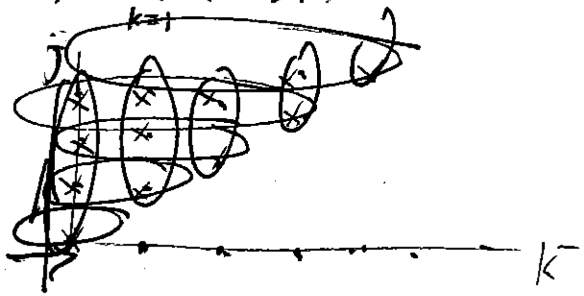
$$= \frac{\mathbb{P}(X_6 \cap (X_2 \cup X_4 \cup X_6))}{\mathbb{P}(X_2 \cup X_4 \cup X_6)}$$

$$= \frac{1/6}{1/2} = \frac{1}{3}$$

Ex 12

$$\sum_{k=0}^{\infty} \overline{F}_X(k) = \sum_{k=0}^{\infty} \mathbb{P}(X>k) = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k)$$

$$= \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \mathbb{P}(X=j)$$



$$= \sum_{j=1}^{\infty} \sum_{k=1}^{j} \mathbb{P}(X=j)$$

$$= \sum_{j=1}^{\infty} \mathbb{P}(X=j) \sum_{k=1}^{j} 1 = \sum_{j=1}^{\infty} j\,\mathbb{P}(X=j) = \sum_{j=0}^{\infty} j\,\mathbb{P}(X=j)!$$

Ex 22

$X \sim \text{geometric}(p) :=$ # of trials until success

$Y \sim \widetilde{\text{geometric}}(p) :=$ # of failures until success

- Stop after first success in both!

$$\underbrace{\underbrace{F\ F\ F}_{Y}\ S}_{X}$$

$$Y = X - 1$$

$$\text{Support} = \{0, 1, 2, \dots\}$$

$$P(Y=k) = P(X-1=k) = \mathbb{P}(X=k+1)$$

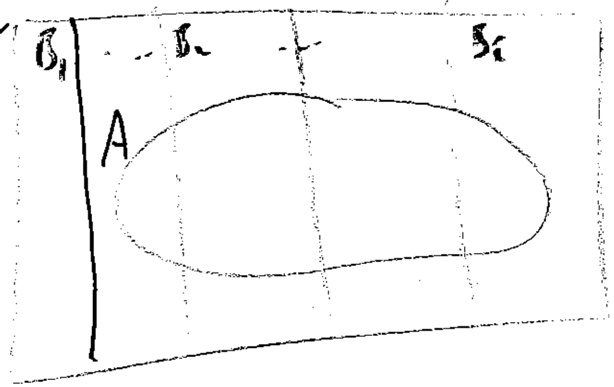
$$= (1-p)^{(k+1)-1} p = \underline{(1-p)^k p}$$

3, 7, 11, 15, 19, 23, 27, 31, 35, 39

Ex 6

(i) Rearrange (1.1) for $P(A \cap B) = P(B)P(A \mid B)$

(ii) Ω



Note $A = \bigcup_i (A \cap B_i)$

and $A \cap B_i$ disjoint

$$\mathbb{P}(A) \overset{P4}{=} \sum_i P(A \cap B_i) \overset{(i)}{=} \sum_i P(A \mid B_i) P(B_i)$$

$\square$

Ex 70

1.) Remember from Ex 54.5

$$r(t) = r(0) P^t$$

so if $r(0) = \pi$

and $\pi = \pi P$

then ~~r(t) =~~ $r(1) = \pi$

and $r(t) = \pi$ also.

2) $\mathbb{P}(r(0) = i, r(t) = j)$

$= \mathbb{P}(r(0) = i)\, \mathbb{P}(r(t) = j \mid r(0) = i)$

$= \pi_i\, e_i [P^t]_j$

and $\mathbb{P}(r(\tau) = i, r(\tau + t) = j)$

$= \mathbb{P}(r(\tau) = i)\, \mathbb{P}(r(\tau + t) = j \mid r(\tau) = i))$

$= \pi_i\, e_i [P^t]_j$ $\qquad e_i = [0, 0, \dots, \overset{\text{ith element}}{1}, 0, 0]$

Ex 66

$$P = \begin{bmatrix} 0.1 & 0.7 & 0.2 & 0 \\ 0.4 & 0.3 & 0 & 0.3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$f_{ij} = \sum_{k \neq j} p_{ik} f_{kj} + p_{ij}$$

$$f_{13} = \sum_{k \neq 3} p_{1k} f_{kj} + p_{13}$$

$$= p_{11} f_{13} + p_{12} f_{23} + p_{13}$$

$$= 0.1 f_{13} + 0.7 f_{23} + 0.2$$

$$f_{23} = \sum_{k \neq 3} p_{2k} f_{k3} + p_{23}$$

$$= p_{21} f_{13} + p_{22} f_{23} + p_{24} f_{243} + p_{23}$$

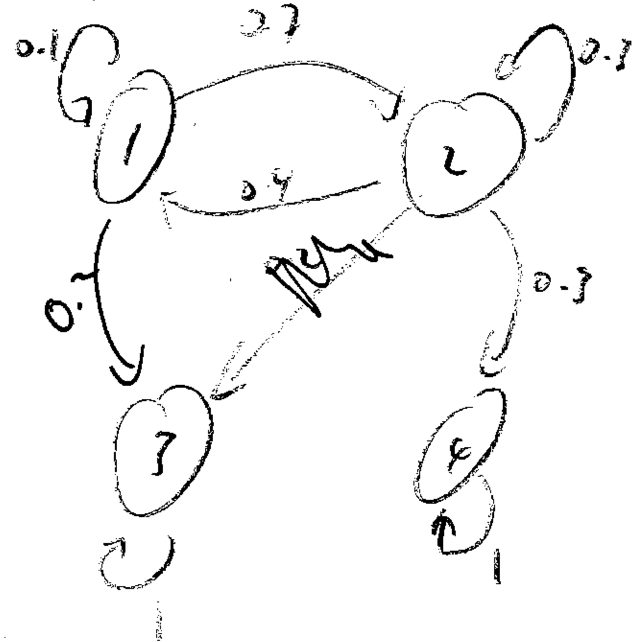
$$= 0.4 f_{13} + 0.3 f_{23} + 0.3 f_{24}$$

$$f_{24} = \sum_{k \neq 4} p_{2k} f_{k4} + p_{24}$$

$$= \cancel{p_{21} f_{14}} +$$

You get the idea!

$f_{13} + f_{14}$



Must be absorbed in 3 or 4.

	Y: 1	2
X: 1	p_{11}	p_{12}
2	p_{21}	p_{22}
3	p_{31}	p_{32}

Ex 30

$$\mathrm{Cov}(X,Y) = \mathbb{E}\left[(X-\mathbb{E}X)(Y-\mathbb{E}Y)\right]$$
$$= \mathbb{E}\left[XY - X\mathbb{E}Y - Y\mathbb{E}X + \mathbb{E}X\mathbb{E}Y\right]$$
$$= \mathbb{E}[XY] - \mathbb{E}[X\mathbb{E}Y] - \mathbb{E}[Y\mathbb{E}X] + \mathbb{E}X\mathbb{E}Y$$
$$= \mathbb{E}XY - \mathbb{E}Y\mathbb{E}X - \mathbb{E}X\mathbb{E}Y + \mathbb{E}X\mathbb{E}Y$$
$$= \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y$$

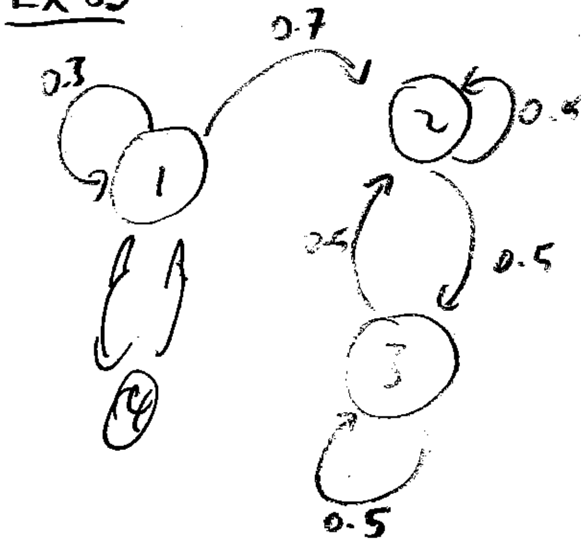
Remember: $\mathrm{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2$

Ex 46

$$\mathbb{P}(Y=k) = \mathbb{P}(k \leq X < k+1)$$
$$= \mathbb{P}(X < k+1) - \mathbb{P}(X \leq k)$$
$$= F_X(k+1) - F_X(k)$$
$$= (1-e^{-\lambda(k+1)}) - (1-e^{-\lambda k})$$
$$= -e^{-\lambda k}e^{-\lambda} + e^{-\lambda k} = e^{-\lambda k}(1-e^{-\lambda})$$
$$= p^k(1-p) \quad \text{with } p = e^{-\lambda}$$

i.e. geometric with fail probability $e^{-\lambda}$

Ex 65



Classes: $\{1\}, \{2,3\}$

Transient: $\{1\}$

Recurrent: $\{2,3\}$

$f_{ij} :=$ probability of ever going state j given start in state i

$$F = \begin{bmatrix} 0.3 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$